The Duplicator-Spoiler Game for an Ordinal Number of Turns

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Abstract

In this paper, we investigate Duplicator-Spoiler Games, a method of comparing the properties of structures. These games are especially significant because they have deep connections to logical expressibility. Traditionally, Duplicator-Spoiler Games were played with a preset, finite number of turns. We do not examine such situations but instead consider games of arbitrary lengths, which we represent with ordinal notation. In our games, the game length is delayed to allow more freedom to distinguish structures, giving previously dead-end games new interest. We determine the number of turns required to win various ordinal Duplicator-Spoiler Games, including those based upon scattered orderings and operations. We then use several techniques to classify large groups of structures and find general game length bounds. These techniques include the inductive construction of large structures, a comparison of games to directed graph isomorphism, an adaption of cantor ordinal and an interpretation of turns as elements of free basis. Additionally, we investigate a relationship between our new ordinal game and logical expressibility, relating our research to the original motivation for the study of Duplicator-Spoiler Games.

1 Introduction

In 1961, Andrzej Ehrenfeucht [3] invented Duplicator-Spoiler Games[1] to formalize the connection between Roland Fraïssé's back-and-forth method [1, 2, 4] and logical expressibility. These games compare the properties of structures, which we will define with base sets and relations. In this paper, we discuss an ordinal variant of Duplicator-Spoiler Games in which the length of games is not predetermined.

Definition 1.1. *Let the m-turn Duplicator-Spoiler Game played with base sets S_1, S_2 and relations $\mathcal{R}_1$ over S_1, $\mathcal{R}_2$ over S_2 of arity z be denoted $\mathcal{G}(S_1, \mathcal{R}_1; S_2, \mathcal{R}_2)$. It is defined as follows.*

1. *There are two players: the spoiler and the duplicator.*

2. *There are m turns. During turn i ($1 \leq i \leq m$) the spoiler selects an element from one of the sets and the duplicator responds with an element from the other set. The element selected from S_1 is called a_i and the element selected from S_2 is called b_i.*

3. *If $(\forall\ 1 \leq j_1, j_2, \ldots, j_z \leq i)$ $[\mathcal{R}_1(a_{j_1}, a_{j_2}, \ldots, a_{j_z}) = \mathcal{R}_2(b_{j_1}, b_{j_2}, \ldots, b_{j_z})]$, then the duplicator wins. Otherwise, the spoiler wins.*

Unless otherwise specified, we will assume the spoiler and the duplicator play optimally. In other words, if one player can implement a strategy that wins every time then the player is assumed to implement said strategy. Furthermore, a game is said to "take" exactly m turns iff the spoiler wins in m turns and the duplicator wins in any number less than m turns.

Definition 1.2. *For any sentence ϕ, a set S models ϕ, denoted $S \models \phi$, iff ϕ is true when interpreted in S.*

The following theorem [3, 5, 6, 7] connects Duplicator-Spoiler Games to logic and is the original motivation for their study.

Theorem 1.3. *Let S_1 and S_2 be sets. Let $\mathcal{R}_1$ be a relation on S_1 and $\mathcal{R}_2$ be a relation on S_2. The following are equivalent.*

[1] Duplicator-Spoiler games are also referred to as Ehrenfeucht-Fraïssé games.

- *The duplicator wins the m-turn game $\mathcal{G}(S_1, \mathcal{R}_1; S_2, \mathcal{R}_2)$.*

- *For all first-order m-quantifier-depth sentences ϕ in the language of standard logical symbols and $\mathcal{R}_1$ and $\mathcal{R}_2$*

$$[S_1 \models \phi \iff S_2 \models \phi].$$

We will now explore linear orderings and their associated Duplicator-Spoiler Games.[2] Recall that linear orderings are transitive, antireflective and total.

We will use $\mathcal{L}$ to denote a linear ordering. We say $\mathcal{L} = (S, <)$ if its base set is S. We use $\mathcal{L}^*$ to denote the reverse of the ordering $\mathcal{L}$. For example, since ω denotes the orderings of the natural numbers, ω^* denotes the ordering of the negative integers. Furthermore, we will use $\mathcal{F}_m$ to denote the finite linear ordering of m elements.

Definition 1.4. *Let $\mathcal{L}_1 = (S_1, <_1), \mathcal{L}_2 = (S_2, <_2)$. The linear ordering $\mathcal{L}_1 + \mathcal{L}_2$ is formed as follows:*

We can assume $S_1 \cap S_2 = \emptyset$ by changing the elements' labels. Let the base set for $\mathcal{L}_1 + \mathcal{L}_2$ be $S_1 \cup S_2$. Let the total order be as follows.

1. $(\forall x, y \in S_1)[x < y \iff x <_1 y]$.

2. $(\forall x, y \in S_2)[x < y \iff x <_2 y]$.

3. $(\forall x \in S_1)(\forall y \in S_2)[x < y]$.

Definition 1.5. *Let $\mathcal{L}_1 = (S_1, <_1), \mathcal{L}_2 = (S_2, <_2)$. The linear ordering $\mathcal{L}_1 * \mathcal{L}_2$ is formed as follows.*

We form the base set of $\mathcal{L}_1 * \mathcal{L}_2$ by replacing every element in S_2 with a copy of S_1. The base set now contains several copies of S_1, one for every element in S_2. We can assume the intersection of all such copies is $\emptyset$ by replacing the elements' labels.

Let the total order be as follows: all elements within a copy of S_1 retain their natural ordering. For all x, y that are members of the original S_2 and for all a, b that are members of copies of S_1 which replaced x and y respectively, $a < b$ iff $x < y$.

Definition 1.6. *Let $\mathcal{L} = (S, <)$. The linear ordering $\mathcal{L}^k$ is formed by iterating multiplication k times on $\mathcal{F}_1$.*

[2] In Section 4 we investigate Duplicator-Spoiler Games played with operations.

