The Infinite Ramsey Theorem and the Large Ramsey Theorem

1 The Infinite Ramsey Theorem

Def 1.1 Let $a, c \in \mathsf{N}$. Let A be a set (A will usually be N or $[n]$ or $\{k, \ldots, n\}$). Let COL: $\binom{\mathrm{A}}{\mathrm{a}} \rightarrow$ [c]. $H \subseteq A$ is *homogenous* if COL is constant on $\binom{H}{a}$.

In this manuscript we will only talk about 2-colorings of $\binom{A}{2}$. Generalizations to any number of colors are trivial. Generalizations to different values of a are fairly easy but may require some thought.

Theorem 1.2 *Every 2-coloring* $\binom{\mathsf{N}}{2}$ *has an infinite homogenous set.*

Proof: Let COL: $\binom{\mathsf{N}}{2} \rightarrow [2]$. We define an infinite sequence of vertices,

$$x_1, x_2, \ldots,$$

and an infinite sequence of sets of vertices,

$$V_0, V_1, V_2, \ldots,$$

that are based on COL.

Here is the intuition: Vertex $x_1 = 1$ has an infinite number of edges coming out of it. Some are RED, and some are BLUE. Hence there are an infinite number of RED edges coming out of x_1, or there are an infinite number of BLUE edges coming out of x_1 (or both). Let c_1 be a color such that x_1 has an infinite number of edges coming out of it that are colored c_1. Let V_1 be the set of vertices v such that $COL(\{v, x_1\}) = c_1$. Then keep iterating this process.

We now describe it formally.

$$\begin{aligned} V_0 &= \mathsf{N} \\ x_1 &= 1 \end{aligned}$$

$$c_1 = \begin{cases} \text{RED} & \text{if } |\{v \in V_0 \mid COL(\{v, x_1\}) = \text{RED}\}| \text{ is infinite;} \\ \text{BLUE} & \text{otherwise} \end{cases} \tag{1}$$

$$V_1 = \quad \{v \in V_0 \mid COL(\{v, x_1\}) = c_1\} \text{ (note that } |V_1| \text{ is infinite)}$$

Let $i \geq 2$, and assume that V_{i-1} is defined. We define x_i, c_i, and V_i:

$$x_i = \quad \text{the least number in } V_{i-1}$$

$$c_i = \begin{cases} \text{RED} & \text{if } |\{v \in V_{i-1} \mid COL(\{v, x_i\}) = \text{RED}\}| \text{ is infinite;} \\ \text{BLUE} & \text{otherwise} \end{cases} \tag{2}$$

$$V_i = \quad \{v \in V_{i-1} \mid COL(\{v, x_i\}) = c_i\} \text{ (note that } |V_i| \text{ is infinite)}$$

How long can this sequence go on for? Well, x_i can be defined if V_{i-1} is nonempty. We an show by induction that, for every i, V_i is infinite. Hence the sequence

$$x_1, x_2, \ldots,$$

is infinite.

Consider the infinite sequence

$$c_1, c_2, \ldots$$

Each of the colors in this sequence is either RED or BLUE. Hence there must be an infinite sequence $i_1, i_2, \ldots$ such that $i_1 < i_2 < \cdots$ and

$$c_{i_1} = c_{i_2} = \cdots$$

Denote this color by c, and consider the vertices

$$H = \{x_{i_1}, x_{i_2}, \cdots\}$$

It is easy to show that H is homog.

∎

2 Finite Ramsey from Infinite Ramsey

Having proved the infinite Ramsey Theorem, we then want to prove the finite Ramsey Theorem. Can we prove the finite Ramsey Theorem *from* the infinite Ramsey Theorem? Yes, we can! This proof will not give any bounds. Other proofs do.

Theorem 2.1 *For all k there exists n such that for all* COL: $\binom{[\mathrm{n}]}{2} \rightarrow [2]$ *there exists a homog set of size k.*

Proof: Suppose, by way of contradiction, that there is some $k \geq 2$ such that no such n exists. For every $n \geq k$, there is some way to color $\binom{[n]}{2}$ so that there is no homog set of size k. Hence there exist the following:

1. COL_0, a 2-coloring of $\binom{[k]}{2}$ that has no homog set of size k.

2. COL_1, a 2-coloring of $\binom{[k+1]}{2}$ that has no homog set of size k.

3. COL_2, a 2-coloring of $\binom{[k+2]}{2}$ that has no homog set of size k.

4. COL_3, a 2-coloring of $\binom{[k+3]}{2}$ that has no homog set of size k.

$\vdots$

j. COL_L, a 2-coloring of $\binom{[k+L]}{2}$ that has no homog set of size k.

