Rectangle Free Coloring of Grids

Stephen Fenner- U of SC William Gasarch- U of MD
Charles Glover- U of MD Semmy Purewal- Col. of Charleston

Credit Where Credit is Due

This Work Grew Out of a Project In the UMCP SPIRAL (Summer Program in Research and Learning) Program for College Math Majors at HBCU's.

One of the students, Brett Jefferson has his own paper on this subject.

ALSO: Zarankiewics [7] asked similar questions.

Square Theorem:

Theorem

For all c, there exists G such that
for every c-coloring of $G \times G$ there exists a monochromatic square.

$$
\begin{array}{ccccc}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & R & \cdots & R & \cdots \\
\cdots & \vdots & \cdots & \vdots & \cdots \\
\cdots & R & \cdots & R & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

Proving the Square Theorem and Bounding $G(c)$

How to prove Square Theorem?

1. Corollary of Hales-Jewitt Theorem [1]. Bounds on G HUGE!
2. Corollary of Gallai's theorem [3,4,6]. Bounds on G HUGE!
3. From VDW directly (folklore). Bounds on G HUGE!
4. Directly (folklore?). Bounds on G HUGE!
5. Graham and Solymosi [2]. $G \le 2^{2^{81}}$. Better but still HUGE.

Best known upper and lower bounds:

1. $G(2) \le 2^{2^{81}}$.
2. $\Omega(c^{4/3}) \le G(c)$. (Upper bound not writable-downable.)

What If We Only Care About Rectangles?

Definition

$G_{n,m}$ is the grid $[n] \times [m]$.

1. $G_{n,m}$ is c-colorable if there is a c-colorings of $G_{n,m}$ such that no rectangle has all four corners the same color.
2. $\chi(G_{n,m})$ is the least c such that $G_{n,m}$ is c-colorable.

Our Main Question

Fix c

Exactly which $G_{n,m}$ are c-colorable?

Two Motivations!

1. Relaxed version of Square Theorem- hope for better bounds.
2. Coloring $G_{n,m}$ without rectangles is equivalent to coloring edges of $K_{n,m}$ without getting monochromatic $K_{2,2}$.

   Our results yield Bipartite Ramsey Numbers!

Obstruction Sets

Definition

$G_{n,m}$ contains $G_{a,b}$ if $a \leq n$ and $b \leq m$.

Theorem

For all c there exists a unique finite set of grids OBS_c such that

$G_{n,m}$ is c-colorable iff
$G_{n,m}$ does not contain any element of OBS_c.

1. Can prove using well-quasi-orderings. No bound on $|\mathrm{OBS}_c|$.
2. Our tools yield alternative proof and show

$$2\sqrt{c}(1 - o(1)) \leq |\mathrm{OBS}_c| \leq 2c^2.$$

Rephrase Main Question

Fix c

What is OBS_c

Rectangle Free Sets and Density

Definition

$G_{n,m}$ is the grid $[n] \times [m]$.

1. $X \subseteq G_{n,m}$ is Rectangle Free if there are NOT four vertices in X that form a rectangle.
2. $\mathrm{rfree}(G_{n,m})$ is the size of the largest Rect Free subset of $G_{n,m}$.

Rectangle Free subset of $G_{21,12}$ of size $63 = \lceil \frac{21 \cdot 12}{4} \rceil$

	01	02	03	04	05	06	07	08	09	10	11	12
1	•	•										
2	•		•									
3		•	•									
4			•	•	•							
5		•		•		•						
6	•				•	•						
7						•	•	•				
8					•		•		•			
9				•				•	•			
10						•				•	•	
11					•					•		•
12				•							•	•
13			•			•			•			•
14			•					•		•		
15			•				•				•	
16		•							•	•		
17		•			•			•			•	
18		•					•					•
19	•								•		•	
20	•							•				•
21	•			•			•			•		

Colorings Imply Rectangle Free Sets

Lemma

Let $n, m, c \in \mathsf{N}$*. If* $\chi(G_{n,m}) \leq c$ *then* $\mathrm{rfree}(G_{n,m}) \geq \lceil mn/c \rceil$*.*

Note: We use to get non-col results as density results!!

PART I: 2-COLORABILITY

We will EXACTLY Characterize which $G_{n,m}$ are 2-colorable!

