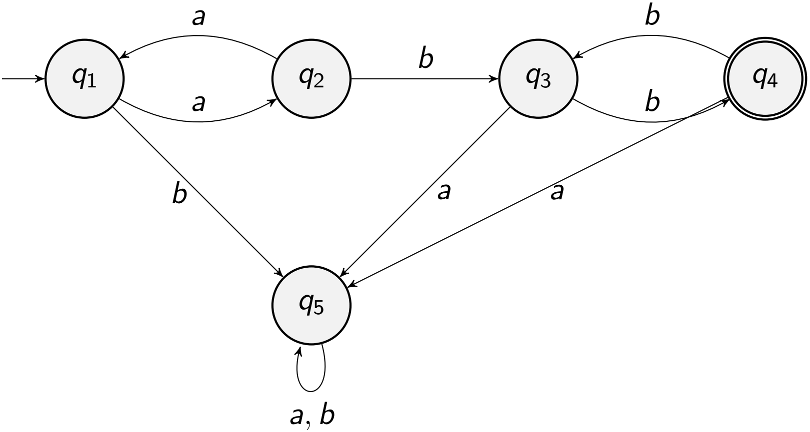
Review for CMSC 452 Midterm

Deterministic Finite Automata (DFA)

DFA Diagram

DFA Diagram



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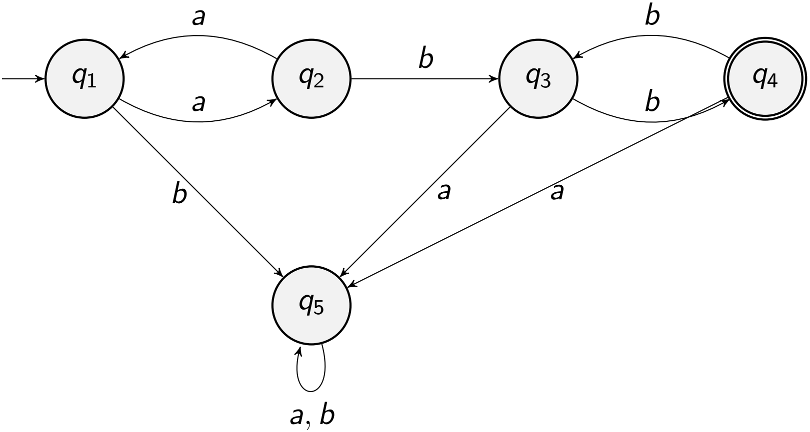

What is the language?

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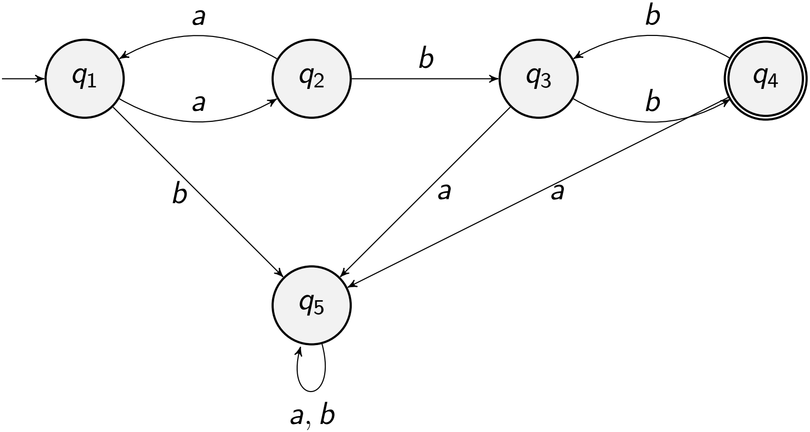

What is the language?