$\vdots$

We will use these 2-colorings to form a 2-coloring COL of $\binom{\mathsf{N}}{2}$ that has no infinite homog set. This contradiction Theorem 1.2.

Let $e_1, e_2, e_3, \ldots$ be a list of every element of $\binom{\mathsf{N}}{2}$. We will color e_1, then e_2, etc.

How should we color e_1? We will color it the way an infinite number of the COL_i's color it. Call that color c_1. Then how to color e_2? Well, first consider ONLY the colorings that colored e_1 with color c_1. Color e_2 the way an infinite number of those colorings color it. And so forth.

We now proceed formally:

$$J_0 = \mathsf{N}$$

$$COL(e_1) = \begin{cases} \text{RED} \ \text{ if } |\{j \in J_0 \mid COL_j(e_1) = \text{RED}\}| \text{ is infinite;} \\ \text{BLUE otherwise.} \end{cases} \quad (3)$$

$$J_1 = \ \{j \in J_0 \mid COL(e_1) = COL_j(e_1)\}$$

Let $i \geq 2$, and assume that $e_1, \ldots, e_{i-1}$ have been colored. Assume, furthermore, that J_{i-1} is infinite and, for every $j \in J_{i-1}$,

$$\begin{aligned} COL(e_1) &= \ COL_j(e_1) \\ COL(e_2) &= \ COL_j(e_2) \\ &\vdots \\ COL(e_{i-1}) &= \ COL_j(e_{i-1}) \end{aligned}$$

We now color e_i:

$$COL(e_i) = \begin{cases} \text{RED if } |\{j \in J_{i-1} \mid COL_j(e_i) = \text{RED}\}| \text{ is infinite;} \\ \text{BLUE otherwise.} \end{cases} \quad (4)$$

$$J_i = \ \{j \in J_{i-1} \mid COL(e_i) = COL_j(e_i)\}$$

One can show by induction that, for every i, J_i is infinite. Hence this process *never* stops.

Claim: Let A be a finite subset of $\{k, k+1, \ldots,\}$. Then there exists an infinite number of i such that COL on $\binom{A}{2}$ agrees with COL_i on $\binom{A}{2}$.

Proof of Claim

Left to the reader.

End of Proof of Claim

We have produced a 2-coloring of $\binom{\mathsf{N}}{2}$. Let By Theorem 1.2 there is an infinite homog set for COL:

$$H = \{x_1 < x_2 < x_3 < \cdots\}.$$

Look at

$$H' = \{x_1 < x_2 < \cdots < x_k\}$$

This is a homog set with respect to COL. By the claim there is an i (in fact, infinitely many) such that COL and COL_i agree on $\binom{H'}{2}$. Clearly H' is a homog set of size k for COL_i. This contradicts the definition of COL_i.
∎

3 Proof of Large Ramsey Theorem

In all of the theorems presented earlier, the labels on the vertices did *not* matter. In this section, the labels *do* matter.

Def 3.1 A finite set $F \subseteq \mathsf{N}$ is called *large* if the size of F is BIGGER than the smallest element of F.

Example 3.2

1. The set $\{1, 2, 10\}$ is large: It has 3 elements, the smallest element is 1, and $3 > 1$.
2. The set $\{5, 10, 12, 17, 20\}$ is NOT large: It has 5 elements, the smallest element is 5, and 5 is NOT strictly greater than 5.
3. The set $\{20, 30, 40, 50, 60, 70, 80, 90, 100\}$ is NOT large: It has 9 elements, the smallest element is 20, and $9 < 20$.
4. The set $\{5, 30, 40, 50, 60, 70, 80, 90, 100\}$ is large: It has 9 elements, the smallest element is 5, and $9 > 5$.
5. The set $\{101, \ldots, 190\}$ is not large: It has 90 elements, the smallest element is 101, and $90 < 101$.

We will be considering monochromatic K_m's where the underlying set of vertices is a large set. We need a definition to identify the underlying set.