$G_{5,5}$ IS NOT 2-Colorable!

Theorem

$G_{5,5}$ is not 2-Colorable.

Proof:

$$\chi(G_{5,5}) = 2 \implies \mathrm{rfree}(G_{5,5}) \geq \lceil 25/2 \rceil = 13$$
$$\implies \text{there exists a column with} \geq \lceil 13/5 \rceil = 3\ R\text{'s}$$

Case 1: There is a column with 5 R's

Case 1: There is a column with 5 R's.

$$\begin{array}{ccccc} R & \circ & \circ & \circ & \circ \\ R & \circ & \circ & \circ & \circ \\ R & \circ & \circ & \circ & \circ \\ R & \circ & \circ & \circ & \circ \\ R & \circ & \circ & \circ & \circ \end{array}$$

Remaining columns have ≤ 1 R so

$$\text{Number of } R\text{'s } \leq 5 + 1 + 1 + 1 + 1 = 9 < 13.$$

Case 2: There is a column with 4 R's

Case 2: There is a column with 4 R's.

$$\begin{array}{ccccc} R & \circ & \circ & \circ & \circ \\ R & \circ & \circ & \circ & \circ \\ R & \circ & \circ & \circ & \circ \\ R & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \end{array}$$

Remaining columns have ≤ 2 R's

$$\text{Number of } R\text{'s } \leq 4+2+2+2+2 \leq 12 < 13$$

Case 3: Max in a column is 3 R's

Case 3: Max in a column is 3 R's.
Case 3a: There are ≤ 2 columns with 3 R's.

$$\text{Number of } R\text{'s} \leq 3+3+2+2+2 \leq 12 < 13.$$

Case 3b: There are ≥ 3 columns with 3 R's.

$$\begin{array}{ccccc}
R & \circ & \circ & \circ & \circ \\
R & \circ & \circ & \circ & \circ \\
R & R & \circ & \circ & \circ \\
\circ & R & \circ & \circ & \circ \\
\circ & R & \circ & \circ & \circ
\end{array}$$

Can't put in a third column with 3 R's!

Case 4: Max in a column is $\leq 2R$'s

Case 4: Max in a column is $\leq 2R$'s.

$$\text{Number of } R\text{'s} \leq 2+2+2+2+2 \leq 10 < 13.$$

No more cases. We are Done! Q.E.D.

$G_{4,6}$ IS 2-Colorable

Theorem

$G_{4,6}$ is 2-Colorable.

Proof.

$$\begin{array}{cccccc} R & R & R & B & B & B \\ R & B & B & R & R & B \\ B & R & B & R & B & R \\ B & B & R & B & R & R \end{array}$$

$\square$

$G_{3,7}$ IS NOT 2-Colorable

Theorem

$G_{3,7}$ is not 2-Colorable.

Proof.

$$\begin{aligned}\chi(G_{3,7}) = 2 &\implies \mathrm{rfree}(G_{3,7}) \geq (\lceil 21/2 \rceil = 11 \\ &\implies \text{there is a row with} \geq \lceil 11/3 \rceil = 4\ R\text{'s}\end{aligned}$$

Proof similar to $G_{5,5}$— lots of cases.

□

Complete Char of 2-Colorability

Theorem

$$\mathrm{OBS}_2 = \{G_{3,7}, G_{5,5}, G_{7,3}\}.$$

Proof.

Follows from results $G_{5,5}$, $G_{7,3}$ not 2-colorable and $G_{4,6}$ is 2-colorable. $\square$

PART II: TOOLS TO SHOW $G_{n,m}$ NOT c-COLORABLE

We show that if A is a Rectangle Free subset of $G_{n,m}$ then there is a relation between $|A|$ and n and m.

Bound on Size of Rectangle Free Sets (new)

Theorem

Let $a, n, m \in \mathbb{N}$. Let q, r be such that $a = qn + r$ with $0 \le r \le n$. Assume that there exists $A \subseteq G_{m,n}$ such that $|A| = a$ and A is rectangle-free.

1. *If $q \ge 2$ then*
$$n \le \left\lfloor \frac{m(m-1) - 2rq}{q(q-1)} \right\rfloor.$$
2. *If $q = 1$ then*
$$r \le \frac{m(m-1)}{2}.$$

Proof of Theorem

$A \subseteq G_{n,m}$, rectangle free.
x_i is number of points in i^{th} column.

