Proving That a Language Is Not Regular

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- **Method 2 (Pumping Lemma (PL)):** Run the DFA on one long word. By the **PHP** the word must visit the same state twice. Then do some **magic**.

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Method 2: Pumping Lemma (PL)

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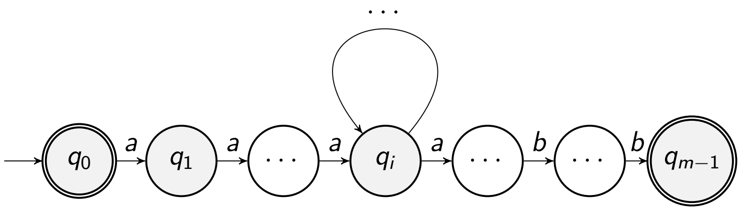

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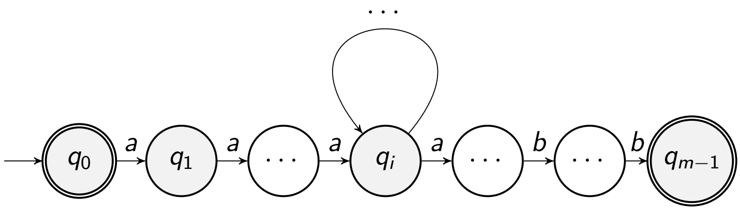

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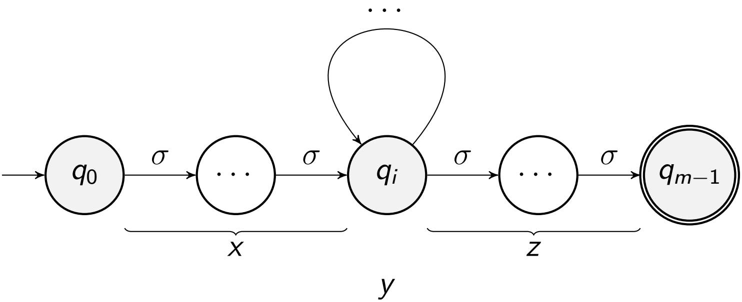

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We then find some i such that $xy^iz \notin L$ for the contradiction.

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$x = a^j$, $y = a^k$, $z = a^{n-j-k} b^n$. Note $k \geq 1$.
By the PL, all of the words

$$a^j \left(a^k\right)^i a^{n-j-k} b^n \;=\; a^{n+k(i-1)} b^n$$

are in L_1.
Take $i = 2$ to get

$$a^{n+k} b^n \in L_1$$

Contradiction since $k \geq 1$.

$L_2 = \{w : \#_a(w) = \#_b(w)\}$ is Not Regular

Proof: Same Proof as L_1 not Reg: Still look at $a^m b^m$.
Key PL says for ALL long enough $w \in L$.

$L_3 = \{w : \#_a(w) \neq \#_b(w)\}$ is Not Regular

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PL Does Not Help. When you increase the number of y's there is no way to control it so carefully to make the number of a's EQUAL the number of b's.

So what do to?

If L_3 is regular then $L_2 = \overline{L_3}$ is regular. But we know that L_2 is not regular. DONE!

$L_4 = \{a^{n^2} : n \in \mathbb{N}\}$ is Not Regular

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Proof
By PL for long enough $a^{n^2} \in L_4$ there exist $x = a^j$, $y = a^k$, $z = a^\ell$ with $xyz = a^{n^2}$. Also $a^j(a^k)^i a^\ell \in L_4$. (Note $k \geq 1$.)

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$$(\forall i \geq 0)[j + ik + \ell = n^2 + ik \text{ is a square}].$$

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So $n^2, n^2 + k, n^2 + 2k, \ldots$ are all squares.

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See slide for exciting finish!

