Image Spaces

What might an image space be

- Map each image to a point in a space.
- Define a distance between two points in that space.
- Maybe also a shortest path (morph).
- We have already seen a simple version of this, in which each n-pixel image is mapped to a point in R^n, and Euclidean distance is used.
  - Also, we've considered linear subspaces (Eigenfaces, Fisherfaces). Each point is mapped to nearest location on the subspace.
  - Shortest path in R^n is just linear interpolation of the images, which doesn't seem like the right morph.
  - What is shortest path with Eigenfaces?

Riemannian Manifold

- More general notion of an image space
  - Can have different topology than Euclidean space.
  - Can represent non-linear subsets of images (more on this is a few weeks).
  - Can provide more general sense of distances.

Manifold

- "A smooth manifold of dimension *m* is a locally compact Hausdorff space *M* together with the following collection of data (henceforth called **atlas** or **smooth structure**) consisting of:
  - An open cover $\{U_i\}$ of *M*
  - Continuous injective maps F_i: U_i ->R^m (called **charts** or **local coordinates**) such that:

if $U_i \cap U_j \neq 0$ the transition map

$$\Psi_j \circ \Psi_i^{-1} : \Psi_i(U_i \cap U_j) \subset \Re^m \to \Psi_j(U_i \cap U_j) \subset \Re^m$$

is smooth.

Lectures on the Geometry of Manifolds by Nicolaescu.

Manifold - informal

- A manifold is:
  - A set (collection of images) with the usual topological properties (eg., not discrete)
  - A mapping from the neighborhood of each point to Euclidean space of fixed dimension.
  - So that these mappings fit together smoothly.
- Example: the surface of a sphere.
- Example: any surface that is topologically a sphere.
- The definition implies: At any point on the manifold, we can construct a tangent space.

Riemann Manifold

- A **Riemann Manifold** is a pair *(M,g)* consisting of a smooth manifold *M* and a metric *g* on the tangent bundle, i.e. a smooth symmetric positive definite tensor field on *M*. *g* is called a Riemann metric on *M*.
- The **tangent bundle** is (informally) the collection of tangent spaces of *M*.

Riemann Manifold - intuition

- We define a local distance on the manifold.
  - At any point, for any direction we move in, there is a local distance defined.
  - This distance is locally linear. If we define the distance for *m* orthogonal directions, the distance in any other direction is a linear combination of these.
- Example: any surface in Euclidean space that is topologically a sphere with the usual Euclidean distance.

Geodesic

- Analogous to lines, geodesics are shortest paths between points.
- Shortest paths locally, but not globally.
  - For any two points very close together on the geodesic, it is the shortest path.
- Example: on a sphere, any two points are connected by two geodesic paths, along two directions of the great circle connecting them.

Why manifolds?

- Seems like correct mathematical notion of an image space with distances.
- Generality greater than Euclidean space.
  - Image space may be topologically like a sphere, not like R^n
  - Offers much more flexible distances

Kendall's shape space

- Two sets of 2D points.
- Mostly we assume there exists a correct one-to-one correspondence
- And this correspondence is given.
  - This is very natural in morphometrics, where points are measured and labeled.
  - In vision we must solve for correspondence. Next class we'll look at papers that do this.

Shape Space

- What is shape? Qualities of points that don't depend on translation, rotation or scale.
- So describe points independent of similarity transformation.

1. Remove translation.
   - Simplest way, translate so point 1 is at origin, then remove point one.
   - More elegant, translate center of mass to origin, remove a point.
2. Scale so that sum ||Xi||^2 = 1.

Resulting set of points is called *pre-shape.*

Pre because we haven't removed rotation yet.

Notation: , U and X denote sets of normalized points. Points called Xi and Ui, with coordinates (xi,yi), (ui, vi).

Pre-shape

- If we started with n points, we now have n-1 so that:
- sum xi^2 + yi^2 = 1.
- So we can think of these coordinates as lying on a unit hypersphere in 2(n-1)-dimensional space.

