CMSC452 Midterm FROM SPRING 2021

1. (20 points) Give an example of each of the following. NO PROOF REQUIRED

   (a) (10 points) A Context Sensitive Language that is not a Context Free Language.

      Give the language AND the Context Sensitive Grammar for it.

      NOTE FROM BILL WRITTEN IN 2024: FOR OUR EXAM CONTEXT SENSTIVE GRAMMARS ARE NOT IN SCOPE. THE NEXT LINE IS BACK TO 2021.

      SOLUTION

      $\{w : \#_a(w) = \#_b(w) = \#_c(w)\}$.

      Here is the CSG for it

      $S \to ABCS$

      $AB \to BA$

      $BA \to AB$

      $AC \to CA$

      $CA \to AC$

      $BC \to CB$

      $CB \to BC$

      $A \to a$

      $B \to b$

      $C \to c$

      $S \to e$.

      END OF SOLUTION

(b) (10 points) A regular language L over the alphabet $\Sigma = \{a\}$ such that any DFA for L *requires* ≥ 100 states.

SOLUTION

You only needed to give ONE example, but we give three for your enlightenment.

$L_1 = \{a^{100}\}$

$L_2 = \{a^i : i \neq 100\}$

$L_3 = \{a^i : i \equiv 0 \pmod{100}\}$

$L_4 = \{a^i : i \not\equiv 0 \pmod{100}\}$

all work.

100 could be replaced by larger numbers

END OF SOLUTION

2. (20 points)

   The alphabet is $\{a\}$. Let

$$L = \{a^i : i \neq 200\}$$

   Does there exist an NFA for L with less than 100 states? If so then draw the NFA; you may use DOT DOT DOT (You DO NOT have to prove that the NFA works.) If not then PROVE there is no such NFA.

   You may answer this problem on this page and the next page.

SOLUTION

If x and y are relatively prime then $xy - x - y$ cannot be written as the sum of x's and y's, but any number larger can be.

We need to find x, y such that $xy - x - y$ is just below 200:

$x = 13$ and $y = 17$. Then $xy - x - y = 191$.

SO we have that

(a) 191 CANNOT be written as $13a + 17b$, where a and b are integers.

(b) Every number ≥ 192 can be written as $13a + 17b$.

(c) Every number ≥ 201 can be written as $13a + 17b + 9$.

This can be used to make a 2-loop NFA M such that

(a) For all $i \geq 201$, M accepts a^i.

(b) a^{200} is not accepted.

(c) For some $i < 200$ M accepts a^i. We are not concerned with these.

(d) This NFA has $9 + 17 = 26$ states.

WE omit the details, but you had to supply them.

We are not done yet.

Note that $2 \times 3 \times 5 \times 7 = 210$.

(a) Create a 2-state DFA for $\{a^i : i \not\equiv 0 \pmod 2\}$. Note that $200 \equiv 0 \pmod 2$.

(b) Create a 3-state DFA for $\{a^i : i \not\equiv 2 \pmod 3\}$. Note that $200 \equiv 2 \pmod 3$.

(c) Create a 5-state DFA for $\{a^i : i \not\equiv 0 \pmod 5\}$. Note that $200 \equiv 0 \pmod 5$.

(d) Create a 7-state DFA for $\{a^i : i \not\equiv 4 \pmod 7\}$. Note that $200 \equiv 4 \pmod 7$.

Have a start state go via e-transitions to M and to these four DFA's.

Number of states is $26 + 2 + 3 + 5 + 7 = 43$.

3. (20 points) Prove that the following language is NOT REGULAR.

$$L = \{ww : w \in \{a,b\}^*\}.$$

You may answer this problem on this page and the next page.

SOLUTION

Assume L is regular. By the pumping lemma there exists n_0, n_1 such that

For all $w \in L$, $|w| \geq n_0$ there exists x, y, z such that

Let $w = a^{2m}ba^{2m}b$.

Let m be large enough so that when you apply the PL all of the a' go into the xy.

$a^{2m}ba^{2m}b = xyz$

$x = a^{k_1}$

$y = a^{k_2}$ Note that $k_2 \neq 0$.

$z = a^{k_3}ba^{2m}b$.

where $k_1 + k_2 + k_3 = 2m$.

By the PL $xy^2z \in L$

$$xy^2z = a^{k_1}a^{2k_2}a^{k_3}ba^{2m}b = a^{2m+k_2}ba^{2m}b$$

But since $k_2 \neq 0$, this string is NOT in L. Contradiction.

END OF SOLUTION

4. (20 points) Let the alphabet be $\{a, b, c, d\}$. Give a Context Free Grammar for

$$\{a^m w d^n : m < n \text{ and } w \in \{b, c\}^*\}$$

No proof required and it DOES NOT have to be in Chomsky Normal Form.

You may answer this problem on this page and the next page.

SOLUTION

$S \rightarrow Td$

$T \rightarrow Td \mid aTd \mid R$

$R \rightarrow bR \mid cR \mid e$

END OF SOLUTION

5. (20 points)

Let $\Sigma = \{a, b\}$. For every $n \in \mathsf{N}$ let

$L_n = \{ww : |w| = n\}$.

For example

$L_2 = \{aaaa, abab, baba, bbbb\}$

Show that any DFA for L_n requires $\geq 2^n$ states.

NOTE FROM BILL FROM 2024: WE DID NOT DO CAREFUL PROOFS OF SHOWING THAT A LANGUAGE REQUIRED BLAH NUMbER OF STATES. AND YOU WERE NEVER TESTED ON THIS. HENCE THIS PROBLEM WOULD NOT BE IN SCOPE FOR OUR EXAM. THE NEXT CAP LETTER PARAGRAPH IS FROM 2021.

(NOTE - DO NOT give me a DFA that uses that many states. That is IRRELEVANT to this problem. You need to show that ANY DFA REQUIRES $\geq 2^n$ states.)

You may answer this problem on this page and the next page.

SOLUTION

Let $M = (Q, \Sigma, \delta, s, F)$ be a DFA for L_n. We extend δ to strings.

Let $x, y \in \Sigma^n$ such that $\delta(s, x) = \delta(s, y)$. We show that $x = y$.

Since $\delta(s, x) = \delta(s, y)$ we have

$\delta(s, xx) = \delta(s, xy)$.

Since $xx \in L_n$, $\delta(s, xx) \in F$, so $\delta(s, xy) \in F$. Hence $xy \in L_n$, so $x = y$.

We have shown that the delta map from Σ^n to Q is 1-1. Hence there are $\geq 2^n$ states

END OF SOLUTION

Scratch Paper