CMSC427/828E Spring 2000
Homework # 5

March 15, 2000

Assume the following notation: R_θ means rotation counterclockwise with angle θ, T_i means translation to the location (x_i, y_i), T_{i+j} means translation to the location of $(x_i + x_j, y_i + y_j)$, $T_{(x,y)}$ means translation to the location (x, y), S_c means uniform scaling with factor c, and $S_{(c_1,c_2)}$ means that non-uniform scaling with c in the x direction, and c_2 in the y direction. This notation will be used through out the homework.

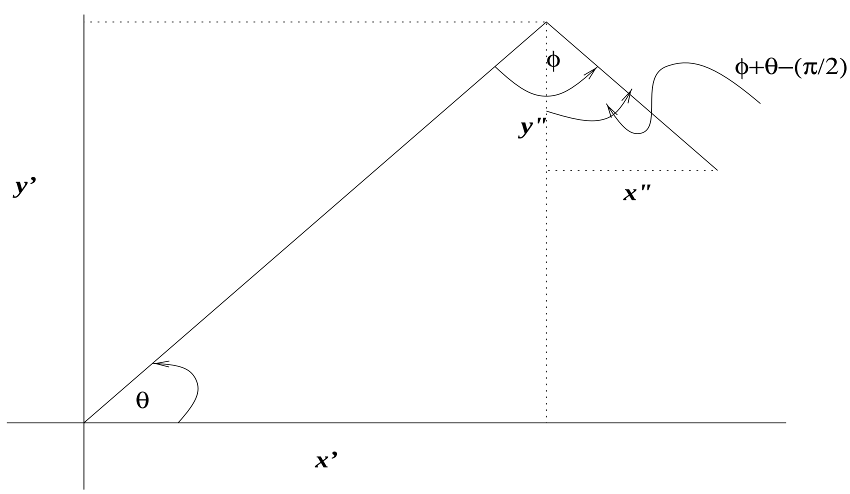

Figure 1: Problem 1 Outline

1.

$$\begin{aligned} x &= x' + x'' \\ &= a_1 \cos\theta + a_2 \sin(\phi + \theta - \frac{\pi}{2}) \\ &= a_1 \cos\theta - a_2 \cos(\phi + \theta) \qquad (1) \\ y &= y' - y'' \\ &= a_1 \sin\theta - a_2 \cos(\phi + \theta - \frac{\pi}{2}) \\ &= a_1 \sin\theta - a_2 \sin(\phi + \theta) \qquad (2) \end{aligned}$$

The previous method is straightforward, but as this is a graphics course, it is better to solve using transformations. It is a transformation of the origin to the point P.

$$\begin{aligned} P &= R_\theta T_{(a_1,0)} R_\phi T_{(-a_2,0)} \hat{O} \\ &= \begin{bmatrix} a_1 \cos\theta - a_2 \cos(\phi+\theta) \\ a_1 \sin\theta - a_2 \sin(\phi+\theta) \\ 1 \end{bmatrix} \end{aligned}$$

2. (a)

$$\begin{aligned} R_\theta S_a &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a\cos\theta & -a\sin\theta & 0 \\ a\sin\theta & a\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3) \\ S_a R_\theta &= \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a\cos\theta & -a\sin\theta & 0 \\ a\sin\theta & a\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (4) \end{aligned}$$

From equation 3 and equation 4, we reach $R_\theta S_a = S_a R_\theta$. Thus, uniform scaling, and rotation are commutative.

(b) Now consider two rotations around θ and ϕ respectively,

$$\begin{aligned} R_\phi R_\theta &= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta & -(\sin\phi\sin\theta - \cos\phi\cos\theta) & 0 \\ \sin\phi\cos\theta - \sin\theta\sin\phi & \cos\phi\cos\theta - \sin\phi\sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) & 0 \\ \sin(\phi+\theta) & \cos(\phi+\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= R_{(\phi+\theta)} \qquad (5) \end{aligned}$$

Since addition is commutative $\Longrightarrow R_{(\phi+\theta)} = R_{(\theta+\phi)}$.