Shape

- If we consider all possible rotations of a set of normalized points, these trace out a closed, 1D curve in pre-shape space.
- Distances between shapes can be thought of as distances between these curves.
  - Notice that to compute distance, without loss of generality we can assume that one set of points (U) does not rotate, since rotating both point sets by the same amount doesn't change distances.

Procrustes Distances

- Full Procrustes Distance. D_F
  - min(s, θ) ||U – sXR(θ).|| That is, we find a scaling and rotation of X that minimizes the euclidean distance to U. (R(θ) means rotate by θ).
- Partial Procrustes Distance. D_P
  - min(θ) ||U – XR(θ)||. That is, rotate X to minimize the euclidean distance to U.
- Procrustes Distance. ρ
  - Rotate X to minimize the geodesic distance on the sphere from X to U.

Linear Pose Solving

- We can linearly find optimal similarity transformation that matches X to U. (ie., minimize sum ||AXi-Ui||^2, where A is a similarity transformation.
  - This is asymmetric between X and U.
- In same way we can linearly compute Full Procrustes Distance.
  - This is symmetric.
  - Leads immediately to other procrustes distances.

Linear Pose: 2D rotation, translation and scale

$$\begin{pmatrix} u_1 & u_2 & . & . & . & u_n \\ v_1 & v_2 & & & & v_n \end{pmatrix} = s \begin{pmatrix} \cos\theta & \sin\theta & t_x \\ -\sin\theta & \cos\theta & t_y \end{pmatrix} \begin{pmatrix} x_1 & x_2 & . & . & . & x_n \\ y_1 & y_2 & & & & y_n \\ 1 & 1 & & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b & t_x \\ -b & a & t_y \end{pmatrix} \begin{pmatrix} x_1 & x_2 & . & . & . & x_n \\ y_1 & y_2 & & & & y_n \\ 1 & 1 & & & & 1 \end{pmatrix}$$

with $\quad a = s\cos\theta, \quad b = s\sin\theta$

- Notice *a* and *b* can take on any values.
- Equations linear in *a, b*, translation.
- Solve exactly with 2 points, or overconstrained system with more.

$$s = \sqrt{a^2 + b^2} \qquad \cos\theta = a/s$$

Similarity Matching

- Given point sets X and U, compare by finding similarity transformation A that minimizes ||AX-U||.
  - X = points X1, …Xn. U = points U1…Un.
  - Find A to minimize sum ||AXi – Ui||^2
  - This is just a straightforward, linear problem.
    - Taking derivatives with respect to four unknowns of A gives four linear equations in four unknowns.

Issues with this approach

- It is asymmetric.
  - Ok when comparing a model to an image.
  - Not so sensible for comparing two shapes.

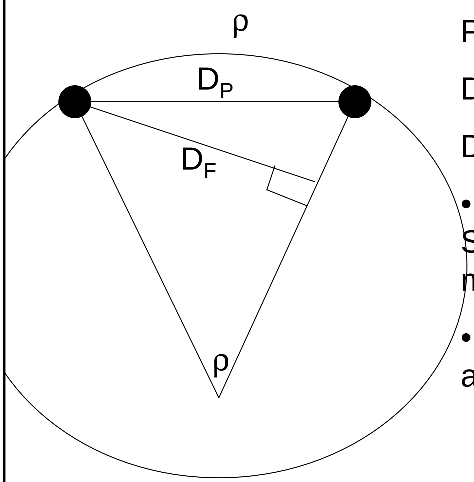
- Note that we now also know how to calculate the Full Procrustes Distance. This is just a least-squares solution to the overconstrained problem:

$$\begin{pmatrix} u_1 & u_2 & . & . & . & u_n \\ v_1 & v_2 & & & & v_n \end{pmatrix} = s \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 & x_2 & . & . & . & x_n \\ y_1 & y_2 & & & & y_n \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x_1 & x_2 & . & . & . & x_n \\ y_1 & y_2 & & & & y_n \end{pmatrix}$$

•It is not obvious that Full Procrustes is symmetric.

Given two points on the hypersphere, we can draw the plane containing these points and the origin.