Definition 1.7. *Consider the game $\mathcal{G}(S_1, \mathcal{R}_1; S_2, \mathcal{R}_2)$. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are both transitive, antireflective and total then the game is called a Linear Ordering Game. It is then denoted $\mathcal{G}(\mathcal{L}_1; \mathcal{L}_2)$.*

In Linear Ordering Games, we imagine lines being drawn from elements of one ordering to another, each line representing a pair of inequalities from a pair of elements. If two lines cross, or the duplicator runs out of elements to select, then the inequalities are not consistent from ordering to ordering and the spoiler wins. Because of this, a line, or turn, can be said to reduce a game into two new ones. We define said reduction.

Definition 1.8. *Consider the Linear Ordering Game $\mathcal{G}(\mathcal{L}_1 + x + \mathcal{L}_3; \mathcal{L}_2 + y + \mathcal{L}_4)$. If $a_1 = x$, the spoiler's first selection, and $b_1 = y$, the duplicator's first selection, then after the first turn the game is said to have been reduced to two new games: $\mathcal{G}(\mathcal{L}_1; \mathcal{L}_2)$ and $\mathcal{G}(\mathcal{L}_3; \mathcal{L}_4)$. With optimal play the spoiler will continue only on the reduced game which can be won with the fewest turns.*

Note that the reduction of games is only a tool to imagine linear orderings visually. It is not needed in the games' formal definition.

Below are known examples of Linear Ordering Games and their results. In some cases we include a first-order sentence that is true in one ordering and false in the other. Note that the *number of turns* needed for the spoiler to win will equal the *quantifier depth* of such sentences (Theorem 1.3).

Theorem 1.9. *For all m, n ($n < m$), the spoiler wins the Linear Ordering Game on finite orderings of size m and n, $\mathcal{G}(\mathcal{F}_m; \mathcal{F}_n)$, in exactly $\lfloor \log_2(n+1) \rfloor + 1$ turns.*

Example 1.10. *The spoiler wins the game $\mathcal{G}(\mathbb{Z}; \mathbb{Q})$ in exactly three turns.*

Note that $\mathbb{Q}$ models the following three-quantifier sentence, but $\mathbb{Z}$ does not.

$$(\forall x)(\forall y)(\exists z)[x < y \implies x < z < y].$$

There is no sentence that can distinguish $\mathbb{Z}$ from $\mathbb{Q}$ with only two quantifiers.

Theorem 1.11. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be dense orderings with no endpoints. The duplicator wins the game $\mathcal{G}(\mathcal{L}_1; \mathcal{L}_2)$ no matter the number of turns.*

Notice this includes orderings of different cardinalities (such as $\mathbb{Q}$ and $\mathbb{R}$).

2 Ordinal Numbers of Turns

Traditionally, Duplicator-Spoiler Games were played with finite, predetermined numbers of turns. We define games in which the declaration of the number of turns is delayed. This delay gives the spoiler the right to reserve an arbitrarily large number of turns. We denote the delays with ordinal notation.

Definition 2.1. *For any ordinal λ, a game that has λ turns remaining will, after one turn, have γ turns remaining for any ordinal γ chosen by the spoiler such that $\gamma < \lambda$.*

Notice that this definition only becomes significant once a limit ordinal is reached. At this point, the spoiler plays a move, the duplicator responds and the spoiler then decrements the number of turns to any of the ordinals less than the given limit ordinal.

Example 2.2. *A game with ω turns remaining has m turns remaining after one turn for any finite m chosen by the spoiler after the first turn is complete.*

Example 2.3. *A game with $\omega+1$ turns remaining has ω turns remaining after one turn is complete.*

Example 2.4. *A game with $\omega * 2$ turns remaining has $\omega + m$ turns remaining after one turn for any finite m chosen by the spoiler after the first turn is complete.*

Example 2.5. *A game with ω^2 turns remaining has $\omega * k + m$ turns remaining after one turn for any finite k and m chosen by the spoiler after the first turn is complete.*

Note that although ordinals are levels of infinity, and our Duplicator-Spoiler Games will last ordinal numbers of turns, they will still terminate after a finite number of turns. This is because ordinals are by definition well-ordered so any ordinal decremented a finite number of times will eventually arrive at 0.

This definition holds for even uncountable ordinal numbers of turns.

Example 2.6. *A game with ω_1 turns remaining has λ turns remaining after one turn is complete for any countable λ chosen by the spoiler after the first turn is complete.*

3 Linear Ordering Games Played on the Ordinals

We now venture into the unknown realm of ordinal Duplicator-Spoiler Games. We will leave with a general form for orderings whose games can take any ordinal number of turns. Consider the game $\mathcal{G}(\omega; \omega + \mathbb{Z})$. For all m, the duplicator wins the m-turn game. Correspondingly, no first-order sentence can distinguish the two orderings. However, this does not mean no finite sentence can distinguish the orderings. Note that $\omega + \mathbb{Z}$ models the following non-first-order sentence, but ω does not. The variable A is quantified over subsets of the ordering.

$$(\exists A)(\exists a \in A)(\forall y \in A)(\neg \exists x \in A)[x < y].$$

Hence, the property of being well-ordered is definable with second-order logic. From now on, we will be exploring games which similarly cannot be won with a present number of turns.

3.1 $\mathcal{G}(\mathbb{Z}^k + \mathbb{Z}^k; \mathbb{Z}^k)$

Theorem 3.1. *The spoiler wins the game $\mathcal{G}(\omega + \mathbb{Z}; \omega)$ in exactly ω turns.*

Proof. Proofs of game-length involve two parts: showing that the spoiler wins in the given number of turns and showing that the duplicator wins with any fewer number of turns.

The spoiler wins with ω turns through the following strategy. The spoiler selects an element a_1 in the integers, the duplicator responds with an element b_1 in the naturals. The game is now reduced to $\mathcal{G}(\omega^* + \omega; \mathcal{F}_{b_1})$ and $\mathcal{G}(\omega; \omega)$. The first option, $\mathcal{G}(\omega^* + \omega; \mathcal{F}_{b_1})$ will take $\lfloor \log_2(m+1) \rfloor + 1$ turns (Theorem 1.9). The spoiler appropriately decrements ω to $\lfloor \log_2(m+1) \rfloor + 1$ and wins.

