

The Infinite Can Ramsey Thm: Mileti's Proof

William Gasarch-U of MD

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- ▶ One used 4-ary Ramsey and 1-d Can Ramsey.
- ▶ One used 3-ary Ramsey, 1-d Can Ram, and Maximal Sets.

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1. His interest: He got a more constructive proof of Can Ramsey.
2. My interest: educational.
3. My interest: better bounds when finitized.
4. This finization has never been written up. Will be an extra credit project.

Min-Homog, Max-Homog, Rainbow

Def: Let $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$. Let $V \subseteq \mathbb{N}$. Assume $a < b$ and $c < d$.

- ▶ V is *homog* if $COL(a, b) = COL(c, d)$ iff $TRUE$.
- ▶ V is *min-homog* if $COL(a, b) = COL(c, d)$ iff $a = c$.
- ▶ V is *max-homog* if $COL(a, b) = COL(c, d)$ iff $b = d$.
- ▶ V is *rainb* if $COL(a, b) = COL(c, d)$ iff $a = c$ and $b = d$.

Can Ramsey Thm for $\binom{\mathbb{N}}{2}$: For all $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$, there exists an infinite set V such that either V is homog, min-homog, max-homog, or rainb.

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$(\exists^\infty x \in A)$ means **for an infinite number of $x \in A$**

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$(\exists^\infty x \in A)$ means **for an infinite number of $x \in A$**

$(\forall^\infty x \in A)$ means **for all but a finite number of $x \in A$**

First Step of Construction

The following notation will make later cases similar to this case.

$$V_1 = \mathbb{N}$$

$$x_1 = 1$$

Have $COL : \binom{V_1}{2} \rightarrow \omega$.

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One of the following happens:

- ▶ $(\exists c \in \omega)(\exists^\infty y \in V_1)[COL(x_1, y) = c]$.
Kill all those who disagree. $COL'(x_1) = (H, c)$.
Similar to 1st step of Inf Ramsey.

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Kill duplicates, so in new set $COL(x_1, ?)$ are all different.
 $COL'(x_1) = (RB, 1)$. Similar to proof of 1-ary Can Ramsey.

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 $COL'(x_1) = (RB, 1)$. Similar to proof of 1-ary Can Ramsey.

In both cases let

V_2 be the new infinite set.

x_2 be the least element of V_2 .

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 - ▶ $COL'(x_2) = (RB, 1)$ if x_1 and x_2 are **similar**.
 $COL'(x_2) = (RB, 2)$ if x_1 and x_2 are **different**.
See next slide.

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When we say (H, j) we think of j as a color.
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We also say $j \in \mathbb{N}$.

Really $\omega = \mathbb{N}$ so they are all numbers.

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$$W = \{w_3, w_4, \dots, \}$$

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- ▶ $\text{COL}(x_1, w_3), \text{COL}(x_1, w_4), \dots$ are all different.
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Third Step, i th Step

V_3 is defined and is infinite. x_1, x_2 are colored.
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After third step

$\text{COL}'(x_3) \in \{(H, j) : j \in \omega\} \cup \{(RB, j) : j \leq 3\}$.

V_4 will be infinite.

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V_{i+1} will be infinite.

With Step, Case 1

Recap We have $X = \{x_1, x_2, x_3, \dots\}$

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 $\text{COL}' : X \rightarrow \omega.$

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Case 1 H occurs inf often as 1st coordinate and

$$(\exists c_0 \in \omega)(\exists^\infty x \in X)[\text{COL}'(x) = (H, c_0)].$$

$$H = \{x \in X : \text{COL}'(x) = (H, c_0)\}$$

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H is homog of color c_0 .

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H is min-homog.

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If neither happens then H only occurs finite often as 1st coordinate.

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If neither happens then H only occurs finite often as 1st coordinate.
Eliminate those finite x such that $\text{COL}'(x) = (H, ?)$.

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For Cases 3,4 assume $(\forall x \in X)[\text{COL}'(x) = (RB, ?)]$.

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Case 3 $(\exists i_0 \in \mathbb{N})(\exists^\infty x \in X)[\text{COL}'(x) = (\text{RB}, i_0)]$.

$$H = \{x \in X : \text{COL}'(x) = (\text{RB}, i_0)\}$$

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H is max-homog.

With Step, Case 4

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$$H = \{h_1, h_2, h_3, \dots\}$$

where $\text{COL}'(h_j) = (\text{RB}, c_j)$ with c_j 's different.

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Let $a < b < c$.

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- All of the edges out of h_a to the right, are different from each other.

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So where are we now?

Let $a < b < c$.

- ▶ All of the edges out of h_a to the right, are different from each other.
- ▶ $\text{COL}(h_a, h_c) \neq \text{COL}(h_b, h_c)$.

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So is H a rainbow set?

ωth Step, Case 4

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So where are we now?

Let $a < b < c$.

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- ▶ $\text{COL}(h_a, h_c) \neq \text{COL}(h_b, h_c)$.

So is H a rainbow set?

No. Counterexample on next slide.

Counterexample Due to Liam Gerst

$$\text{COL} : \binom{N}{2} \rightarrow \omega$$

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With Step, Case 4 (cont)

Recap

$$H = \{h_1, h_2, h_3, \dots\}$$

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With Step, Case 4 (cont)

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Recap

$$H = \{h_1, h_2, h_3, \dots\}$$

Let $a < b < c$.

- ▶ All of the edges out of h_a to the right are different from each other.
- ▶ $\text{COL}(h_a, h_c) \neq \text{COL}(h_b, h_c)$.

Claim For all $i \in \mathbb{N}$, c a color, $\deg_c(h_i) \leq 2$.

Proof Assume, BWOC that $\deg_c(h_i) \geq 3$.

Case 1 There two vertices x, y to the right of h_i such that $\text{COL}(h_i, x) = \text{COL}(h_i, y) = c$. This contradicts that all the edges coming out of h_i are different.

Case 2 There two vertices x, y to the left of h_i such that $\text{COL}(x, h_i) = \text{COL}(y, h_i) = c$. This contradicts that x and y disagree.

End of Proof of Claim

Last Step

Recall

Lemma Let X be infinite. Let $COL : \binom{X}{2} \rightarrow \omega$. Let $d \in \omega$. If for every $x \in X$ and $c \in \omega$, $\deg_c(x) \leq d$ then there is an infinite rainbow set.

We apply this to our set H with $d = 2$ to get a rainbow set.