	1	$\cdots$	m
1		$\cdots$	
$\vdots$		$\vdots$	
n		$\cdots$	
	x_1 points $\binom{x_1}{2}$ pairs of points	$\cdots$ $\cdots$	x_m points $\binom{x_m}{2}$ pairs of points

Proof of Theorem

$A \subseteq G_{n,m}$, rectangle free.
x_i is number of points in i^{th} column.

	1	$\cdots$	m
1		$\cdots$	
$\vdots$		$\vdots$	
n		$\cdots$	
	x_1 points $\binom{x_1}{2}$ pairs of points	$\cdots$ $\cdots$	x_m points $\binom{x_m}{2}$ pairs of points

$$\sum_{i=1}^{m} \binom{x_i}{2} \leq \binom{n}{2}.$$

Proof of Theorem (cont)

$$\sum_{i=1}^{m} \binom{x_i}{2} \leq \binom{n}{2}.$$

Sum minimized when $x_1 = \cdots = x_m = x$

$$m\binom{x}{2} \leq \binom{n}{2}.$$

$$x \leq \frac{m + \sqrt{m^2 + 4m(n^2 - n)}}{2m}$$

$$|A| \leq xm \leq \frac{m + \sqrt{m^2 + 4m(n^2 - n)}}{2}$$

PART III: TOOLS TO SHOW $G_{n,m}$ IS c-COLORABLE

We define and use Strong c-Colorings to get c-Colorings

Strong c-Colorings

Definition

Let $c, n, m \in \mathbb{N}$. $\chi : G_{n,m} \to [c]$. χ is a strong c-coloring if the following holds: CANNOT have a rectangle with the two right most corners are same color and the two left most corners the same color.

Example: A strong 3-coloring of $G_{4,6}$.

R	R	G	R	G	G
B	G	R	G	R	G
G	B	B	G	G	R
G	G	G	B	B	B

Strong Coloring Lemma

Let $c, n, m \in \mathbb{N}$. If $G_{n,m}$ is strongly c-colorable then $G_{n,cm}$ is c-colorable.

Example:

R	R	G	R	G	G	B	B	R	B	R	R	G	G	B	G	B	B
B	G	R	G	R	G	G	R	B	R	B	R	R	B	G	B	G	B
G	B	B	G	G	R	R	G	G	R	R	B	B	R	R	B	B	G
G	G	G	B	B	B	R	R	R	G	G	G	B	B	B	R	R	R

Combinatorial Coloring Theorem

Let $c \geq 2$.

1. There is a strong c-coloring of $G_{c+1,\binom{c+1}{2}}$.
2. There is a c-coloring of $G_{c+1,m}$ where $m = c\binom{c+1}{2}$.

Example: Strong 5-coloring of $G_{6,15}$.

O	O	O	O	O	R	R	R	R	R	R	R	R	R	R
O	R	R	R	R	O	O	O	O	B	B	B	B	B	B
R	O	B	B	B	O	B	B	B	O	O	O	G	G	G
B	B	O	G	G	B	O	G	G	O	G	G	O	O	P
G	G	G	O	P	G	G	O	P	G	O	P	O	P	O
P	P	P	P	O	P	P	P	O	P	P	O	P	O	O

Coloring Using Primes!

Theorem

Let p be a prime.

1. *There is a strong p-coloring of $G_{p^2,p+1}$.*
2. *There is a p-coloring of G_{p^2,p^2+p}.*

Proof.

Uses geometry over finite fields. $\square$

Note: Have more general theorem.

Tournament Graph Coloring Theorem

Let $c \geq 2$.

1. There is a strong c-coloring of $G_{2c,2c-1}$.
2. There is a c-coloring of $G_{2c,2c^2-c}$.

Proof.

Uses tourament graphs. $\square$

PART IV: 3-COLORABILITY

We will EXACTLY Characterize which $G_{n,m}$ are 3-colorable!

Easy 3-Colorable Results

Theorem

1. *The following grids are not 3-colorable.*
   $G_{4,19}$, $G_{19,4}$, $G_{5,16}$, $G_{16,5}$, $G_{7,13}$, $G_{13,7}$, $G_{10,12}$, $G_{12,10}$, $G_{11,11}$.
2. *The following grids are 3-colorable.*
   $G_{3,19}$, $G_{19,3}$, $G_{4,18}$, $G_{18,4}$, $G_{6,15}$, $G_{15,6}$, $G_{9,12}$, $G_{12,9}$.