We now provide the proof that the duplicator wins with some finite number of turns. The spoiler selects an element a_1 in $\mathbb{Z}$ because any selection in ω would not further reduce the sets. The duplicator selects an element b_1 such that b_1 is the 2^mth elements in the naturals. The spoiler now needs $\lfloor \log_2(2^m + 1) \rfloor + 1$ turns, but has only m turns ($m < m + 1 \leq \lfloor \log_2(2^m + 1) \rfloor + 1$).

The spoiler wins the ω-turn game and the duplicator wins the m-turn game for some finite m so the spoiler wins the game $\mathcal{G}(\omega + \mathbb{Z}; \omega)$ in exactly ω turns.

$\square$

Theorem 3.2. *The spoiler wins the game $\mathcal{G}(\mathbb{Z}+\mathbb{Z};\mathbb{Z})$ in exactly $\omega+1$ turns.*

Proof. After one turn the game will either be reduced to $\mathcal{G}(\omega+\mathbb{Z};\omega)$ or to $\mathcal{G}(\mathbb{Z}+\omega^*;\omega^*)$. The first takes exactly ω turns (Theorem 3.1). The second is composed of the reverse orderings of the first and will thus take the same number of turns. This spoiler needs ω turns to win this game after one turn is complete. It is thus an $\omega+1$-turn game. □

Theorem 3.3. *The spoiler wins the (k=2)-game $\mathcal{G}(\mathbb{Z}^2+\mathbb{Z}^2;\mathbb{Z}^2)$ in exactly $\omega*2+1$ turns.*

Proof. Consider the game $\mathcal{G}(\mathbb{Z}^2+\mathbb{Z}^2;\mathbb{Z}^2)$. The spoiler attempts to reduce the game to $\mathcal{G}(\mathbb{Z}+\mathbb{Z};\mathbb{Z})$ and retain $\omega+1$ turns. In doing so, the spoiler treats each integer copy in $\mathbb{Z}$ as its own element. We will now prove that the spoiler wins the $(\omega*2+1)$-turn game.

Move 1: The spoiler selects an element a_1 in the lesser $\mathbb{Z}^2$. The duplicator selects an element b_1.

Move 2: The spoiler selects an element a_2 in the greater $\mathbb{Z}^2$. The duplicator must select an element b_2 such that b_2 is m_1 integer copies greater than b_1. The game is now reduced to $\mathcal{G}(\omega+\mathbb{Z}*\omega+\mathbb{Z}*\omega^*+\omega^*;\omega+\mathbb{Z}*m_1+\omega^*)$, which is game-length-equivalent to $\mathcal{G}(\mathbb{Z}*\omega+\mathbb{Z}*\omega^*;\mathbb{Z}*m_1)$. The spoiler decrements the first ω to $\lfloor\log_2(m_1+1)\rfloor+1$.

Move 3: The spoiler now implements the strategy seen in Theorem 3.2, using $\lfloor\log_2(m_1+1)\rfloor+1$ turns to bisect integer sets and reduce the problem to $\mathcal{G}(\omega+\omega^*;\mathcal{F}_{m_2})$. After this, the spoiler decrements the second ω to $\lfloor\log_2(m_2+1)\rfloor+1$, the minimum number of turns required to win (Theorem 1.9).

The proof that the duplicator wins the $(\omega*2)$-turn game is similar to that in Theorem 3.2. □

Theorem 3.4. *The spoiler wins the game $\mathcal{G}(\mathbb{Z}^k+\mathbb{Z}^k;\mathbb{Z}^k)$ in exactly $\omega*k+1$ turns.*

Proof. Let us first prove that the spoiler wins the game $\mathcal{G}(\mathbb{Z}^k+\mathbb{Z}^k;\mathbb{Z}^k)$ with $\omega*k+1$ turns.

In similar fashion to Theorem 3.3 the spoiler treats each $\mathbb{Z}^{k-1}$ as its own element, making the game look like $\mathcal{G}(\mathbb{Z}+\mathbb{Z};\mathbb{Z})$. This takes $\omega+1$ turns and reduces the game to $\mathcal{G}(\omega+\mathbb{Z}*\omega+\ldots+\mathbb{Z}^{k-1}*\omega+\mathbb{Z}^{k-1}*\omega^*+\ldots+\mathbb{Z}*\omega^*+\omega^*;\omega+\mathbb{Z}*\omega+\ldots+\mathbb{Z}^{k-2}*\omega+\mathbb{Z}^{k-1}*(m-1)+\mathbb{Z}^{k-2}*\omega^*+\ldots+\mathbb{Z}*\omega^*+\omega^*)$ which is turn-equivalent to $\mathcal{G}(\mathbb{Z}^{k-1}+\mathbb{Z}^{k-1};\mathbb{Z}^{k-1})$. Done inductively, the game will take $\omega_1 + 1 + \omega_2 + 1 + \ldots + \omega_k + 1$ turns. In our game, as in standard ordinal notation, this means $\omega * k + 1$.

Again, the proof that the duplicator wins the $(\omega * k)$-turn game is similar. $\square$

3.2 Upper Bounds on Simple Structures

All of the pairs of linear orderings in this section have resulted in the spoiler winning in strictly less than ω^2 turns. This is because we have only examined games with orderings below the complexity of $\mathbb{Z}^\omega$. Given this set of orderings it is impossible to form a game that requires ω^2 turns.

Definition 3.5. *Let the class of orderings polyω_a be defined as such. $\emptyset$ and $\mathcal{F}_1$ are members of polyω_0. ω and ω^* are members of polyω_1. If the orderings $\mathcal{L}_1$ and $\mathcal{L}_2$ are members of polyω_a and polyω_b then $\mathcal{L}_1 + \mathcal{L}_2$ is a member of poly$\omega_{\max(a,b)}$, and $\mathcal{L}_1 * \omega$ and $\mathcal{L}_1 * \omega^*$ are members of polyω_{a+1}.*

Notice that orderings such as ω^*, ω^2 and $\mathbb{Z}$ can be formed easily from such constructions.