Odd number of *a*'s followed by an even number of *b*'s, but at least two.

$$\{w : \#_a(w) \equiv 1 \pmod 2 \wedge \#_b(w) \equiv 2 \pmod 3\}$$

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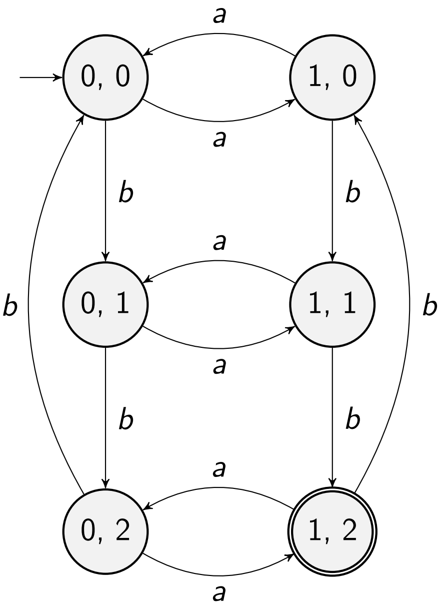

Transition Table

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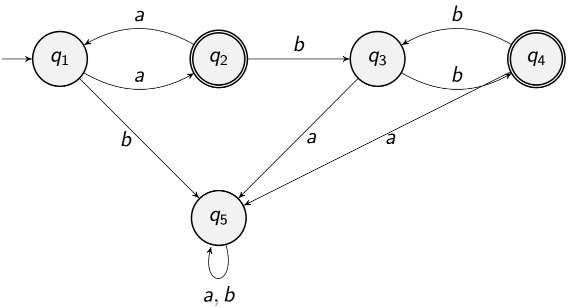

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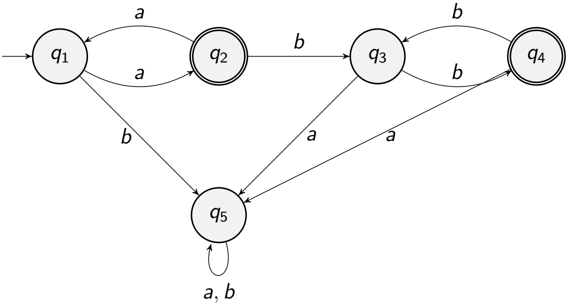

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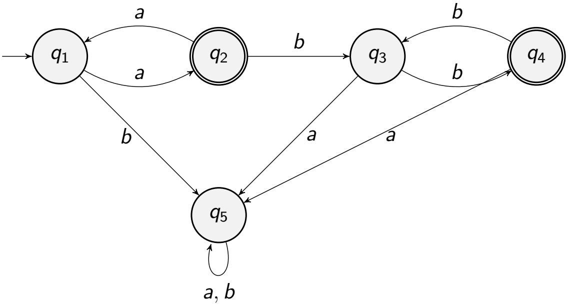

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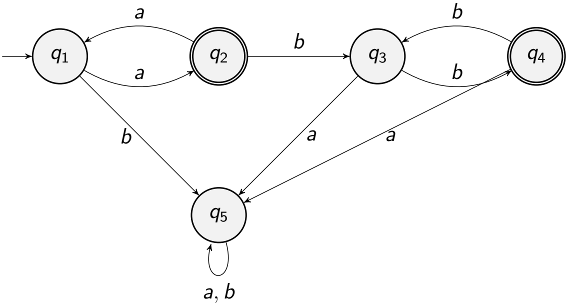

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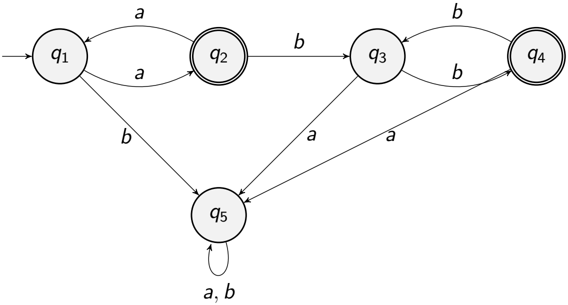

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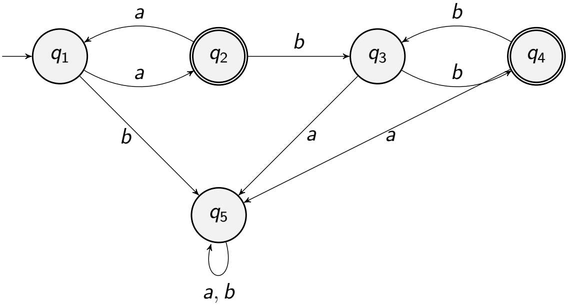