Proof.

Follows from tools. $\square$

$G_{10,10}$ is 3-colorable

Theorem

$G_{10,10}$ is 3-colorable.

Proof.

UGLY! TOOLS DID NOT HELP AT ALL!!

R	R	R	R	B	B	G	G	B	G
R	B	B	G	R	R	R	G	G	B
G	R	B	G	R	B	B	R	R	G
G	B	R	B	B	R	G	R	G	R
R	B	G	G	G	B	G	B	R	R
G	R	B	B	G	G	R	B	B	R
B	G	R	B	G	B	R	G	R	B
B	B	G	R	R	G	B	G	B	R
G	G	G	R	B	R	B	B	R	B
B	G	B	R	B	G	R	R	G	G

$\square$

$G_{10,11}$ is not 3-colorable

Theorem

$G_{10,11}$ is not 3-colorable.

Proof.

You don't want to see this. UGLY case hacking. $\square$

Complete Char of 3-colorability

Theorem

$\mathrm{OBS}_3 =$

$$\{G_{4,19}, G_{5,16}, G_{7,13}, G_{10,11}, G_{11,10}, G_{13,7}, G_{16,5}, G_{19,4}\}.$$

Proof.

Follows from above results on grids being or not being 3-colorable. $\square$

PART V: 4-COLORABILITY

We will EXACTLY Characterize which $G_{n,m}$ are 4-colorable!

Easy NOT 4-Colorable Results

Theorem

The following grids are NOT 4-colorable:

1. $G_{5,41}$ *and* $G_{41,5}$
2. $G_{6,31}$ *and* $G_{31,6}$
3. $G_{7,29}$ *and* $G_{29,7}$
4. $G_{9,25}$ *and* $G_{25,9}$
5. $G_{10,23}$ *and* $G_{23,10}$
6. $G_{11,22}$ *and* $G_{22,11}$
7. $G_{13,21}$ *and* $G_{21,13}$
8. $G_{17,20}$ *and* $G_{20,17}$
9. $G_{18,19}$ *and* $G_{19,18}$

Follows from tools for proving grids are NOT colorable.

Easy IS 4-Colorable Results

Theorem

The following grids are 4-colorable:

1. $G_{4,41}$ *and* $G_{41,4}$.
2. $G_{5,40}$ *and* $G_{40,5}$.
3. $G_{6,30}$ *and* $G_{30,6}$.
4. $G_{8,28}$ *and* $G_{28,8}$.
5. $G_{16,20}$ *and* $G_{20,16}$.

Follows from tools for proving grids are colorable.

Theorems with UGLY Proofs

Theorem

1. $G_{17,19}$ *is NOT 4-colorable: Used some tools.*
2. $G_{24,9}$ *is 4-colorable: Used strong coloring of* $G_{9,6}$.

Theorems with UGLY Proofs

Theorem

1. *$G_{17,19}$ is NOT 4-colorable: Used some tools.*
2. *$G_{24,9}$ is 4-colorable: Used strong coloring of $G_{9,6}$.*

P	R	R	P	R	R
P	B	B	R	P	B
P	G	G	B	B	P
R	P	G	P	G	B
B	P	R	B	P	G
G	P	B	G	R	P
G	B	P	P	B	G
R	G	P	G	P	R
B	R	P	R	G	P

4-coloring of $G_{21,11}$ Due to Brad Loren

	1	2	3	4	5	6	7	8	9	10	11
1	G	B	B	G	R	P	R	G	P	B	P
2	B	G	G	P	B	G	P	R	R	B	R
3	R	R	B	P	B	P	B	P	G	G	R
4	P	R	P	G	B	B	R	P	R	G	B
5	R	P	G	B	B	P	P	B	R	G	G
6	B	R	P	R	G	P	B	R	G	P	B
7	P	G	B	R	G	B	R	G	P	P	R
8	P	P	G	B	R	B	G	R	G	B	P
9	R	B	R	B	G	G	R	P	P	G	B
10	R	P	P	R	G	R	B	B	P	B	G
11	B	P	R	R	P	B	G	G	R	P	G
12	R	B	P	P	P	B	B	R	G	R	G
13	G	G	B	B	R	R	P	P	R	P	G
14	G	B	R	P	B	G	G	R	B	P	P
15	G	P	G	P	G	R	R	R	B	B	B
16	B	B	R	G	P	G	P	B	P	R	G
17	P	G	B	G	P	P	R	B	G	R	B
18	B	P	B	G	G	R	G	P	B	R	R
19	P	G	R	P	R	B	G	B	B	G	R
20	B	R	P	B	R	G	P	G	G	R	P
21	G	R	R	B	P	R	B	P	B	G	P