Theorem 3.6. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be members of polyω_a and polyω_b $(a > b)$. The game $\mathcal{G}(\mathcal{L}_1; \mathcal{L}_2)$ is either won by the spoiler in fewer than $\omega * (b+1)$ turns or is won by the duplicator with any game length.*

Proof. Following the treatment of Theorem 3.4, the spoiler wins games that include orderings with only multiplication and up to one finite addition, such as $\mathcal{L}_1 + \mathcal{L}_2$ and $\mathcal{L}_3$ in $(\omega * b + 1)$-maximum time. If there are more than two large exponentiated structures then the spoiler uses the bisection method. The game is then reduced to the previously mentioned games in strictly finite time. Thus, the game takes $\omega * b + 1+$finite time, which is itself $\omega * b+$finite time, which is less than $\omega * (b+1)$. $\square$

3.3 $\mathcal{G}(\mathbb{Z}^\lambda + \mathbb{Z}^\lambda; \mathbb{Z}^\lambda)$

Given the bounds on simple structures above it would seem plausible that there do not exist Linear Ordering Games which require ω^2 turns. However, we have not yet considered the ordering $\mathbb{Z}^\omega$. This ordering is complicated. What do we mean by $\mathbb{Z}^\omega$? The most natural definition is $\lim_{x\to\infty} \mathbb{Z}^x$. However, this is not a full definition. We construct the ordering inductively.

Definition 3.7. $\mathbb{Z}^\omega = \lim_{x\to\infty} \mathbb{Z}^x = \ldots (\mathbb{Z}^3)(\omega^*) + (\mathbb{Z}^2)(\omega^*) + (\mathbb{Z})(\omega^*) + \mathcal{F}_1 + (\mathbb{Z})(\omega) + (\mathbb{Z}^2)(\omega) + (\mathbb{Z}^3)(\omega) + \ldots$ *Notice that the middle* $(\mathbb{Z})(\omega^*) + (\mathbb{Z})(\omega)$ *"absorbs" to form* $\mathbb{Z}^2$ *and the next group to* $\mathbb{Z}^3$ *and so on. Thus, as we get farther from the center, the* x *of* $\mathbb{Z}^x$ *goes to* ∞*.*

Now, with this definition, we may tackle ω^2 games.

Theorem 3.8. *The game* $\mathcal{G}(\mathbb{Z}^\omega + \mathbb{Z}^\omega; \mathbb{Z}^\omega)$ *takes* $\omega^2 + 1$ *turns.*

Proof. Because it takes two turns to reduce this game to the complexity of $\mathbb{Z}^k$ for any finite k the game requires a 2-turn delay before decrementing to the form $\omega * k + m$ for any finite k and m. If the spoiler could do any better then the spoiler would declare k and m before reducing the game to the equivalent of $\mathbb{Z}^k$. The duplicator then chooses to reduce the game to $\mathbb{Z}^{k+1}$ and wins (Theorem 3.4). □

The next step is to extend this definition in order to find games which can require any ordinal number of turns. We begin by generalizing the previous definition of $\mathbb{Z}^\omega$ to any ordinal [5].

Definition 3.9. $\mathbb{Z}^\beta$ *for any ordinal* β *is defined as such:*

1. $\mathbb{Z}^0 = \mathcal{F}_1$.
2. *For any ordinal* γ, $\mathbb{Z}^\gamma * \mathbb{Z} = \mathbb{Z}^{\gamma+1}$.
3. *For any limit ordinal* λ, $\mathbb{Z}^\lambda = (\sum (\mathbb{Z}^\gamma * \omega | \gamma < \lambda))^* + \sum (\mathbb{Z}^\gamma * \omega | \gamma < \lambda)$.

Notation 3.10. *Let* $x_\lambda = \sum (\mathbb{Z}^\gamma * \omega | \gamma < \lambda)$ *and let* $\mathcal{D}_\lambda$ *denote any linear ordering that is not divisible by* $\mathbb{Z}^\lambda$ $(\neg\exists\mathcal{L})(\mathcal{D}_\lambda = \mathcal{L} * \mathbb{Z}^\lambda)$.

Theorem 3.11. *For any ordinal* $\lambda \geq 1$, *the game* $\mathcal{G}(\mathbb{Z}^\lambda + \mathbb{Z}^\lambda; \mathbb{Z}^\lambda)$ *takes* $\omega * \lambda + 1$ *turns.*

Proof. We prove this inductively in a similar fashion to Definition 3.9.

1. The spoiler wins the base case of $\lambda = 1$, $\mathcal{G}(\mathbb{Z} + \mathbb{Z}; \mathbb{Z})$, in exactly $\omega * 1 + 1 = \omega + 1$ turns (Theorem 3.2).

2. We will now prove that if the spoiler wins the game $\mathcal{G}(\mathbb{Z}^\gamma + \mathbb{Z}^\gamma; \mathbb{Z}^\gamma)$ in exactly $\omega * \gamma + 1$ turns then the spoiler wins the game $\mathcal{G}(\mathbb{Z}^{\gamma+1} + \mathbb{Z}^{\gamma+1}; \mathbb{Z}^{\gamma+1})$ in exactly $\omega * (\gamma + 1) + 1 = \omega * \gamma + \omega + 1$ turns.

   (a) We first show that the spoiler wins with $\omega * \gamma + \omega + 1$ turns.

   After two turns, the spoiler can reduce the game to $\mathcal{G}(x_{\gamma+1} + x^*_{\gamma+1}; \mathcal{D}_\gamma + \mathbb{Z}^\gamma * m + \mathcal{D}_\gamma)$. Now the spoiler decrements $\omega * \gamma + \omega + 1$ to $\omega * \gamma + \lfloor \log_2(m + 1) \rfloor$. The spoiler now employs a bisection approach (Theorem 1.9) and after $\lfloor \log_2(m+1) \rfloor - 1$ turns the game is reduced to the equivalent (discounting extra complexity in the larger structure and irrelevant tails and in the form of $\mathcal{D}_\beta$ for $\beta < \gamma$) of $\mathcal{G}(\mathbb{Z}^\gamma + \mathbb{Z}^\gamma; \mathbb{Z}^\gamma)$ and there are $\omega * \gamma + 1$ turns remaining.

   (b) We now prove that the duplicator wins if the game has only $\omega * \gamma$ turns.

