Homework 7, MORALLY Due March 25
Recall: Spring Break is March 18-22

1. (30 points-10 points each) Write the following sequences in closed form: For example, if the sequence was

$$1, 1, 2, 2, 3, 3, \ldots$$

(So $a_1 = 1,\ a_2 = 1,\ a_3 = 2,\ a_4 = 2,\ \ldots$.)

then the answer is

$$a_n = \begin{cases} (n+1)/2 & \text{if } n \equiv 1 \pmod 2 \\ n/2 & \text{if } n \equiv 0 \pmod 2 \end{cases} \tag{1}$$

   (a) $2, 5, 10, 17, 26, \ldots$.

   (b) $1, 1, 1, 2, 2, 2, 3, 3, 3, \ldots$

   (c) $1, -4, 9, -16, \ldots$.

2. (25 points) We define a sequence as follows:

   $a_1 = 1$. For all $n \geq 2$, $a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$.

   (a) (0 points but you will need it for the next part.) Write a program that will, given n, find the following:

      i.
         - For $1 \leq i \leq n$,

           $$\frac{\text{The number of } a_i \equiv 0 \pmod 2}{n}$$

         - For $1 \leq i \leq n$,

           $$\frac{\text{The number of } a_i \equiv 1 \pmod 2}{n}$$

      ii.
         - For $1 \leq i \leq n$,

           $$\frac{\text{The number of } a_i \equiv 0 \pmod 3}{n}$$

         - For $1 \leq i \leq n$,

           $$\frac{\text{The number of } a_i \equiv 1 \pmod 3}{n}$$

         - For $1 \leq i \leq n$,

           $$\frac{\text{The number of } a_i \equiv 2 \pmod 3}{n}$$

      iii. **Similar** for mod 5,7,11,13,17,19.

   (There is more to this problem on the next page)

(b) (15 points) Run your program with n=1000. Report the results neatly.

(c) (10 points) Based on your data give conjectures about the following

   i. Is $a_n \equiv 0 \pmod 2$ infinitely often? Do the cases $a_n \equiv 0 \pmod 2$ and Do the cases $a_n \equiv 1 \pmod 2$ occur about the same amount of time?

   ii. **Similar** for Mod 3,5,7,11,13,17,19.

3. (25 points) In this problem we guide you through a proof that number of the form $4^n(8k+7)$ cannot be written as the sum of 3 squares.

   (a) (0 points but you will need this for later) Find the following set

   $$X = \{x^2 \pmod 8 : x \in \{0,1,2,3,4,5,6,7\}\}.$$

   (b) (7 points) Show that if $x \equiv 7 \pmod 8$ then x is NOT the sum of three squares.

   (*Hint:* Show that any three elements of X from Part a can never sum to 7 mod 8.)

   (c) (8 points) Prove that if $x^2 + y^2 + z^2 \equiv 0 \pmod 4$ then x, y, z are all even. (*Hint:* Look at what happens when:

   1 of $\{x, y, z\}$ is odd,

   2 of $\{x, y, z\}$ is odd,

   3 of $\{x, y, z\}$ is odd.

)

   (d) (9 points) Prove the following by induction on n.

   Theorem Let $n \geq 0$. Let $k \in \mathsf{N}$. Show that $4^n(8k+7)$ cannot be written as the sum of 3 squares.

4. (20 points) On HW 4 problem 3 you were asked for a conjecture of the form

$$(\forall a \in \mathsf{Z})[a^m \equiv a \pmod{m}] \text{ iff BLANK}(m).$$

You based your conjecture on a program that you wrote and ran up to 200.

ALL of your had the answer BLANK(m) is m *is prime*.

Run your program up to 1000.

Did you find any value m such that:

- $(\forall a \in \mathsf{Z})[a^m \equiv a \pmod{m}]$
- m is NOT prime.

If you DID then output the smallest such m.

If you DID NOT then state a Theorem in Mathematics, it can be a little one, that proves your conjecture.