An Application of Ramsey's Theorem to Logic

William Gasarch

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We will assume E is symmetric and not reflexive.

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There are x_1, x_2 such that x_1 connects to EVERY other vertex, and x_2 connects to NO other vertex.
For all $n \geq 2$ there is G with n vertex that satisfies this sentence.

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4. $(\forall x_1)\cdots(\forall x_n)$ means they are DISTINCT.

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Definition If ϕ is a sentence in the language of graphs then $\mathbf{spec(\phi)}$ is the set of all n such that there is G on n vertices such that $G \models \phi$.

Spectrum: Examples

$$\phi = (\exists x_1, x_2, x_3)[E(x_1, x_2) \wedge E(x_1, x_3)]$$

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$\mathrm{spec}(\phi) = \{3, 4, 5, \ldots\}$

Spectrum: Examples

$$\phi = (\forall x)(\exists y \neq x)[E(x, y) \wedge (\forall z \neq y)[\neg E(x, z)]]$$

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$(\exists G$ on 3 vertices$)[G \models \phi]$? NO. Discuss.

$\mathrm{spec}(\phi) = \{0, 2, 4, 6, \ldots, \}$

Spectrum: Another Example

$(\forall x_1, x_2, x_3)$
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$\neg(E(x_1, x_2) \wedge E(x_1, x_3) \wedge E(x_2, x_3))$

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This is asking for a graph without a 3-clique or 3-ind set.

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By Ramsey's Theorem we know that all graphs of size ≥ 6 have a 3-clique or 3-ind set.

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$\mathrm{spec}(\phi) = \{0, 1, 2, 3, 4, 5\}$.

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$\mathrm{spec}(\phi) = \{2\}$.

Note how Simple Those Spectrum's Were

$\phi = (\forall x)(\forall y)[E(x, y)]$.
$\mathrm{spec}(\phi) = \mathbb{N}$.

$\phi = (\exists x, y, z)(\forall w \notin \{x, y, z\})[E(w, x) \wedge E(w, y) \wedge E(w, z)]$.
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Proof Use brute force.
We will use Lemma without comment.

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Note For many (ϕ, G) can do much better than brute force.

Main Theorem

Theorem The following function is computable: Given ϕ, an $\exists^*\forall^*$ sentence in the theory of graphs, output $\mathrm{spec}(\phi)$. ($\mathrm{spec}(\phi)$ will be a finite or cofinite set; hence it will have an easy description.)

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Proof of Claim 1 Let $G = (V, E)$ and $H = (V', E')$ where $V' \subseteq V$. Since $G \models \phi$

$$G \models (\forall y_1 \in V) \cdots (\forall y_m \in V)[\psi(v_1, \ldots, v_n, y_1, \ldots, y_m)]$$

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$$G \models (\forall y_1 \in V)\cdots(\forall y_m \in V)[\psi(v_1,\ldots,v_n,y_1,\ldots,y_m)]$$

H is just G with less vertices, and the vertices that remain have the same edges. And $v_1,\ldots,v_n$ are in H. Hence we DO have

Claim 1

$$\phi = (\exists x_1)\cdots(\exists x_n)(\forall y_1)\cdots(\forall y_m)[\psi(x_1,\ldots,x_n,y_1,\ldots,y_m)]$$

Let $G \models \phi$ with witnesses $v_1,\ldots,v_n$. Let H be an induced subgraph of G that contains $v_1,\ldots,v_n$. Then $H \models \phi$.

Proof of Claim 1 Let $G = (V, E)$ and $H = (V', E')$ where $V' \subseteq V$. Since $G \models \phi$

$$G \models (\forall y_1 \in V)\cdots(\forall y_m \in V)[\psi(v_1,\ldots,v_n,y_1,\ldots,y_m)]$$

H is just G with less vertices, and the vertices that remain have the same edges. And $v_1,\ldots,v_n$ are in H. Hence we DO have
$(\forall y_1 \in V')\cdots(\forall y_m \in V')[\psi(v_1,\ldots,v_n,y_1,\ldots,y_m)]$, SO
$H \models (\forall y_1)\cdots(\forall y_m)[\psi(v_1,\ldots,v_n,y_1,\ldots,y_m)]$
End of Proof of Claim 1