   After one turn the game is reduce to $\mathcal{G}(x_{\gamma+1} + \mathbb{Z}^{\gamma+1}; x_{\gamma+1})$ and the number of turns must be decremented to some ordinal β such that $\beta = \omega * \gamma + m < \omega * \gamma + \omega$. Now the spoiler's only strategy is to select an element in $\mathbb{Z}^{\gamma+1}$, and the duplicator responds in $x_{\gamma+1}$ such that the game is reduced to $\mathcal{G}(x_{\gamma+1} + x^*_{\gamma+1}; \mathcal{D}_\gamma + \mathbb{Z}^\gamma * k + \mathcal{D}_\gamma)$ for $k \geq 2^m$. Because the duplicator chose k to be slightly larger than 2^{m-1} the spoiler will only have $\omega * \gamma$ turns once the game is reduced to the equivalent of $\mathcal{G}(\mathbb{Z}^\gamma + \mathbb{Z}^\gamma; \mathbb{Z}^\gamma)$. However, we know that the game needs exactly $\omega * \gamma + 1$ turns and thus $\omega * \gamma$ is not enough.

3. We will finally prove that for all limit ordinals λ, if for all $\gamma < \lambda$ the spoiler wins the game $\mathcal{G}(\mathbb{Z}^\gamma + \mathbb{Z}^\gamma; \mathbb{Z}^\gamma)$ in exactly $\omega * \gamma + 1$ turns then the spoiler also wins the game $\mathcal{G}(\mathbb{Z}^\lambda + \mathbb{Z}^\lambda; \mathbb{Z}^\lambda)$ in exactly $\omega * \lambda + 1$ turns.

   (a) We first show that the spoiler wins with $\omega * \lambda + 1$ turns.

   The spoiler selects two elements in the two exponentiated integers in the first set. The game is then reduced to $\mathcal{G}(x_\gamma + x^*_\gamma; \mathcal{D}_\beta + \mathbb{Z}^\gamma * m + \mathcal{D}_\beta)$ for finite m and $\beta, \gamma < \lambda$. The spoiler now decrements the the number of turns to be at least $\omega * \gamma + \lfloor \log_2(m + 1) \rfloor$ and according to the inductive assumption has enough turns to win.

(b) Now let us suppose the spoiler has only $\omega * \lambda$ turns. We will show that in this case the duplicator wins.

After one turn the game is reduced to $\mathcal{G}(x_\lambda + \mathbb{Z}^\lambda; x_\lambda)$, but there are only $\gamma * \omega$ ($\gamma < \lambda$) turns remaining. With this many turns the spoiler could win the game $\mathcal{G}(\mathbb{Z}^\beta + \mathbb{Z}^\beta; \mathbb{Z}^\beta)$ ($\beta < \gamma$), but not the game $\mathcal{G}(\mathbb{Z}^\gamma + \mathbb{Z}^\gamma; \mathbb{Z}^\gamma)$. However, we know that the game $\mathcal{G}(x_\lambda + \mathbb{Z}^\lambda; x_\lambda)$ takes more turns than the game $\mathcal{G}(\mathbb{Z}^\gamma + \mathbb{Z}^\gamma; \mathbb{Z}^\gamma)$ because $\gamma < \lambda$. Therefore the spoiler does not have enough turns for either game and the duplicator wins.

□

The following theorem is a direct consequence of Theorems 1.9 and 3.11.

Theorem 3.12. *Let $n \in \mathcal{N}$. The game $\mathcal{G}(x_\lambda + \mathbb{Z}^\lambda * 2^n + \mathbb{Z}^\lambda; x_\lambda + \mathbb{Z}^\lambda * 2^n)$ takes exactly $\omega * \lambda + n$ turns ($n \geq 0$).*

Notice that, as we may select any natural number n, Theorem 3.12 becomes no less general if we restrict λ to a limit ordinal. Any ordinal α can be written in the form $\lambda + n$ for some limit ordinal λ and finite number n. Also, any limit ordinal λ can be written in terms of another ordinal β such that $\omega * \beta = \lambda$. We now have a general form for finding the a game which can any ordinal number of turns.

Suppose we want a game to require exactly α turns for some ordinal $\alpha \geq 1$. We find β and n such that $\omega * \beta + n = \alpha$. Recall that $x_\beta = \sum (\mathbb{Z}^\gamma * \omega | \gamma < \beta)$. We play the game $\mathcal{G}(x_\beta + \mathbb{Z}^\beta * 2^n + \mathbb{Z}^\beta; x_\beta + \mathbb{Z}^\beta * 2^n)$. This can also be written in the form $\mathcal{G}\{x_\beta + (x_\beta^* + x_\beta) * (2^n + 1); x_\beta + (x_\beta^* + x_\beta) * (2^n)\}$.

4 Operation Games

We define Operation Games to be games played with the relations that perform the functions of operations. A game played with the operation R is a game played with a relation $\mathcal{R}$ such that for all arguments and results, $(R(arguments) = results) = \mathcal{R}(arguments, results)$.

With this definition, Theorem 1.3 holds for games played with operations. The logical sentence of Theorem 1.3, ϕ, is equivalent to a sentence in the language of standard logical symbols and the operations of the game.

In this section we will explore games with operations that require ordinal numbers of turns. In addition to finding specific games, we will find general bounds on large groups of operations.

4.1 $\mathcal{G}(\mathcal{R},+;\mathcal{Q},+)$

Let us consider the Operation Game played over the reals and the rationals with binary addition, denoted $\mathcal{G}(\mathcal{R},+;\mathcal{Q},+)$. We will show that the duplicator wins this game with any predetermined number of turns, then show that the duplicator still wins with ω turns and lastly show that the spoiler wins with $\omega+1$ turns.

We now wish to prove that, for all m, the duplicator wins the m-turn game. Notice this proof is equivalent to proving that no first-order sentence with standard logical symbols and the operation of addition can distinguish the reals and the rationals. Although this idea in logic is known, we need this proof to extend our idea to show that the duplicator also wins in ω turns, which is not covered with the known idea in logic. We find that this proof falls from a more complicated one involving higher arity.

We define the more complicated Duplicator-Spoiler Game.