4-coloring of $G_{22,10}$ Due to Brad Loren

	1	2	3	4	5	6	7	8	9	10
1	P	G	R	R	G	G	P	P	B	B
2	G	P	B	G	B	B	P	R	P	R
3	B	G	B	R	P	P	G	R	P	B
4	P	P	G	G	R	R	B	B	G	P
5	P	B	P	P	G	R	R	G	G	R
6	P	B	R	B	R	P	G	R	G	G
7	G	P	G	P	B	P	R	B	R	G
8	P	R	R	B	P	B	G	G	B	R
9	P	B	B	R	R	G	R	G	P	G
10	R	R	B	B	P	G	R	B	G	P
11	R	G	G	P	R	B	B	G	P	R
12	R	B	R	G	G	P	P	B	B	G
13	B	R	G	B	G	R	B	R	P	P
14	G	G	P	B	B	P	R	R	G	B
15	R	G	P	R	B	R	B	P	P	G
16	B	B	P	G	P	B	P	G	R	R
17	G	P	B	R	P	G	B	P	B	R
18	R	B	G	P	B	G	P	R	R	P
19	G	B	R	P	P	R	B	G	R	B
20	B	R	P	G	R	G	G	B	R	P
21	B	R	G	R	B	P	G	P	B	P
22	G	P	P	R	G	B	G	B	R	B

Absolute Results

Theorem

1. *The following grids are in* OBS_4*:* $G_{5,41}$, $G_{6,31}$, $G_{7,29}$, $G_{9,25}$, $G_{10,23}$, $G_{11,22}$, $G_{22,11}$, $G_{23,10}$, $G_{25,9}$, $G_{29,7}$, $G_{31,6}$, $G_{41,5}$.
2. *For each of the following grids it is not known if it is 4-colorable. These are the only such.* $G_{17,17}$, $G_{17,18}$, $G_{18,17}$, $G_{18,18}$. $G_{21,12}$, $G_{22,10}$.
3. *Exactly one of these is in OBS*$_4$*:* $G_{21,11}$, $G_{21,12}$.
4. *Exactly one of these is in OBS*$_4$*:* $G_{17,19}$, $G_{17,18}$, $G_{17,17}$.
5. *If* $G_{19,17} \in \mathrm{OBS}_4$ *then it is possible that* $G_{18,18} \in \mathrm{OBS}_4$.

Rectangle Free Conjecture

Recall the following lemma:

Lemma

Let $n, m, c \in \mathsf{N}$*. If* $\chi(G_{n,m}) \le c$ *then* $\mathrm{rfree}(G_{n,m}) \ge \lceil nm/c \rceil$*.*

Rectangle Free Conjecture

Recall the following lemma:

Lemma

Let $n, m, c \in \mathbb{N}$. If $\chi(G_{n,m}) \leq c$ then $\text{rfree}(G_{n,m}) \geq \lceil nm/c \rceil$.

Rectangle-Free Conjecture (RFC) is the converse:

Let $n, m, c \geq 2$. If $\text{rfree}(G_{n,m}) \geq \lceil nm/c \rceil$ then $G_{n,m}$ is c-colorable.

Rectangle Free Subset of $G_{22,10}$ of Size of size $55 = \left\lceil \frac{22 \cdot 10}{4} \right\rceil$

	01	02	03	04	05	06	07	08	09	10
1	●						●			
2		●					●			
3			●				●			
4				●			●			
5					●		●			
6						●	●			
7	●	●						●		
8			●	●				●		
9					●	●		●		
10		●	●						●	
11				●	●				●	
12	●					●			●	
13	●			●						●
14		●				●				●
15			●		●					●
16		●			●					
17	●		●							
18				●		●				
19			●			●				
20		●		●						
21	●				●					
22							●	●	●	●

If RFC is true then $G_{22,10}$ is 4-colorable.