$L_4 = \{a^{n^2} : n \in \mathbb{N}\}$ is Not Regular (cont)

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$n^2 + k \geq (n+1)^2 = n^2 + 2n + 1.$

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$n^2 + 2k \geq (n+2)^2 = n^2 + 4n + 4$.

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So k is bigger than any natural number!

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Contradiction.

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By PL, for large p, $a^p \in L_5$ $\exists$ $x = a^j$, $y = a^k$, $z = a^\ell$ such that

$$a^j (a^k)^i a^\ell \in L_5$$

$$(\forall i \geq 0)[j + ik + \ell \text{ is prime}].$$

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So, $p, p+k, p+2k, \ldots, p+pk$ are all prime.
But $p + pk = p(k+1)$. Contradiction.

$L_6 = \{\#_a(w) > \#_b(w)\}$ is Not Regular

We will be brief here.

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We will be brief here.
Take $w = b^n a^{n+1}$, long enough so the y-part is in the b's.

$L_6 = \{\#_a(w) > \#_b(w)\}$ is Not Regular

We will be brief here.
Take $w = b^n a^{n+1}$, long enough so the y-part is in the b's.
Pump the y to get more b's than a's.

$L_7 = \{a^n b^m : n > m\}$ is Not Regular

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Can **also** bound $|yz|$ by same proof.

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Can **also** bound $|yz|$ by same proof.

Do that and then you can get y to be all b's, pump b's, and get out of the language.

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$L_8 = \{a^{n_1} b^m c^{n_2} : n_1, n_2 > m\}$ is Not Regular (Cont)

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For all i $xy^i z = a^{j+ik+(n-j-k)} b^{n-1} c^n \in L_8$.

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Key We are used to thinking of i large.

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But we can also take $i = 0$.
cut out that part of the word. We take $i = 0$ to get

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$$xy^0 z = a^{n-k} b^{n-1} c^n$$

Since $k \geq 1$, we have that $\#_a(xy^0 z) < n \leq n - 1 = \#_b(xy^0 z)$.
Hence $xy^0 z \notin L_8$.

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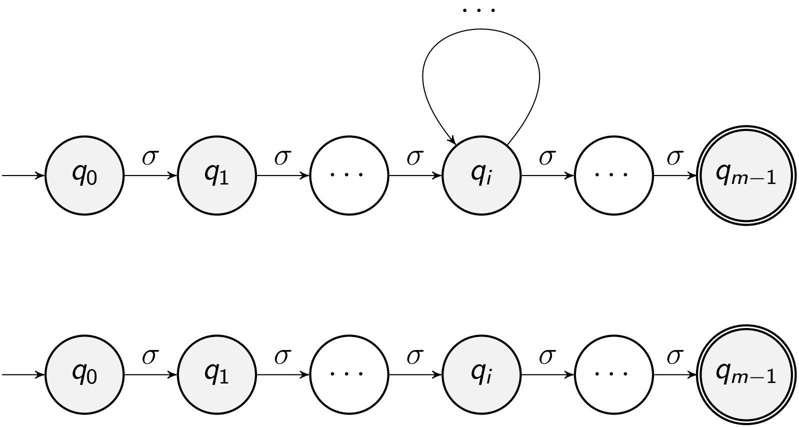
Hence $xy^0 z \notin L_8$. Contradiction.

$i = 0$ Case as a Picture



Lower Bounds: Looking Ahead

1. DFA's are simple enough devices that we can actually prove languages are not regular
2. We will later see that Context Free Grammars are simple enough devices that we can prove Languages are not Context Free.
3. Poly-bounded Turing Machines seem to be complicated devices, so proving P$\neq$NP seems to be hard.

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2. We will later see that Context Free Grammars are simple enough devices that we can prove Languages are not Context Free.
3. Poly-bounded Turing Machines seem to be complicated devices, so proving P≠NP seems to be hard. However, I expect Isaac, Adam, and Sam will work it out by the end of the semester.
4. Proving problems undecidable is surprisingly easy since such proofs do not depend on the details of the model of computation.