Definition 4.1. *Let $r \in \mathcal{N}$. The m-turn r-game $\mathcal{G}(\mathcal{R},+;\mathcal{Q},+)$ is similar to the m-turn Duplicator-Spoiler Game $\mathcal{G}(\mathcal{R},+;\mathcal{Q},+)$ with the exception of the final winning condition. The duplicator wins if and only if the following conditions are met.*

For all A, B such that A and B are multisets of $\{1,2,\ldots,m\}$ and $|A|,|B|\leq r$,

$$\sum_{i\in A} a_i = \sum_{j\in B} a_j \iff \sum_{i\in A} b_i = \sum_{j\in B} b_j.$$

The game we are truly concerned with is the case of $r=2$. However, it is easier to show that the duplicator always wins the m-turn r-game rather than just the m-turn two-game.

The proof is inductive, consisting of two parts.

1. For all r, the duplicator wins the zero-turn r-game.

2. Assume that for all r, the duplicator wins the m-turn r-game. From this we show that for all

r, the duplicator wins the $(m+1)$-turn r-game.

Lemma 4.2. *For all r, the duplicator wins the zero-turn, r-game.*

Proof. All of the multisets A and B must be the empty set so all iff conditions are satisfied. $\square$

Theorem 4.3. *If the $(r \cdot r^{2^{mr}})$-game $\mathcal{G}(\mathcal{R}, +; \mathcal{Q}, +)$ with m turns is won by the duplicator then the r-game $\mathcal{G}(\mathcal{R}, +; \mathcal{Q}, +)$ with $m+1$ turns is also won by the duplicator.*

Proof. For the first m turns the duplicator follows the strategy of the m-turn $r \cdot r^{2^{mr}}$-game. At this point, since the arity of this game is no higher, all multisets are still consistent. The spoiler now selects a_{m+1} in $\mathcal{R}$. The game is now divided into two cases. Either a_{m+1} is linearly consistent with at least one of the other elements in $\mathcal{R}$ (Case 1) or independent (Case 2).

Case 1: There exist multisets of $\{1, 2, \ldots, m\}$ $A_1, A_2, \ldots, A_i, B_1, B_2, \ldots, B_i$ and natural numbers $n_1, n_2, \ldots, n_i$ such that $|A_1|, |A_2|, \ldots, |A_i|, |B_1|, |B_2|, \ldots, |B_i|, n_1, n_2, \ldots, n_i \leq r$ and

$$n_i * a_{m+1} + \sum_{x \in A_i} a_x = \sum_{y \in B_i} a_y.$$

The spoiler will pick b_{m+1} such that the same can be said for the second set. We will find a_{m+1} in terms of its relationship with the other elements and then duplicate that relationship in b_{m+1}.

2^{mr} bounds the different number of multisets of A_i and B_i so $i \leq 2^{mr}$. Let $N = n_1 \cdot n_2 \cdot \ldots \cdot n_i$. $n \leq r$ and there are at most an i number of n's so $N \leq r^i \leq r^{2^{mr}}$. We multiply each side by $\frac{N}{n_i}$ and are left with

$$N a_{m+1} + \frac{N}{n_i} \sum_{x \in A_i} a_x = \frac{N}{n_i} \sum_{y \in B_i} a_y.$$

Let $A_i' = A_i * \frac{N}{n_i}$. Each element in A_i' can be rewritten as $\frac{N}{n_i}$ times each element in A_i. Due to this,

$$\frac{N}{n_i} \sum_{x \in A_i} a_x = \sum_{x \in A_i} \frac{N}{n_i} a_x = \sum_{x \in A_i'} a_x.$$

If the same is done to B_i then $\left|A_i^{'}\right|, \left|B_i^{'}\right| \leq r\frac{N}{n_i} \leq rN$. We are then left with

$$Na_{m+1} + \sum_{x \in A_i^{'}} a_x = \sum_{y \in B_i^{'}} a_y.$$

$$Na_{m+1} = \sum_{y \in B_i^{'}} a_y - \sum_{x \in A_i^{'}} a_x.$$

Na_{m+1} is equal to the difference of the summations for all i. This means that

$$\sum_{y_1 \in B_1^{'}} a_{y_1} - \sum_{x_1 \in A_1^{'}} a_{x_1} = \sum_{y_2 \in B_2^{'}} a_{y_2} - \sum_{x_2 \in A_2^{'}} a_{x_2} = \ldots = \sum_{y_i \in B_i^{'}} a_{y_i} - \sum_{x_i \in A_i^{'}} a_{x_i}.$$

The game is $r \cdot r^{2^{mr}}$-equivalent and $\left|A_i^{'}\right|, \left|B_i^{'}\right| \leq rN$ (note that $rN \leq r \cdot r^{2^{mr}}$), so the same must be true of

$$\sum_{y_1 \in B_1^{'}} b_{y_1} - \sum_{x_1 \in A_1^{'}} b_{x_1} = \sum_{y_2 \in B_2^{'}} b_{y_2} - \sum_{x_2 \in A_2^{'}} b_{x_2} = \ldots = \sum_{y_i \in B_i^{'}} b_{y_i} - \sum_{x_i \in A_i^{'}} b_{x_i}.$$

The duplicator then selects b_{m+1} such that the previous properties hold. This means that the following property must also hold also.

$$Nb_{m+1} = \sum_{y \in B_i^{'}} b_y - \sum_{x \in A_i^{'}} b_x$$

so

$$b_{m+1} = \frac{\sum_{y \in B_i^{'}} b_y - \sum_{x \in A_i^{'}} b_x}{N}.$$

With this selection the game is r-equivalent.

If the spoiler had selected $b_{m+1} \in \mathcal{Q}$ the same properties would hold and the duplicator would select

$$a_{m+1} = \frac{\sum_{y \in B_i^{'}} a_y - \sum_{x \in A_i^{'}} a_x}{N}.$$

Case 2: The conditions in Case 1 are not met so the spoiler's selction is linearly independent of the other elements. If the spoiler selected a_{m+1}, then the duplicator selects b_{m+1} such that $b_{m+1} > r * r^{2^{mr}} * b_c$ such that b_c is the largest selected element selected in $\mathcal{Q}$. We follow the same treatment for elements in $\mathcal{R}$.