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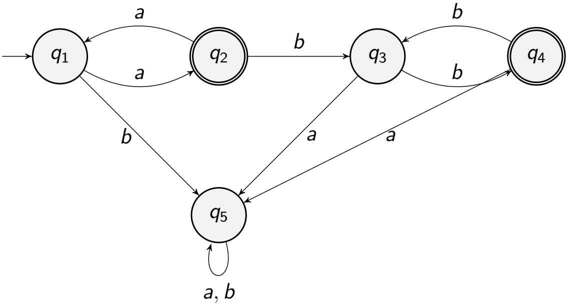

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- Transition function

	a	b
q_1	q_2	q_5
q_2	q_1	q_3
q_3	q_5	q_4
q_4	q_5	q_3
q_5	q_5	q_5

Divisibility

Divisibility

We get a DFA (a trick?) for Mod 11.

Trick for Mod 11. All $\equiv$ are Mod 11

Is there a trick for mod 11?

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Pattern is $1, -1, 1, -1, \ldots.$

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Thm $d_n \cdots d_0 \equiv d_0 - d_1 + d_2 - \cdots \pm d_n$.

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Pattern is $1, -1, 1, -1, \ldots$.

Thm $d_n \cdots d_0 \equiv d_0 - d_1 + d_2 - \cdots \pm d_n$.

Proof may be on HW or Midterm or Final or some combination.

DFA for Mod 11

Need to keep track of both the running weighted sum mod 11 and if you are reading an even or odd place.

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$$\delta((i,j),\sigma) \begin{cases} (i+\sigma \pmod{11}, j+1 \pmod{2}) \text{ if } j=0 \\ (i-\sigma \pmod{11}, j+1 \pmod{2}) \text{ if } j=1 \end{cases} \tag{1}$$

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Classifier If end in $(i, 0)$ or $(i, 1)$ then number is $\equiv i$.

Nondeterministic Finite Automata (NFA)

NFA's Intuitively

1. An NFA is a DFA that can guess.
2. NFAs do not really exist.
3. Good for $\cup$ since can guess which one.
4. An NFA accepts iff SOME guess accepts.

Every NFA-lang a DFA-lang!

Thm If L is accepted by an NFA then L is accepted by a DFA.
Pf Sketch L is accepted by NFA $(Q, \Sigma, \Delta, s, F)$ where

1. Get rid of e-transitions using reachability.
2. Get rid of non-determinism by using power sets. Possibly 2^n blowup.

Regular Expressions

Examples

1. $b^*(ab^*ab^*)^*ab^*$
2. $b^*(ab^*ab^*ab^*)^*$
3. $(b^*(ab^*ab^*)^*ab^*) \cup (b^*(ab^*ab^*ab^*)^*)$

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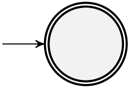

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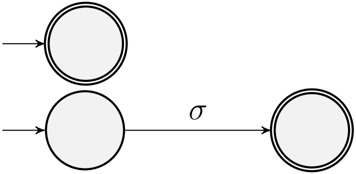

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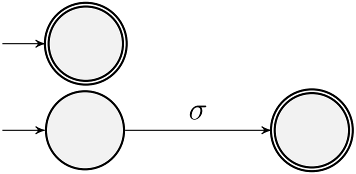

We skip rest of the proof.