Let COL be a 2-coloring of $\binom{[n]}{2}$. Consider the set $\{1, 2\}$. It is clearly both homogeneous and large (using our definition of large). Hence the statement

"for every $n \geq 2$, every 2-coloring of K_n has a large homogeneous set"

is true but trivial.

What if we used $V = \{k, k + 1, \ldots, n\}$ as our vertex set? Then a large homogeneous set would have to have size at least k.

Notation 3.3 $LR(k)$ is the least n, if it exists, such that every 2-coloring of $\binom{\{k,\ldots,n\}}{2}$ has a large homogeneous set.

Theorem 3.4 *For every $k \geq 2$ there exists n such that for all 2-colorings of $\binom{\{k,\ldots,n\}}{2}$ there exists a large homog set.*

Proof: This proof is similar to our proof of the finite Ramsey Theorem *from* the infinite Ramsey Theorem (the proof of Theorem 2.1).

Suppose, by way of contradiction, that there is some $k \geq 2$ such that no such n exists. For every $n \geq k$, there is some way to color $\binom{\{k,\ldots,n\}}{2}$ so that there is no large homog sets. Hence there exist the following:

1. COL_1, a 2-coloring of $\binom{\{k,k+1\}}{2}$ that has no large homog set.
2. COL_2, a 2-coloring of $\binom{\{k,k+1,k+2\}}{2}$ that has no large homog set.
3. COL_3, a 2-coloring of $\binom{\{k,\ldots,k+3\}}{2}$ that has no large homog set.

   $\vdots$

j. COL_L, a 2-coloring of $\binom{\{k,\ldots,k+L\}}{2}$ that has no large homog set.

   $\vdots$

We will use these 2-colorings to form a 2-coloring COL of $\binom{\{k,k+1,\ldots\}}{2}$. This coloring will have an infinite homog set by Theorem 1.2. This will give us a contradiction to the definition of one of the COL_i.

Let $e_1, e_2, e_3, \ldots$ be a list of every element of $\binom{\{k,k+1,\ldots\}}{2}$. We will color e_1, then e_2, etc.

How should we color e_1? We will color it the way an infinite number of the COL_i's color it. Call that color c_1. Then how to color e_2? Well, first consider ONLY the colorings that colored e_1 with color c_1. Color e_2 the way an infinite number of those colorings color it. And so forth.

We now proceed formally:

$$J_0 = \mathsf{N}$$

$$COL(e_1) = \begin{cases} \text{RED} & \text{if } |\{j \in J_0 \mid COL_j(e_1) = \text{RED}\}| \text{ is infinite;} \\ \text{BLUE otherwise.} \end{cases} \quad (5)$$

$$J_1 = \{j \in J_0 \mid COL(e_1) = COL_j(e_1)\}$$

Let $i \geq 2$, and assume that $e_1, \ldots, e_{i-1}$ have been colored. Assume, furthermore, that J_{i-1} is infinite and, for every $j \in J_{i-1}$,

$$\begin{aligned} COL(e_1) &= COL_j(e_1) \\ COL(e_2) &= COL_j(e_2) \\ &\vdots \\ COL(e_{i-1}) &= COL_j(e_{i-1}) \end{aligned}$$

We now color e_i:

$$COL(e_i) = \begin{cases} \text{RED if } |\{j \in J_{i-1} \mid COL_j(e_i) = \text{RED}\}| \text{ is infinite;} \\ \text{BLUE otherwise.} \end{cases} \quad (6)$$

$$J_i = \{j \in J_{i-1} \mid COL(e_i) = COL_j(e_i)\}$$

One can show by induction that, for every i, J_i is infinite. Hence this process *never* stops.

Claim: Let A be a finite subset of $\{k, k+1, \ldots, \}$. Then there exists an infinite number of i such that COL on $\binom{A}{2}$ agrees with COL_i on $\binom{A}{2}$.

Proof of Claim

Left to the reader.

End of Proof of Claim

By Theorem 1.2 there is an infinite homog set for COL:

$$H = \{x_1 < x_2 < x_3 < \cdots\}.$$

Look at

$$H' = \{x_1 < x_2 < \cdots < x_{x_1+1}\}$$

This is a homog set with respect to COL. By the claim there is an i (in fact, infinitely many) such that COL and COL_i agree on $\binom{H'}{2}$. Clearly H' is a large homog set for COL_i. This contradicts the definition of COL_i.

■