Procrustes Distances is ρ.

$D_P = 2 \sin(\rho/2)$

$D_F = \sin \rho$.

• These are all monotonic in ρ. So the same choice of rotation minimizes all three.

• D_F is easy to compute, others are easy to compute from D_F.

Why Procrustes Distance?

- Procrustes distance is most natural. Our intuition is that given two objects, we can produce a sequence of intermediate objects on a ‘straight line’ between them, so the distance between the two objects is the sum of the distances between intermediate objects. This requires a geodesic.

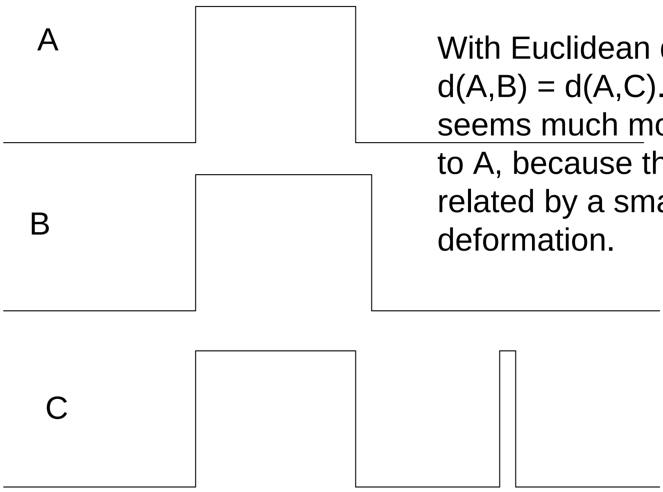
Tangent Space

- Can compute a hyperplane tangent to the hypersphere at a point in preshape space.
- Project all points onto that plane.
- All distances Euclidean. Average shape easy to find.
- This is reasonable when all shapes similar.
- In this case, all distances are similar too.
  – Note that when ρ is small, ρ, $2\sin(\rho /2)$, $\sin(\rho)$ are all similar.

Trouve and Younes

- Image space is the space of all images.
- To get a manifold, we must describe a tangent space, that gives a cost to small image changes.
- Normal Euclidean distance in this space allows intensity of a point to change, with a cost = square of the change.
- T&Y also allow image to deform, with a cost based on the smoothness of deformation.

Intuition



With Euclidean distance, d(A,B) = d(A,C). But B seems much more similar to A, because they are related by a small deformation.

Local Cost

- Let *v* be a deformation. This is a diffeomorphism (smooth, continuous one-to-one transformation).
  - We might be interested in non-smooth deformations, but these are not as nice mathematically.
- I(v(x)) is the deformed image.
- We can't expect two images to be identical up to a deformation.
  - So, add cost ||J(x)-I(v(x))||.
  - Use Euclidean norm here.
  - Total cost also requires a norm on *v*, a vector field (see paper)

$$\min_{v} \lambda \left\| J(x) - I(v(x)) \right\|_{L_2} + \left\| v \right\|_g$$

Tangent Space

- This assigns a cost to any combination of deformation and intensity change.
- However, we need a cost for changes in the image.
- Any image change is consistent with any deformation + some intensity change.
  - Ie, there are an infinite number of ways to create an image change.
  - We define the cost of the image change to be the infinum of all of these.
- This defines an image manifold

Geodesics

- We have defined a local cost on infinitesimal image changes.
  - For tiny changes, we can minimize over deformations fairly easily, because everything is linear.
- To find the distance between two images, we must compute a geodesic path between them.
- This also gives us a morph between them.

Computing geodesics

- Gradient descent
  - Start with some path between images, discretizing time.
  - Path is represented by discrete representation over deformation at each time.
    - Durrleman describes use of kernels here.
  - And by changes in intensity at each time.
  - Compute derivatives and minimize path.
  - Lots of variables, but this can work in practice.
- Geodesic shooting
  - Geodesic path is entirely determined by initial change.
  - Pick some change, calculate path, measure distance to target image, and correct.

Example of Geodesic