$$R_\phi R_\theta = R_\theta R_\phi \qquad (6)$$

(c) Now consider two translation T_1 followed by T_2 :

$$\begin{aligned} T_2T_1 &= \begin{bmatrix} 1 & 0 & x2 \\ 0 & 1 & y2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & x1 \\ 0 & 1 & y1 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & x_2 + x_1 \\ 0 & 1 & y_2 + y_1 \\ 0 & 0 & 1 \end{bmatrix} \\ &= T_{2+1} \end{aligned} \tag{7}$$

Since addition is commutative $\Longrightarrow T_{2+1} = T_{1+2}$.

$$T_2T_1 = T_1T_2 \tag{8}$$

3. In order to reflect along line $y = mx + h$, We need to do the following: $T_{(0,h)} \Rightarrow R_{-\theta} \Rightarrow reflect \Rightarrow R_\theta \Rightarrow T_{(0.-h)}$, where $\theta = \arctan m$. Reflection is done by $S_{(1,-1)}$.

$$\begin{aligned} P_{ref} &= T_{(0,-h)} R_\theta S_{(1,-1)} R_{-\theta} T_{(0,h)} P \\ &= \begin{bmatrix} \cos(2\theta) & \sin(2\theta) & h\sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) & -h(1+\cos(2\theta)) \\ 0 & 0 & 1 \end{bmatrix} P \end{aligned}$$

There are many other solutions. For example, one can calculate the perpendicular to the line, and then double the distance of that perpendicular.

4. We need two functions, one to check the collinearity of three points, wheras the other checks if there are any three points in a set of points which are collinear.

```
boolean ptsCollinear (p1, p2, p3) {
        if          (p2 - p1) × (p3 - p1) = 0
                    then return true;
                    then return false;
}


boolean setCollinear (p_{i=1,2,...,n})


for  i  = 1 → n - 2
   for j  = i + 1 → n - 1;
     for k = j + 1 → n
        if checkCollinear (p_i, p_j, p_k)
        return true;
return false;
```

5. In the left-handed system, positive rotations are *clockwise* when looking from a positive axis toward the origin. This definition of the positive rotations allows the same matrices of the right-handed system to be used in the left-handed system without any modifications.

6. Let's find the matrix representation of a rotation followed by a translation:

$$T_{(x,y)}R_\phi = \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

Now let's represent a translation followed by a rotation:

$$R_\phi T_{(x,y)} = \begin{bmatrix} \cos\phi & -\sin\phi & x\cos\phi - y\sin\phi \\ \sin\phi & \cos\phi & x\sin\phi + y\cos\phi \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

From equations 9 and 10, RT can be represented by by a rotation followed by translation where

$$\begin{aligned} R_\phi T_{(x,y)} &= T_{(x',y')}R_\phi \\ &= \begin{bmatrix} 1 & 0 & x\cos\phi - y\sin\phi \\ 0 & 1 & x\sin\phi + y\cos\phi \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & x' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{11}$$

where $x' = x\cos\phi - y\sin\phi$, and $y' = x\sin\phi + y\cos\phi$.

Also, recall from problem # 2 that a sequence of rotations can be replaced by a single rotation, and also, a sequence of translations can be replaced by a single translation. Note that in the following, T and R represent abstract translation, and abstract rotation, respectively.

So assume that there is a general sequence $S_1, S_2, \ldots, S_n$, where each S_i can be either a R or a T. We want to prove that $M = S_n S_{n-1} \ldots S_1 = TR$. The proof will be done by induction.

At $t = 0$, $M = I$, where I is the identity matrix. $M = I = TR$ where the rotation is by angle 0, and the translation to the origin $(0, 0)$.

At $t = 1$, $M = S_1 M_{old} = S_1 TR$, if $S_1 = T \Longrightarrow M = TTR = TR$ due to equation 7. Else, if $S_1 = R \Longrightarrow M = RTR = T'RR = TR$ due to equation 11.

Assume proof is held for $t = n$, i.e.; $M = S_n S_{n-1} \ldots S_1 = TR$, then at $t = n+1$, If $S_{n+1} = T \Longrightarrow M = TTR = TR$, and similarly, if $S_1 = R \Longrightarrow M = RTR = T'RR = TR$ due to equation 11. So, it holds for $t = n + 1$.

Thus, any sequence of rotations, and translations can be represented by a rotation followed by a translation.