Decidability of WS1S and S1S: An Exposition

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Credit Where Credit is Due

Buchi proved that WS1S was decidable.
I don't know off hand who proved S1S decidable.

Part I

We Define WS1S And Prove It's Decidable

Formulas and Sentences

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   $(\forall y)(\exists x)[x + y = 7]$— **F** if domain is $\mathbb{N}$, the naturals.

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Weak second order means quantify over finite sets.

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WS2S is also decidable but we will not prove this.

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   This means that $X = \{y + c : y \in Y\}$.

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3. $(Q_1 x)[\phi_1(x)] \vee (Q_2 y)[\phi_2(y)]$ is equiv to $\neg((\neg Q_1 x)[\phi_1(x)] \wedge (\neg Q_2 y)[\phi_2(y)])$.

Key Definition

Def If $\phi(x_1, \ldots, x_n, X_1, \ldots, X_m)$ is a WS1S Formula then $\mathrm{TRUE}(\phi)$ is the set

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This is the set of $(a_1, \ldots, a_n, A_1, \ldots, A_m)$ that make ϕ TRUE.

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We want to say that $\mathrm{TRUE}(\phi)$ is regular. Need to represent $(a_1, \ldots, a_n, A_1, \ldots, A_m)$.

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The triple $(3, 4, \{0, 1, 2, 4, 7\})$ is represented by

	0	1	2	3	4	5	6	7
x	0	0	0	1	$*$	$*$	$*$	$*$
y	0	0	0	0	1	$*$	$*$	$*$
X	1	1	1	0	1	0	0	1

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Note After we see 0001 for x we **do not care** what happens next. The *'s can be filled in with 0's or 1's and the string of symbols from $\{0,1\}^3$ above would still represent $(3, 4, \{0, 1, 2, 4, 7\})$.

Representation–More Formal

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Finite set X is represented by a string in $\{0,1\}^*$ which is its bit-vector.

Example And Our Alphabet

Consider the set

$$\{(x, y, X) : (x = y + 1) \land (y \in X)\}$$

We want to show that it's regular. Here is an example of how we **represent** a tuple (number,number,finite set):

	0	1	2	3	4	5	6	7
x	0	0	0	0	0	1	0	0
y	0	0	0	0	1	1	0	1
X	1	1	1	0	1	0	0	1

This string is IN our lang since $x = 5$, $y = 4$, and $X = \{0, 1, 2, 4, 7\}$.

Alphabet is $\{000, 001, 010, 011, 100, 101, 110, 111\}$ though we think of it vertically rather than horizontally.

Stupid Strings

What does

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represent?

This string is **Stupid!** There is no value for x. This string does not represent anything!

Our DFA's will have 3 kinds of states: **accept, reject,** and **stupid. Stupid** means that the string did not represent anything because it has a number-variable be all 0's. (It is fine for a set var to of all 0's- that would be the empty set.)

Key Theorem

Thm For all WS1S formulas ϕ the set $\mathrm{TRUE}(\phi)$ is regular.

We prove this by induction on the formation of a formula. If you prefer- induction on the length of a formula.

Theorem for Atomic Formulas

Lemma For all WS1S atomic formulas ϕ the set $\mathrm{TRUE}(\phi)$ is regular.

On the next few slides we give the DFA for some Atomic Formulas.

The ones we do not may be HW or on the Final.

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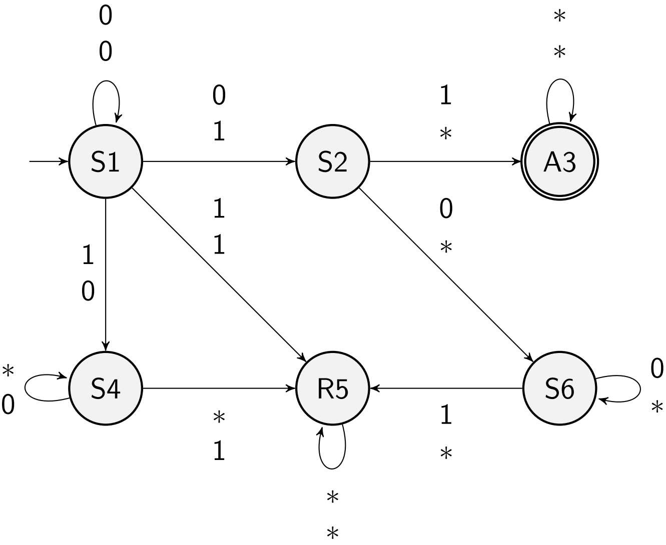
The DFA for $x = y + c$ is similar. Might be on a HW or the Final.

$x = y + 1$



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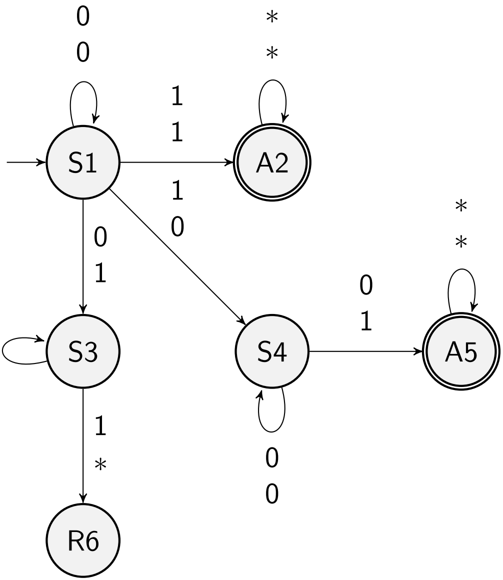