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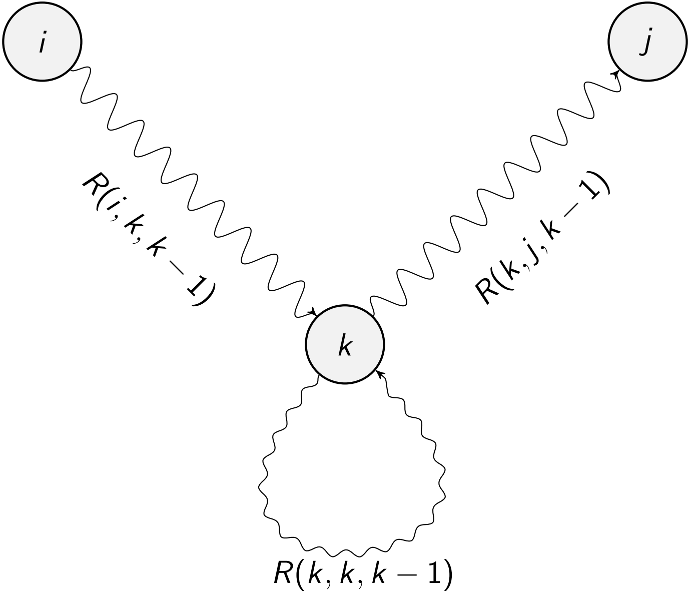
$R(i, j, k) = \{w : \delta(i, w) = j$ but only use states in $\{1, \ldots, k\}$ $\}$.

Inductive Step $R(i, j, k)$ as a Picture



Complete Proof on One Slide

For all $1 \leq i, j \leq n$:

$$R(i,j,0) = \begin{cases} \{\sigma : \delta(i,\sigma) = j\} & \text{if } i \neq j \ \} \\ \{\sigma : \delta(i,\sigma) = j\} \cup \{e\} & \text{if } i = j \ \} \end{cases} \tag{2}$$

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For all $1 \leq i, j \leq n$ and all $0 \leq k \leq n$

$$R(i,j,k) = R(i,j,k-1) \bigcup R(i,k,k-1) R(k,k,k-1)^* R(k,j,k-1)$$

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All $R(i,j,0)$ are Regex.

For all $1 \le i, j \le n$ and all $0 \le k \le n$

$$R(i,j,k) = R(i,j,k-1) \bigcup R(i,k,k-1) R(k,k,k-1)^* R(k,j,k-1)$$

If ALL $R(i,j,k-1)$ are Regex, then ALL $R(i,j,k)$ are Regex.

Textbook Regular Expressions

We allow numbers as exponents. For example the following is not a regex but is a trex:

$$\{a, b\}^* a \{a, b\}^n.$$

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Often the trex is shorter than the regex.

Closure Properties

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Property	DFA	NFA	regex
$L_1 \cup L_2$	Prod	*e*-trans	Def
$L_1 \cap L_2$	Prod	Prod	X
$\overline{L}$	Swap	X	X
$L_1 \cdot L_2$	X	*e*-trans	Def
L^*	X	*e*-trans	Def

Summary of Blowup for Closure Properties

X means **Can't Prove Easily**

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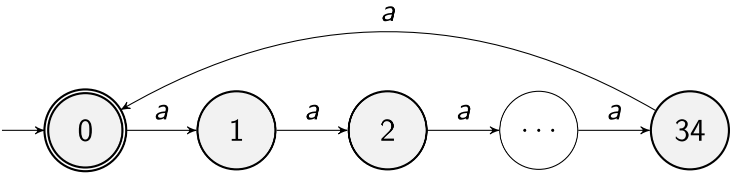
Closure Property	DFA	NFA	Regex
$L_1 \cup L_2$	$n_1 n_2$	$n_1 + n_2$	$\ell_1 + \ell_2$
$L_1 \cap L_2$	$n_1 n_2$	$n_1 n_2$	X
$L_1 \cdot L_2$	X	$n_1 + n_2 + 1$	$\ell_1 + \ell_2$
$\overline{L}$	n	X	X
L^*	X	$n + 1$	$\ell + 1$