Rectangle Free subset of $G_{21,12}$ of size $63 = \left\lceil \frac{21\cdot 12}{4} \right\rceil$

	01	02	03	04	05	06	07	08	09	10	11	12
1	•	•										
2	•		•									
3		•	•									
4			•	•	•							
5		•		•		•						
6	•				•	•						
7						•	•	•				
8					•		•		•			
9				•				•	•			
10						•				•	•	
11					•					•		•
12				•							•	•
13			•			•			•			•
14			•					•		•		
15			•				•				•	
16		•							•	•		
17		•			•			•			•	
18		•					•					•
19	•								•		•	
20	•							•				•
21	•			•			•			•		

If RFC is true then $G_{21,12}$ is 4-colorable.

Rectangle Free subset of $G_{18,18}$ of size $81 = \lceil \frac{18 \cdot 18}{4} \rceil$

	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18
1		●		●										●		●	●	
2	●	●								●	●		●					
3	●								●						●	●		●
4						●			●			●	●	●				
5		●	●			●												●
6	●			●		●	●											
7							●	●		●				●				●
8			●				●		●		●						●	
9		●			●		●					●			●			
10				●							●	●						●
11	●		●		●									●				
12			●	●				●					●		●			
13					●	●		●			●					●		
14	●							●				●					●	
15				●	●				●	●								
16						●				●					●		●	
17			●							●		●				●		
18					●								●				●	●

If RFC is true then $G_{18,18}$ is 4-colorable. NOTE: If delete 2nd column and 5th Row have 74-sized RFC of $G_{17,17}$.

Assuming RFC...

Theorem

If RFC then

$$\mathrm{OBS}_4 = \{G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,10}, G_{22,11}, G_{21,13}, G_{19,17}\} \bigcup$$

$$\{G_{13,21}, G_{11,22}, G_{10,23}, G_{9,25}, G_{7,29}, G_{6,31}, G_{5,41}\}.$$

Proof.

Follows from known 4-colorability, non-4-colorability results, and Rect Free Sets above. □

CASH PRIZE!

The first person to email me both (1) plaintext, and (2) LaTeX, of a 4-coloring of the 17×17 grid that has no monochromatic rectangles will receive \$289.00.

CASH PRIZE!

The first person to email me both (1) plaintext, and (2) LaTeX, of a 4-coloring of the 17×17 grid that has no monochromatic rectangles will receive $289.00.

Also want to know about 18×18.

CASH PRIZE!

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So OBS_4 is known!

PART VI: OPEN QUESTIONS

1. What is OBS_5?
2. Prove or disprove Rectangle Free Conjecture.
3. Have $\Omega(\sqrt{c}) \leq |\mathrm{OBS}_c| \leq O(c^2)$. Get better bounds!
4. Refine tools so can prove ugly results cleanly.

Bibliography

1 R. Graham, B. Rothchild, and J. Spencer. *Ramsey Theory*. Wiley, 1990.

2 R. Graham and J. Solymosi. Monochromatic equilateral right triangles on the integer grid. *Topics in Discrete Mathematics, Algorithms and Combinatorics*, 2006. `www.math.ucsd.edu/~/ron/06_03_righttriangles.pdf` or `www.cs.umd.edu/~/vdw/graham-solymosi.pdf`.

3 R. Rado. Studien zur kombinatorik. *Mathematische Zeitschrift*, pages 424–480, 1933. `http://www.cs.umd.edu/~gasarch/vdw/vdw.html`.

4 R. Rado. Notes on combinatorial analysis. *Proceedings of the London Mathematical Society*, pages 122–160, 1943. `http://www.cs.umd.edu/~gasarch/vdw/vdw.html`.

Bibliography (continued)

5 I. Reiman. Uber ein problem von k.zarankiewicz. *Acta. Math. Acad. Soc. Hung.*, 9:269–279, 1958.

6 Witt. Ein kombinatorischer satz de elementargeometrie. *Mathematische Nachrichten*, pages 261–262, 1951. `http://www.cs.umd.edu/~gasarch/vdw/vdw.html`.

7 K. Zarankiewicz. Problem P 101. *Colloq. Math.*