Homework 01, MORALLY Due Feb 5 at 10:00AM
DEAD-CAT DAY Feb 7, 10:00AM

1. (20 points) Give a Propositional Formula on four variables that has exactly three satisfying assignments. Give the satisfying assignments.

   SOLUTION TO PROBLEM ONE

   We give a formula of the form $C_1 \vee C_2 \vee C_3$ so that the three assignments are those that make C_1 true, C_2 true, C_3 true, and make sure that no assignment satsifies two of those

   Vars are w, x, y, z

   $$(w \wedge x \wedge y \wedge z) \vee (w \wedge x \wedge y \wedge \overline{z}) \vee (w \wedge x \wedge \overline{y} \wedge \overline{z})$$

   EXERCISE FOR YOU: Do more of these. How many such formulas are there?

   END OF SOLUTION TO PROBLEM ONE

2. (20 points) Use truth table so show that

$$(x \vee y) \wedge z$$

is not equivalent to

$$x \vee (y \wedge z).$$

INDICATE which rows they differ on.

SOLUTION TO PROBLEM TWO

Below is the truth table. Here is how I did it with some shortcuts.

Look at the first formula $(x \vee y) \wedge z$. If $z = F$ then its false. So I filled in those four entries. For those entries left $z = T$, so the formula is really $x \vee y$. So thats T unless $x = y = F$.

Look at the second formula $x \vee (y \wedge z)$. If x is true then its true. So I filled in those four entries. For those entries left $x = F$, so the formula is really $y \wedge z$. So thats F unless $y = z = T$.

We put a * on the evaluation when the formulas give different values.

x	y	z	$(x \vee y) \wedge z$	$x \vee (y \wedge z)$
F	F	F	F	F
F	F	T	F	F
F	T	F	F	F
F	T	T	T	T
T	F	F	$F*$	$T*$
T	F	T	T	T
T	T	F	$F*$	$T*$
T	T	T	T	T

END OF SOLUTION TO PROBLEM TWO

3. (30 points) n has the *emily property* if there is a formula on n variables with exactly $n^2 + 100$ satisfying assignments.

   (a) (15 points) Fill in the BLANK in the following sentence

   n has the emily property IFF BLANK(n).

   The condition BLANK has to be simple, for example, n is divisible by 5 (thats not the answer).

   (b) (15 points) Prove the statement you made in the first part. Note that this means you have to show that
   If BLANK(n) then n has the emily property
   and
   If NOT(BLANK(n)) then n DOES NOT have the emily property

SOLUTION TO PROBLEM THREE

(a) n has the emily property IFF BLANK(n).
So we need that there is a boolean formula with exactly $n^2 + 100$ satisfying assignments. Here is how you would construct such a formula: Make a Truth Table where the first $n^2 + 100$ rows are T and the rest are F, and then make a formula from that truth table (as shown in class). SO you might thing you can do this for ALL n. But you would be wrong. There are 2^n rows in a truth table. So we need

$$n^2 + 100 \le 2^n$$

So we need $2^n - n^2 - 100 \ge 0$.
There are two ways to do this:
METHOD ONE: Plug $n = 1, 2, 3, \ldots$ until you get $2^n - n^2 - 100 \ge 0$. Then assume this is true for all larger n (this is not rigorous, but its true and we're fine with it).
$n = 1$: $2^1 - 1^2 - 100 < 0$ so NO
$n = 2$: $2^2 - 2^2 - 100 < 0$ so NO
WAIT- we need to have $2^n \ge 100$. This might not suffice but we should start there. Thats $n = 7$ since $2^6 = 64 < 100$ but $2^7 = 128 > 100$.

$n = 7$: $2^7 - 7^2 - 100 = 128 - 149 < 0$.

$n = 8$: $2^8 - 8^2 - 100 = 256 - 164 > 0$. YEAH.

So BLANK(n) is $n \geq 8$.

METHOD TWO: Let $f(x) = 2^x - x^2 - 100$. We need to know when this is always positive. Lets take the derivative and find max and min

$f'(x) = (\ln 2)2^x - 2x$. One can find that their are two roots, one close to 1 and one close to 3. Evaluating the function in the intervals before and between the roots, one can find out tht being 4 the function is increasing.

Now look at the original f. Its positive for the first time (at an integer) at 8. Since the deriviative is positive fom 4 on, f is increasing and hence positive from 8 on.

BLANK(n) is $n \geq 8$.

METHOD TWO is messier than METHOD ONE; however, METHOD TWO is more rigorous. If that does not impress you, you are not alone.

(b) I did the prove while doing the problem.

END OF SOLUTION TO PROBLEM THREE

4. (30 points) (NOTE: 0 and 1 are NOT prime. You will need that for this problem.)

   (a) (15 points) View the input x, y, z as the number in binary xyz which we denote (xyz). For example, 100 is 4.

   Write a Truth Table for the following function with 3 inputs x, y, z and 3 outputs a, b, c.

$$f(x,y,z) = \begin{cases} 0 & \text{if } (xyz) \text{ is NOT PRIME.} \\ 1 & \text{if } (xyz) \text{ is PRIME.} \end{cases}$$

   (b) (15 points) Convert your truth table into formulas. DO NOT SIMPLIFY.

   (c) (0 points- DO NOT HAND IN) Draw a circuit that computes that truth table.

SOLUTION TO PROBLEM FOUR

(a) Truth Table for IS IT A PRIME

a	b	c	prime?
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

(b) Formula. Look at the rows that evaluate to 1. For each one obtain a mini-fml. Then OR then together.

$$(\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge b \wedge c) \vee (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge c)$$

END OF SOLUTION TO PROBLEM FOUR