Number of States for DFAs and NFAs

Minimal DFA for $L_1 = \{a^i : i \equiv 0 \pmod{35}\}$



Min DFA for $L_2 = \{a^i : i \not\equiv 0 \pmod{35}\}$

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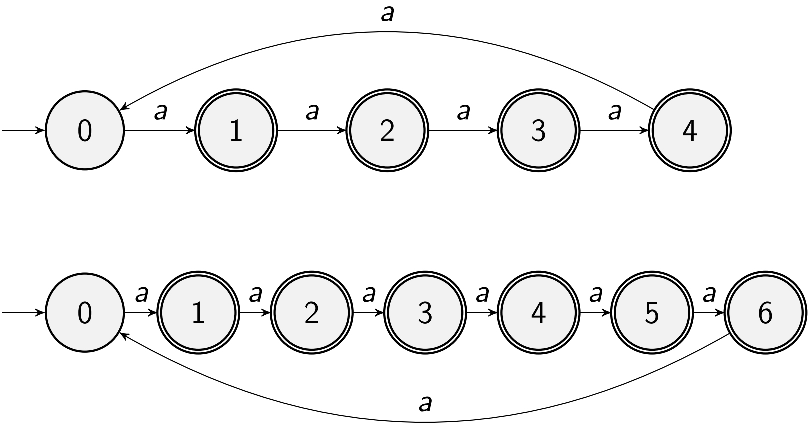
$\exists$ DFA for L_2: 35 states: swap final-$\overline{\text{final}}$ states in DFA for L_1.

Small NFA for $L_2 = \{a^i : i \not\equiv 0 \pmod{35}\}$

Need these two NFA's.



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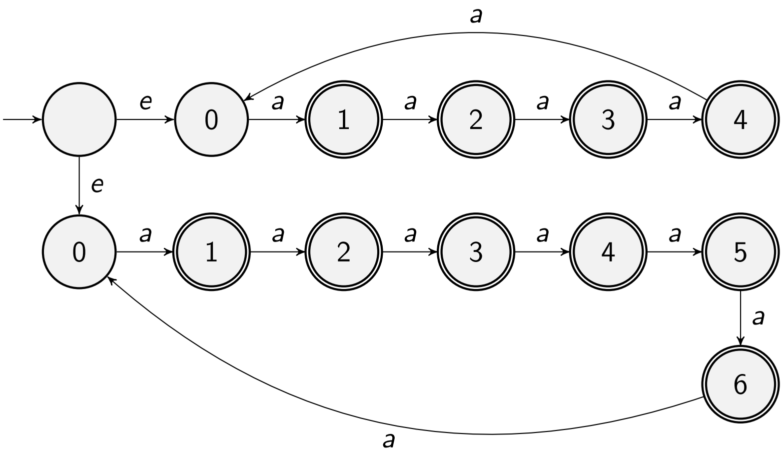

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DFA for L_2 requires 35 states.

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NFA for L_2 can be done with $1 + 5 + 7 = 13$ states.

DFA for $L_4 = \{a^i : i \neq 1000\}$

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1. There is a DFA for L_4 that has 1000 states.
2. Any DFA for L_3 has ≥ 1000 states.

Small NFA for $L_4 = \{a^n : n \neq 1000\}$

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Two NFA's:

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NFA A:

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- Does NOT accept a^{1000}.

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- Accepts all words longer than 1000.
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Small NFA for $L_4 = \{a^n : n \neq 1000\}$

Two NFA's:

NFA A:

- Does NOT accept a^{1000}.
- Accepts all words longer than 1000.
- Do not care about words shorter than 1000.

NFA B:

Small NFA for $L_4 = \{a^n : n \neq 1000\}$

Two NFA's:

NFA A:

- Does NOT accept a^{1000}.
- Accepts all words longer than 1000.
- Do not care about words shorter than 1000.

NFA B:

- Does NOT accept a^{1000}.

Small NFA for $L_4 = \{a^n : n \neq 1000\}$

Two NFA's:

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- Accepts all words longer than 1000.
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NFA B:

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- Accepts all words shorter than 1000.

Small NFA for $L_4 = \{a^n : n \neq 1000\}$

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Small NFA for $L_4 = \{a^n : n \neq 1000\}$

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NFA B:

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Create the union of NFA's A and B.

Sums of 32's and 33's

Thm

1) $(\forall n \geq 1001)(\exists x, y \in \mathbb{N})[n = 32x + 33y + 9]$.