□

In this proof we only assumed that the sets were closed under addition and divisible by the natural numbers. It will therefore work for several other games and is not limited to $\mathcal{G}(\mathcal{R}, +; \mathcal{Q}, +)$. We will use this proof again in the next section.

Theorem 4.4. *The duplicator wins the ω-turn game $(\mathcal{R}, +; \mathcal{Q}, +)$.*

Proof. By changing the base case to involve a game where one move has already been played, we show that the duplicator wins the game with ω turns. Notice that this approach does not work for a game where two moves have already been played because the spoiler only needs an arbitrarily large arity to employ the method seen in Theorem 4.5.

If the spoiler selects zero then the duplicator selects zero. If the spoiler selects a non-zero element then the duplicator selects a non-zero element. After each case, all multisets are for our purposes identical and the duplicator still wins. □

Theorem 4.5. *The spoiler wins the game $\mathcal{G}(\mathcal{R}, +; \mathcal{Q}, +)$ with $\omega + 1$ turns.*

Proof. In this proof, we will show that any two rational numbers have a common multiple, but the same cannot be said for the reals.

Move 1: The spoiler selects a non-zero element a_1. The duplicator selects a non-zero rational number $b_1 = \frac{p_1}{q_1}$.

Move 2: The spoiler selects another non-zero element a_2 such that $lcm(a_1, a_2)$ does not exists. The duplicator selects another non-zero rational number $b_2 = \frac{p_2}{q_2}$. The spoiler now decrements the the number of turns to $\max(|p_1 q_2| + 1, |p_2 q_1| + 1)$. Technically, the spoiler could reduce the number of turns to a number on the order of $\log_2 (lcm(b_1, b_2) + 1)$, which would be smaller, but either method works so we provide the simplest.

Move 3: The spoiler now selects $b_3 = p_1 p_2$. This is a common multiple $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$, but nothing is an integral multiple of both a_1 and a_2. The duplicator cannot select an element that is a multiple of each so the duplicator chooses only one.

Move 4: If the spoiler selected a_3 to be a multiple of a_1 then the spoiler begins selecting $b_4 = b_2 + b_2 = 2 * b_2$ then $b_5 = b_4 + b_2 = 3 * b_2 \ldots$ until $b_{p_1 q_2 + 2} = p_1 q_2 b_2 = b_3$. However, $a_{p_1 q_2 + 2} = p_1 q_2 a_2 \neq a_3$. The same holds for a_3 being a multiple of a_2 instead of a_1.

□

Thus, the game $\mathcal{G}(\mathcal{R}, +; \mathcal{Q}, +)$ takes exactly $\omega + 1$ turns. Notice that through considering games of higher arity we have also shown that the same game with higher arity addition requires $\omega + 1$ turns.

4.2 $\mathcal{G}(\mathcal{Q}^n, +; \mathcal{Q}^m, +)$ and the Free Basis Approach

The game $\mathcal{G}(\mathcal{R}, +; \mathcal{Q}, +)$ was a specific case of a much broader theme. The spoiler's winning strategy was to pick two numbers that did not have a common multiple in the real numbers. The spoiler could also have looked for a common divisor, instead of a common multiple. In both cases, these numbers were related, or had something in common, with respect to addition. They were linearly dependent elements. The spoiler's strategy worked because the free basis of the rationals over the scalar field of the rationals was of size one. However, the size of the free basis of the reals over the scalar field of the rationals is infinite. Thus, after two independent selections in the reals, the duplicator was forced to select two dependent elements in the rationals. It could take the spoiler an arbitrarily large number of turns to show that two numbers are dependent. However, once the two numbers are known, the game will take a set number of turns. Thus, the spoiler delays decrementing the number of turns to a finite ordinal for two turns and the game requires $\omega + 1$ turns.

Games in this section demonstrate the theme of using linear independence and a difference in the size of the free basis different structures. We will let addition in multiple dimensions denote vector addition.

Theorem 4.6. *The game $\mathcal{G}(\mathcal{Q} \times \mathcal{Q}, +; \mathcal{Q}, +)$ is an $(\omega + 1)$-turn game.*

Proof. It takes three turns to find the first linearly dependent element, ignoring the identity element. It therefore takes three turns to declare the length of the game. The third element in $\mathcal{Q}$ is therefore a combination of the first two, unlike in $\mathcal{Q}^2$. □

Theorem 4.7. *The game $\mathcal{G}(\mathcal{Q}^n, +; \mathcal{Q}^m, +)$ $(n > m)$ is an $(\omega + m)$-turn game.*

Proof. As in Theorem 4.6, the spoiler selects $m + 1$ elements in $\mathcal{Q}^n$ and forces the spoiler to select an element in $\mathcal{Q}^m$ that is linearly dependent upon at least one of the other elements selected. At this point, the spoiler decrements the number of turns to a finite number and shows that the linear independence of the relative subsets is different.

Conversely, the spoiler may not win in any fewer turns due to Theorem 4.3. The spoiler cannot win the higher arity game in m turns and therefore needs $\omega + m$ turns to win. □

As in the game $\mathcal{G}(\mathcal{R}, +; \mathcal{G}, +)$, Theorem 4.7 will also hold for addition of higher arity. These two games are in fact closely related. Since the real numbers with respect to linear independence are equivalent to $\mathcal{Q}^\infty$, the real numbers can be substituted for the rational numbers:

Theorem 4.8. *The game $\mathcal{G}(\mathcal{R}, +; \mathcal{Q}^m, +)$ (finite m) is an $(\omega + m)$-turn game.*

In general, Theorem 4.3 and an analysis of the free basis of the sets imply the following theorem:

Theorem 4.9. *Suppose S_1 and S_2 form vector spaces over the operation of addition and a scalar field the rational numbers. Further suppose the size of their free bases are B_1 and B_2 respectively and $B_1 \neq B_2$. The game $\mathcal{G}(S_1, +; S_2, +)$ will take $\omega + \min(B_1, B_2)$ turns.*

4.3 Bounds on Unary Operation Games From Directed Subgraph Isomorphism

In this section we will discuss unary operation games and find general bounds.

