250 MIDTERM-UNTIMED PART
Morally Due Monday Feb 26, 10:00AM
Submit on Gradescope Just Like a HW

READ THE INSTRUCTIONS

1. This is an OPEN BOOK, OPEN WEB, TAKE HOME exam If you have a question, email or post on piazza.
2. There are 3 problems which add up to 50 points. The exam is MORALLY due Feb 26, 10:00AM. Since the timed exam is on Feb 29 you are well advised to finish the untimed exam by the Moral Due date to give you time to study for the timed exam.
3. This part must be **typed**
4. For each question show all of your work

1. (25 points) Let $n \in \mathsf{N}$. NSQ(n) is the least number such that n can be written as the sum of NSQ(n) squares. Clearly NSQ(n) $\leq n$ since

$$n = 1^2 + \cdots + 1^2.$$

It is known that NSQ(n) ≤ 4.

In this problem you will write and run two programs that will, given $n \in \mathsf{N}$, find a bound on NSQ(n).

(a) (0 points but you need this for later problems) Write a program that will, given n, does the following:

- Find the largest n_1 such that $n_1^2 \leq n$. If $n - n_1^2 = 0$ then you are done: $n = n_1^2$ so output 1. If not then go to the next step.
- Find the largest n_2 such that $n_2^2 \leq n - n_1^2$. If $n - n_1^2 = 0$ then you are done: $n = n_1^2 + n_2^2$ so output 2. If not then ...
- Keep going like this until there is an output.

(b) (0 points but you need this for later problems) Write a program that will, given n, find, for all $1 \leq i \leq n$, a number $A[i]$ such that i can be written as the sum of $A[i]$ squares.

- $A[0] \leftarrow 0$ (0 can be written as the sum of 0 squares).
- $A[1] \leftarrow 1$ (1 can be written as the sum of 1 square).
- For $i \leftarrow 2$ to n

$$A[i] \leftarrow 1 + \min\{A[i - j^2] \colon i - j^2 \geq 0\}$$

(c) (5 points) Run the first program on $n = 1, 2, \ldots, 1000$. List out all of the n where the answer was ≥ 5.

(d) (0 points) How would you fill in the following sentence

The first program on input n outputs a number ≥ 5 iff BLANK.

(e) (5 points) Run the second program on $n = 1000$ (so we now know the answers for $1, \ldots, 1000$). List all of the n where $A[n]$ differs from the first program on n.

(f) (0 points) How would you fill in the following sentence
The first and second differ on n iff BLANK.

(g) (5 points) List all of the n such that $A[n] = 4$.

(h) (5 points) How would you fill in the following sentence with a SIMPLE property BLANK.
if n has property BLANK then $A[n] = 4$
(It does not need to be the case that if $A[n] = 4$ then it has property BLANK.)

(i) (0 points) Use your second program, and the PRIMES function in Python, to list out all odd primes $p \leq 200$ such that $A[p] = 2$.

(j) (5 points) How would you fill in the following sentence with a SIMPLE property BLANK.
if n is prime and has property BLANK then $A[n] = 2$
(It does not need to be the case that if $A[n] = 2$ then it has property BLANK.)

2. (12 points–3 points each) For each of the following sentences say if they are TRUE or FALSE and JUSTIFY your answer. ADVICE: The FIRST thing your answer should have is either the word TRUE or the word FALSE.

   Let $\mathsf{I} = \mathsf{R} - \mathsf{Q}$, the set of irrationals, for this problem.

   (a) $(\forall x, y \in \mathsf{I})[x < y \to \frac{x+y}{2} \in \mathsf{I})]$.

   (b) $(\forall x, y \in \mathsf{I})[x < y \to (\exists z \in \mathsf{I})[x < z < y]$.

   (c) $(\forall x, y \in \mathsf{I})[x < y \to (\exists z \in \mathsf{Q})[x < z < y]$.

   (d) $(\forall x, y \in \mathsf{Q})[x < y \to (\exists z \in \mathsf{I})[x < z < y]$.

3. (13 points) Consider the following arithmetic function:

$$f(x_1, x_2, x_3, x_4, x_5) = \begin{cases} 1 & \text{if exactly two of the inputs is 1 AND } x_5 = 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) (3 points) How many rows are in the Truth Table for f?

(b) (0 points) Do you want to do the Truth Table for f (Hint: The answer is NO!!!!)

(c) (2 points) How many rows of the truth table have output 1? (HINT: don't get fancy here. Just list them out as you will need that for a later problem anyway.)

(d) (0 points) Do you want to write down just those rows of the Truth Table? (Hint: The answer better be YES since that's what the next part asks you to do, and really, there just aren't that many rows.)

(e) (2 points) Write down all of the rows of the Truth Table that output 1.

(f) (2 points) Write a DNF formula for f using the partial truth table in the last part. (Hint: Don't get fancy. Don't try to make it shorter.)

(g) (2 points) You used a trick to avoid writing down that TT. Name a function $g(x_1, \ldots, x_n)$ where the trick would save you LOTS of time.

(h) (2 points) Name a function $h(x_1, \ldots, x_n)$ where the trick would NOT save LOTS of time. Explain why.