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1) $(\forall n \geq 1001)(\exists x, y \in \mathbb{N})[n = 32x + 33y + 9]$.

2) $(\neg \exists x, y \in \mathbb{N})[1000 = 32x + 33y + 9]$.

NFA A

Idea Start state, then 8 states, then a loop of size 33 with a shortcut at 32.

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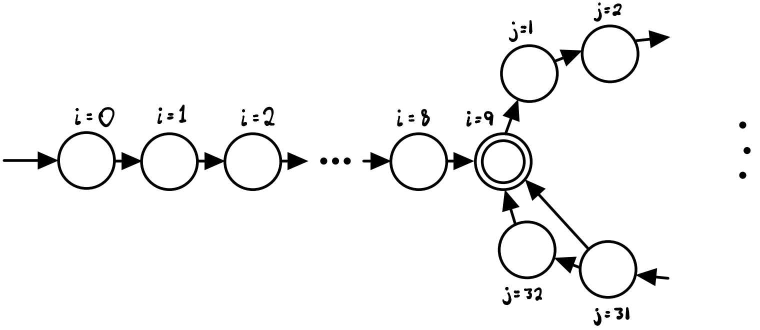

Why Works for $\{a^i : i \geq 1001\}$ and More

By the loop Theorem for 32, 33, the NFA

1. Accepts $\boldsymbol{\{a^i : i \geq 1001\}}$.
2. Might accept more.
3. DOES NOT accept a^{1000}.

Number of States for $\{a^i : i \geq 1001\}$

1. Start state

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Total number of states: $9 + 33 = 42$.

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$1000 \equiv 1 \pmod{3}$ 3-state DFA for $\{a^i : i \not\equiv 1 \pmod{3}\}$.

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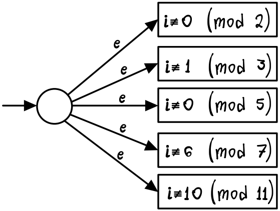
Could go on to 13,17, etc. But we will see we can stop here.

Machine B

Machine B



e
e
e
e
e
$i \neq 0 \pmod 2$
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NFA for $\{a^i : i \le 999\}$ AND More, but NOT a^{1000}

Thm Let M be the NFA from the last slide with the Mods.
$M(a^{1000})$ is rejected. This is obvious.

NFA for $\{a^i : i \le 999\}$ AND More, but NOT a^{1000}

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We omit the proof that it works but note that we use that the product of the mods

$$2 \times 3 \times 5 \times 7 \times 11 = 2310 > 1000.$$

How Many States for $\{a^i : i \le 999\}$ AND More, but NOT a^{1000}?

$2 + 3 + 5 + 7 + 11 = 28$ states.
Plus the start state, so 29.

NFA for $\{a^i : i \neq 1000\}$

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Take NFA of union using e-transitions for an NFA and do not count start state twice, so have

$$42 + 29 - 1 = 70 \text{ states.}$$

Can We Do Better than 70 States?

YES–59 states:

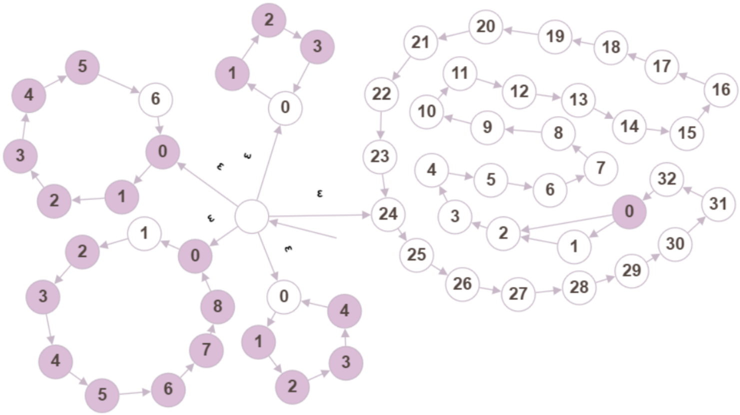

Figure: 59 State NFA for L_4

Math Needed for $\{a^i : i \neq n\}$ I

Frobenius Thm (aka The Chicken McNugget Thm)

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$xy - x - y + t = n + 1$.
This leads to loops and tail that are roughly $\leq 2\sqrt{n}$ states.