Theorem 4.10. *All Unary Operation Games are either won by the spoiler in fewer than $\omega * 2$ turns or are won by the duplicator with any ordinal game length.*

Proof. All games played on unary operations will henceforth be represented with graphs. If the iterative operation of an element can result in the element itself then instead of showing a loop in our graph, we will label the element with the number of iterations of the operations needed to return to the element. Otherwise, we will show a directed edge from one element to the result of its operation. This means that if an element is labeled with a number it cannot be directed towards another element.

With this labeling, each set accompanied by an operation forms a set of trees, also known as a forest. We will only consider individual connected components, which are all in form of trees, though some are not rooted. When we play the Duplicator-Spoiler Game on these trees, we will be allowed to select elements and make sentences about the relative subgraph isomorphism of the trees. If the elements selected in both sets form isomorphic graphs, the the duplicator wins. Otherwise, the spoiler wins. As a result, we can then inch up and down trees to show isomorphism.

The spoiler's strategy will be one of labeling. First, a set of trees which have no isomorphic duplication will be labeled with a finite number of individual turns, else the graphs are isomorphic. The graphs cannot be isomorphic else the duplicator wins. Once this is done, it can only take a finite number of moves to show that these graphs are in fact not isomorphic. Thus, the game is limited by $\omega + finite$ turns. This is because the spoiler may not know the number of turns required to show the graphs are not isomorphic until the individual elements have been labeled. □

Interpreting unary operations as graphs we find the following.

Definition 4.11. *Let $U{+_a}$ denote the unary operation of adding a to another number. $U +_a (x) = x + a$.*

Theorem 4.12. *Let $a \in \mathcal{N}$. The game $\mathcal{G}(\mathcal{Z}, U{+_a}; \mathcal{Z}, U{+_1})$ will take $\omega + a$ turns.*

In Theorem 4.12, unary operations represent forests that have components that are simply straight lines. Their trees have no branches. The spoiler labels each component and shows that one forest has more components than the other.

Notice that, in each of these cases, the fewest number of elements needed to span one of the sets plus one is the number of turns needed to declare the remaining finite number of turns..

We are yet to find bounds on Binary Operation Games or games with even higher arity. Such a bound would involve interpreting the games as hypergraphs, instead of standard graphs. Should these bounds exist, they are either similar or form forming a pattern dependent on arity.

5 The Ordinal Game's Meaning in Formal Logic

Now that several ordinal games have been investigated we turn to the ordinal-turn game's meaning in formal logic. The question arises: how many quantifiers, or sets of quantifiers, do we need to distinguish orderings from ordinal-turn games?

One might assume ordinal games have an exact connection to higher-order logic, but our early examples prove otherwise. The duplicator wins the game $\mathcal{G}(\mathbb{R}, <; \mathbb{Q}, <)$ (Theorem 1.11), but the Least Upper Bound Axiom allows us to create the following second-order sentence that is modeled by the reals but not the rationals.

$$(\forall A)(\exists b)(\forall a \in A)[(a < b) \implies (\nexists c)(a < c < b)].$$

In fact, there are already game-variants that are connected to higher-order and weak higher-order logic. They involve selecting sets of elements and finite sets of elements, respectively. In our game, ω is more powerful than simply selecting a finite set of elements because the spoiler gets feedback from the duplicator as the elements are selected.

Therefore a new system of logic is needed to quantify the unique characteristics of ordinal numbers of turns. However, our game is nonetheless connected to finite numbers of logical quantifiers and will thus resemble weak higher-order logic. The system could involve quantification over natural numbers that will determine the number of quantifiers used. We call this omega notation. For example, let us consider the game $\mathcal{G}(\mathbb{Z} + \mathbb{Z}, <; \mathbb{Z}, <)$. The following sentence is an example of the type of logic needed. This sentence is true of $\mathbb{Z}$, but not of $\mathbb{Z} + \mathbb{Z}$. It also contains two first-order quantifiers before quantifying over a natural number which determines the size of the omega notation. This corresponds to a two-turn delay in the spoiler's decrementation ($\omega + 1$ turns).

$$(\forall x)(\forall y)(\exists \mathbf{i})(\exists a_1, a_2 \dots a_i)[(x < a_i < y) \implies (a_i = a_1) \vee (a_i = a_2) \vee \dots \vee (a_i = a_{i-1})]$$

Similarly, this can be expanded for higher ordinals. Two natural numbers can denote $\omega * 2$, a natural number of natural numbers ω^2 and so on.

6 Conclusion and Open Problems

We have introduced ordinal-turn games and investigated their properties, have found a general form for orderings whose games can take any number of ordinal turns and have investigated several operation games with ordinal numbers of turns. We have found bounds on certain types of Binary Operations Games using elements of free bases and bounds on Unary Operations Games using directed subgraph isomorphism. Lastly, we investigated a possible connection between ordinal numbers of turns and logical expressibility.

The meaning of ordering games can be further investigated. What is the deeper meaning of having large ordinal numbers of turns such as uncountable ordinals? Is it merely a large recursive structure or does reaching ϵ_0, ω_1 or ω_2 carry special significance?

Since games with scattered ordering that can require any ordinal number of turns have been found , we next want to know, can the spoiler win any game played on scattered orderings given some ordinal number of turns?

As bounds have been found for unary operations it seems natural to assume Operation Games of higher arity will also have bounds. They may also be $\omega * 2$, or they may follow a pattern based upon the arity. One approach for finding bounds is to adjust the graphs of unary operations to hypergraphs to account for higher arity.

Although we can see a relationship between our ordinal game variant and formal logic, a full system is yet to be developed. As of now it seems that this system will include quantifying over natural numbers that determine the number of quantifiers, as well as standard logical symbols. One natural number seems to represent ω turns, two natural number $\omega * 2$ turns and so on.

Once this system is developed, a proof of its connection to the ordinal games is needed. Such a proof might involve an adaptation of Ehrenfeucht's original proof of the first-order connection to predetermined numbers of turns [3]. Proving this connection is an important next step.

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