HW 01 Some Solutions

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   say
   Either an inf numb of R_1 *or* $\cdots$ *or* R_c *edges come out of* x_i.

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   SKETCH $c = 1$ trivial. $c = 2$ is the proof your saw in class. Assume $c \geq 3$ Assume theorem true for all $c' < c$. We will only be using $c - 1$ and 2.

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VOTE Which proof did you like better.

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So I am using the theorem
$(\forall)$ $\mathrm{COL}\colon \binom{N}{2} \rightarrow [2]$ $(\exists)$ *inf homog set.*

NO, I am not using that! The set I am coloring is an infinite subset of $\mathbb{N}$. So I am really using the following trivial corollary of the above theorem:
$(\forall)$ *inf* $A \subseteq \mathbb{N}$, $(\forall)$ $\mathrm{COL}\colon \binom{A}{2} \rightarrow [2]$ $(\exists)$ *inf homog set.*

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Proof for a-ary c-color Ramsey.
SKETCH Given $\mathrm{COL}\colon \binom{\mathbb{N}}{a} \to [c]$, form $\mathrm{COL}' : \binom{N}{a-1} \to [c]$ via

$$\mathrm{COL}'(z_1, \ldots, z_{a-1}) = \mathrm{COL}(x_1, z_1, \ldots, z_{a-1}).$$

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Find homog set inductively and kill all vertices not in that set.
x_2 is min element of homog set.
Lather, Rinse, Repeat to get $x_1, x_2, \ldots$.

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For $i < j$.

$$COL(i,j) = \begin{cases} RED & \text{if } x_i < x_j \\ BLUE & \text{if } x_i > x_j \\ GREEN & \text{if } x_i = x_j \end{cases} \tag{1}$$

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Apply Ramsey Theory to get a theorem.
If homog RED then get subseq set $x_{i_1} < x_{i_2} < \ldots$
If homog BLUE then get subseq set $x_{i_1} > x_{i_2} > \ldots$
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Thm Every inf seq of R has either an inf $\uparrow$ seq, an inf $\downarrow$ seq, or an inf $=$ seq.

Problem 4 Extra

Can generalize to R^n by either applying Ramsey with 3-colors n times, or applying Ramsey with 3^n colors.
Thm Every inf seq of R^n has an inf subseq where, for each coordinate, either $\uparrow$ seq, or $\downarrow$ or $=$.

Problem 4 Extra

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This is a part of the proof of the Bolzano-Weierstrass Theorem. Next Slide.

Bolzano-Weierstrass Theorem

Lemma

1. Any increasing sequence bounded sequence of reals converges to a real.
2. Any decreasing sequence bounded sequence of reals converges to a real.

This is not obvious. This depends on the construction of the Reals.

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BW Thm If $p_1, p_2, p_3, \ldots$ is an inf sequence of points in R^n that is contained in a box, then there exists a subsequence that converges to a point in R^n.

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Proof

Problem 4 yields that there is a subsequence in each coordinate that is either $\downarrow$, $\uparrow$, or $=$. Lemma yields each coord converges.

Problem 5- History

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It is the worst math novelty song ever. Listen for yourself:
`https://www.youtube.com/watch?v=dfO18klwKHg`

Problem 5

$$p_1, p_2, p_3, \ldots,$$

be an infinite sequence of points in R^2.
Consider the following coloring of $\binom{N}{2}$.

$$COL(i,j) = \begin{cases} RED & \text{if } d(p_i, p_j) > 1 \\ BLUE & \text{if } d(p_i, p_j) < 1 \end{cases} \quad (2)$$

Apply Ramsey Theorem. What do you get?
SOLUTION
Thm Given an infinite sequence of points in R^2 there exists an infinite subset so that either (a) they are all within 1 of each other, or (b) they are all more than 1 apart.

Problem 4 and 5 thoughts

The proofs of the theorems in Problem 4 and 5 are FAR EASIER with Ramsey Theory. The proofs without Ramsey end up doing Ramsey in context.

Problem 6 (Extra Credit)

Prove or disprove:
For every 2-coloring of the edges of $K_{\mathbb{N},\mathbb{N}}$ *there exists* H_1, H_2 *infinite such that* (H_1, H_2) *is a homog set.*

Problem 6 (Extra Credit)

Prove or disprove:
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Discuss and Vote

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Discuss and Vote

SOLUTION FALSE. Color with

$$\mathrm{COL}(i,j) = \begin{cases} RED & \text{if } i < j \\ BLUE & \text{if } i \geq j \end{cases} \tag{3}$$

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Thought What if we use 100 colors? The same counterexample works but you end up with an (H_1, H_2) homog set that only has TWO colors. We will call that a 2-homog set.

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For every 100-coloring of the edges of $K_{\mathbb{N},\mathbb{N}}$ there exists H_1, H_2 infinite such that (H_1, H_2) is a 2-homog set. 3-homog set(?). Some c-homog with $c < 100$?

Problem 7 (Extra Credit)

Prove or disprove:
For all colorings $\mathrm{COL} : \binom{\mathsf{Z}}{2} \rightarrow [2]$ *there exists a set* $H \subseteq \mathsf{Z}$ *that is order-equiv to* Z *and is homogenous.*

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SOLUTION FALSE. Color with

$$\mathrm{COL}(i,j) = \begin{cases} RED & \text{if } i,j \geq 0 \\ BLUE & \text{if } i,j < 0 \\ BLUE & \text{if one is } \geq 0 \text{ and the other is } < 0 \end{cases} \tag{4}$$

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