Math Needed for $\{a^i : i \ne n\}$ II

Thm Let $n \in \mathbb{N}$. Let $q_1, \ldots, q_k$ be rel prime such that $\prod_{i=1}^{k} q_i \ge n$. Then the set of all i such that
$i \not\equiv n \pmod{q_1}$.
$\vdots$
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Contains $\{1, \ldots, n-1\}$ and **does not contain n**

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So can use this to get NFA for $\{a^i : i \leq n-1\}$ (and other stuff but not a^n) with $\leq (\log n)^2 \log\log n$ states.

Proving That a Language Is Not Regular

Pumping Lemma (PL)

$L_1 = \{a^n b^n : n \geq 0\}$ is Not Regular

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Proof

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Proof Assume L_1 is regular via DFA M with m states.

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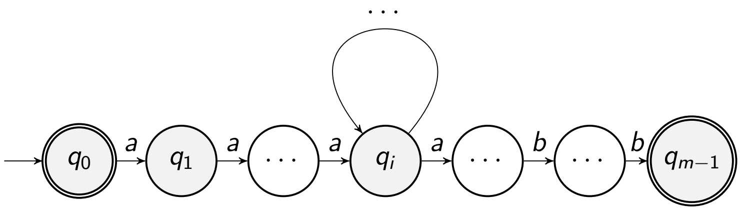

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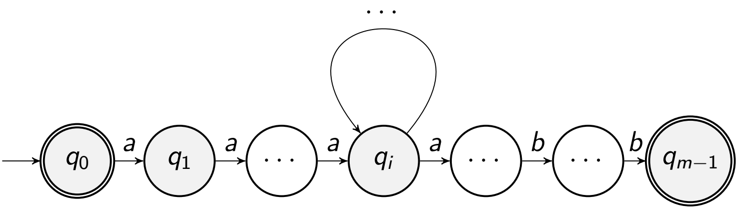

$a^{n+k} b^n$ is accepted by following the loop again. Contradiction.

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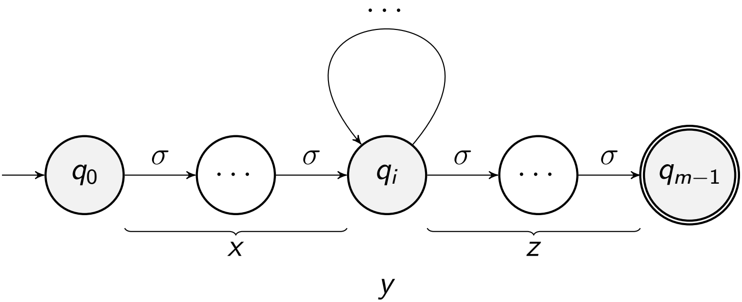

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We then find some i such that $xy^iz \notin L$ for the contradiction.

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Contradiction since $k \geq 1$.

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If L_3 is regular then $L_2 = \overline{L_3}$ is regular. But we know that L_2 is not regular. DONE!

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Proof
By PL for long enough $a^{n^2} \in L_4$ there exist $x = a^j$, $y = a^k$, $z = a^\ell$ with $xyz = a^{n^2}$. Also $a^j(a^k)^i a^\ell \in L_4$. (Note $k \geq 1$.)

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PL says you can always add some constant k to produce a word in the language.

Proof
By PL for long enough $a^{n^2} \in L_4$ there exist $x = a^j$, $y = a^k$, $z = a^\ell$ with $xyz = a^{n^2}$. Also $a^j(a^k)^i a^\ell \in L_4$. (Note $k \geq 1$.)

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So $n^2, n^2 + k, n^2 + 2k, \ldots$ are all squares. Omit the rest.

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Take $w = b^n a^{n+1}$, long enough so the y-part is in the b's.
Pump the y to get more b's than a's.

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Use PL with bound on $|yz|$.

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$i = 0$ Case as a Picture