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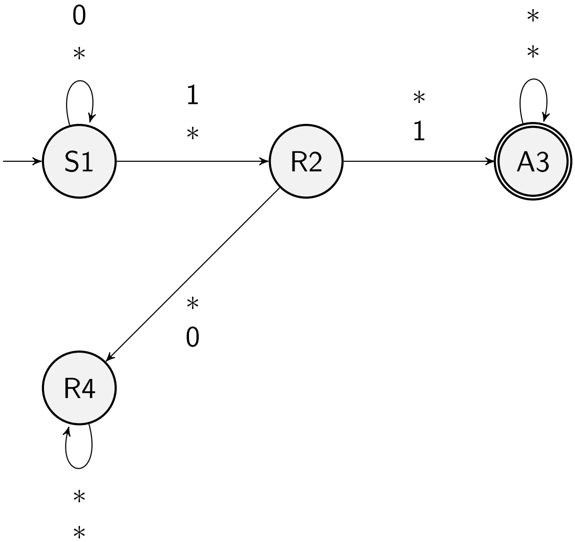
The DFA for $x + c \in X$ is similar. Might be on a HW or the Final.

$x + 1 \in X$



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Getting DFA's for those atomic formulas, or special cases, might be on a HW or the Final.

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Next slides for what to do about quantifiers.

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We want $\mathrm{TRUE}((\exists x_1)[\phi(x_1, \ldots, x_n, X_1, \ldots, X_m)])$ is regular.
Ideas?

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Ideas?
Use nondeterminism.
Will show you in class.

DFA Decidability Theorem

We need the following easy theorem:

Thm The following problem is decidable: given a DFA determine if **there exists** a string it accepts.

DFA Decidability Theorem Proof

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Might be on HW.

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6. If $L(M) \neq \emptyset$ then $(\exists X)[\phi(X)]$ is TRUE.
   If $L(M) = \emptyset$ then $(\exists X)[\phi(X)]$ is FALSE.

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Note No De Morgans Law–we complement the DFA.

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If YES then the original sentence is TRUE.
If NO then the original sentence is FALSE.

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Given a sentence

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And the answer is: Can do better: $\mathbf{2^{2^{n^3 \log n}}}$. **This is provably the best you can do (roughly).**

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Are there interesting problems that can be STATED in WS1S?
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NO There are no interesting MATH problems that can be expressed in WS1S.

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Presb Arith is decidable by TRANSFORMING Pres Arith Sentences into WS1S sentences.
Presb Arithmetic has been used in Code Optimization (using a better dec procedure than reducing to WS1S).

S1S

PART II OF THIS TALK:
WE DEFINE S1S AND PROVE IT'S DECIDABLE

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Question Can we still use finite automata?

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KEY We never actually RAN a DFA on any string.
Def A ***B*-NFA** is an NFA. If $x \in \Sigma^\omega$ then x is accepted by B-NFA M if there is a path such that $M(x)$ hits a final state inf often.
Good News: (PROVE IN GROUPS)

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2. Emptyness problem for DFA's is decidable.

KEY We never actually RAN a DFA on any string.

Def A ***B*-NFA** is an NFA. If $x \in \Sigma^\omega$ then x is accepted by B-NFA M if there is a path such that $M(x)$ hits a final state inf often.

Good News: (PROVE IN GROUPS)

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Need B-reg closed under complementation.

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Odd News Proof Uses **Ramsey Theory,** yet I never proved it in my Ramsey Theory course.

B-Reg and Mu-Reg

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Easy (IN GROUPS) Mu-reg Closed: UNION, INTER, COMP.

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Recap and Plan

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- B-reg easily closed: $\cup$, $\cap$, PROJ, but COMPLEMENT hard.
- Mu-reg easily closed: $\cup$, $\cap$, COMPLEMENT. But PROJ hard.
- How to prove? Show B-reg $=$ Mu-reg.

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4. Construct B-NFA M for $\{X : \phi(X) \text{ is true}\}$.
5. Test if $L(M) = \emptyset$.
6. If $L(M) \neq \emptyset$ then $(\exists X)[\phi(X)]$ is TRUE.
   If $L(M) = \emptyset$ then $(\exists X)[\phi(X)]$ is FALSE.

COMPLEXITY OF THE DECISION PROCEDURE

Given a sentence

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How long will the procedure above take in the worst case?
$2^{2^{\cdots n}}$ steps since we do n nondet to det transformations.

Anything Interesting STATABLE IN S1S?

Are there interesting problems that can be STATED in S1S?
YES Verification of programs that are supposed to run forever like operating systems. Verification of security protocols.

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NO There are no interesting MATH problems that can be expressed in S1S.

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ω-Reg

Def A language L is **ω-reg** if there exists regular langs $U_1, U_2, \ldots, U_n, V_1, V_2, \ldots, V_n$ such that

$$L = \bigcup_{i=1}^{n} U_i V_i^{\omega}.$$

Thm B-reg $= \omega$-reg
Work with Neighbors

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$$\mathrm{ioPrefix(V)} = \{x = \sigma_1\sigma_2 \cdots \in \Sigma^\omega : (\exists^\infty i)[\sigma_1 \cdots \sigma_i \in V]\}$$

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2. A language L is **ioPrefix-reg** if there exists regular langs $U_1, U_2, \ldots, U_n,\ V_1, V_2, \ldots, V_n$ such that

$$L = \bigcup_{i=1}^{n} U_i \cdot \mathrm{ioPrefix}(V)$$

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William Gasarch-U of MD