1 Erdős-Rado Proof of 3-hypergraph Ramsey

Def 1.1 $R_a(k)$ is the least number n such that, for all $\mathrm{COL}\colon \binom{[n]}{a} \rightarrow [2]$ there exists a homog set of size k.

In our proof in class that $R_3(k)$ exists the construction used 2-ary Ramsey many times, and then 1-ary Ramsey once. Each use of 2-ary Ramsey cuts the set from size X to size $\log X$. The one use of 1-ary Ramsey cuts the set from size X to size $\frac{X}{2}$. All of those uses of 2-ary Ramsey force us to start with a very big set.

To be able to start with a smaller set we will do a different construction. This one will use 1-ary Ramsey many times, but 2-ary Ramsey only once.

Theorem 1.2 *For almost all k, $R_3(k) \le 2^{2^{4k}}$.*

Proof:

Let n be a number to be determined. Let COL be a 2-coloring of $\binom{[n]}{3}$. We define a sequence of vertices,

$$x_1, x_2, \ldots, x_{R_2(k)}.$$

Here is the intuition: Let $x_1 = 1$. Let $x_2 = 2$. The vertices x_1, x_2 induces the following coloring of $\{3, \ldots, n\}$.

$$\mathrm{COL}^*(y) = \mathrm{COL}(x_1, x_2, y).$$

Let V_1 be a 1-homogeneous for COL^* of size at least $\frac{n-2}{2}$. Let $\mathrm{COL}^{**}(x_1, x_2)$ be the color of V_1. Let x_3 be the least vertex left (bigger than x_2).

The number x_3 induces *two* colorings of $V_1 - \{x_3\}$:

$$(\forall y \in V_1 - \{x_3\})[\mathrm{COL}_1^*(y) = \mathrm{COL}(x_1, x_3, y)]$$

$$(\forall y \in V_1 - \{x_3\})[\mathrm{COL}_2^*(y) = \mathrm{COL}(x_2, x_3, y)]$$

Let V_2 be a 1-homogeneous for COL_1^* of size $\frac{|V_1|-1}{2}$. Let $\mathrm{COL}^{**}(x_1, x_3)$ be the color of V_2. Restrict COL_2^* to elements of V_2, though still call it COL_2^*. We reuse the variable name V_2 to be a 1-homogeneous for COL_2^* of size at least $\frac{|V_2|}{2}$. Let $\mathrm{COL}^{**}(x_1, x_3)$ be the color of V_2. Let x_4 be the least element of V_2. Repeat the process.

We describe the construction formally.

CONSTRUCTION

$$\begin{aligned} x_1 = &\quad 1 \\ V_1 = &\quad [n] - \{x_1\} \end{aligned}$$

Let $2 \le i \le R_2(k)$. Assume that $x_1, \ldots, x_{i-1}, V_{i-1}$, and

$$\mathrm{COL}^{**} \colon \binom{\{x_1, \ldots, x_{i-1}\}}{2} \to [2]$$

are defined.

$$\begin{aligned} x_i = &\quad \text{the least element of } V_{i-1} \\ V_i = &\quad V_{i-1} - \{x_i\} \text{ (We will change this set without changing its name).} \end{aligned}$$

We define $\mathrm{COL}^{**}(x_1, x_i)$, $\mathrm{COL}^{**}(x_2, x_i)$, ..., $\mathrm{COL}^{**}(x_{i-1}, x_i)$. We will also define smaller and smaller sets V_i. We will keep the variable name V_i throughout.

For $j = 1$ to $i - 1$

1. $\mathrm{COL}^* \colon V_i \to [2]$ is defined by $\mathrm{COL}^*(y) = \mathrm{COL}(x_j, x_i, y)$.

2. Let V_i be redefined as the largest 1-homogeneous set for COL^*. Note that $|V_i|$ decreases by at most half.

3. $\mathrm{COL}^{**}(x_j, x_i)$ is the color of V_i.

KEY: For all $1 \le i_1 < i_2 \le i$, for all $y \in V_i$, $\mathrm{COL}(x_{i_1}, x_{i_2}, y) = \mathrm{COL}^{**}(x_{i_1}, x_{i_2})$.

END OF CONSTRUCTION

When we derive upper bounds on n we will show that the the construction can be carried out for $R_2(k)$ stages. For now assume the construction ends.

We have vertices

$$X = \{x_1, x_2, \ldots, x_{R_2(k)}\}$$

and a 2-coloring COL^{**} of $\binom{X}{2}$. By the definition of $R_2(k)$ there exists a Homog-for-COL^{**}:

$$H = \{x_{i_1}, \ldots, x_{i_k}\}.$$

Let the color of this homogenous set be 1. We show that H is a homogenous set for COL. Let $1 \le i_1 < i_2 < i_3 \le k$. We show that $\mathrm{COL}(x_{i_1}, x_{i_2}, x_{i_3}) = 1$.

By the definition of COL^{**} for all $y \in V_{i_2}$, $\mathrm{COL}(x_{i_1}, x_{i_2}, y) = \mathrm{COL}^{**}(x_{i_1}, x_{i_2}) = 1$. In particular $\mathrm{COL}(x_{i_1}, x_{i_2}, x_{i_3}) = 1$.

We now see how large n must be so that the construction be carried out. Note that in stage i $|V_i|$ be decreases by at most half, i times. Hence $|V_{i+1}| \ge \frac{|V_i|}{2^i}$.

Therefore

$$|V_i| \ge \frac{|V_1|}{2^{1+2+\cdots+(i-1)}} \ge \frac{n-1}{2^{(i-1)^2}}.$$

We want $|V_{R_2(k)}| \ge 1$. It suffice so take $n = 2^{R_2(k)^2}$.

We know that $R_2(k) \le 2^{2k}$. Hence

$$R_3(k) \le 2^{R_2(k)^2} \le 2^{2^{4k}}.$$

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