Claim 2, The Main Claim

$$\phi = (\exists x_1)\cdots(\exists x_n)(\forall y_1)\cdots(\forall y_m)[\psi(x_1,\ldots,x_n,y_1,\ldots,y_m)]$$

If $(\exists N \ge n + 2^n R(m))[N \in \mathrm{spec}(\phi)]$ then

$$\{n+m,\ldots,n+2^n R(m),\ldots\} \subseteq \mathrm{spec}(\phi).$$

Proof of Claim 2

Since $N \in \mathrm{spec}(\phi)$ there exists $G = (V,E)$, a graph on N vertices such that $G \models \phi$. Let $v_1,\ldots,v_n$ be such that:

$$(\forall y_1)\cdots(\forall y_m)[\psi(v_1,\ldots,v_n,y_1,\ldots,y_m)].$$

(Proof continued on next slide)

Proof of Claim 2 Continued

$$(\forall y_1)\cdots(\forall y_m)[\psi(v_1,\ldots,v_n,y_1,\ldots,y_m)].$$

Let $X=\{v_1,\ldots,v_n\}$ and $U=V-X$.
Note that $|U-X|\ge 2^nR(m)$. We use $2^nR(m)$ elements of it which we denote

$$u_1,\ldots,u_{2^nR(m)}.$$

Proof of Claim 2 Continued

$$(\forall y_1) \cdots (\forall y_m)[\psi(v_1, \ldots, v_n, y_1, \ldots, y_m)].$$

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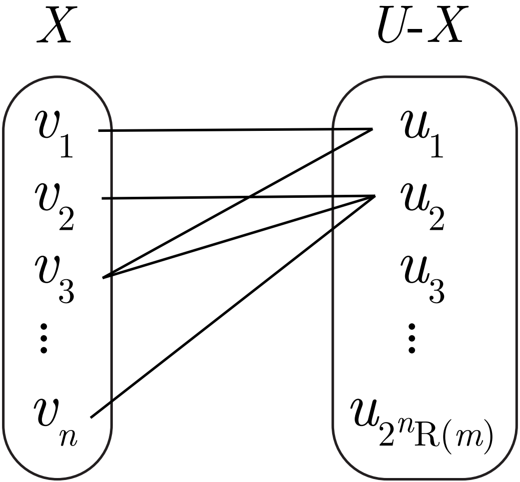
bigskip
Picture on next slide.

X and $U - X$



Proof of Claim 2 Cont: Pigeohole

We define a 2^n-Coloring of U. $u \in U$ maps to $(b_1, \ldots, b_n)$:

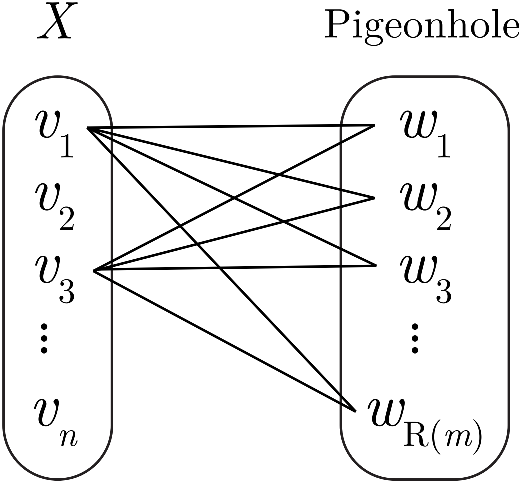
$$b_i = \begin{cases} 0 \text{ if } (u, v_i) \notin E \\ 1 \text{ if } (u, v_i) \in E \end{cases} \tag{1}$$

Hence every $u \in U$ is mapped to a description of how it relates to every element in X. Since $|U| \geq 2^n R(m)$ there exists $R(m)$ vertices,

$$\{w_1, \ldots, w_{R(m)}\}$$

that map to the same vector. So they all look the same to U. (Picture on the next slide.)

w_i's Look the Same to U



Proof of Claim 2 Cont: Ramsey

Apply Ramsey's Theorem to the graph on

$$\{w_1, \ldots, w_{R(m)}\}.$$

to obtain homog set

$$z_1, \ldots, z_m.$$

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We will assume the $z_1, \ldots, z_m$ form an ind set.
(The case where the form a clique is similar.)

Proof of Claim 2 Cont: Ramsey

Apply Ramsey's Theorem to the graph on

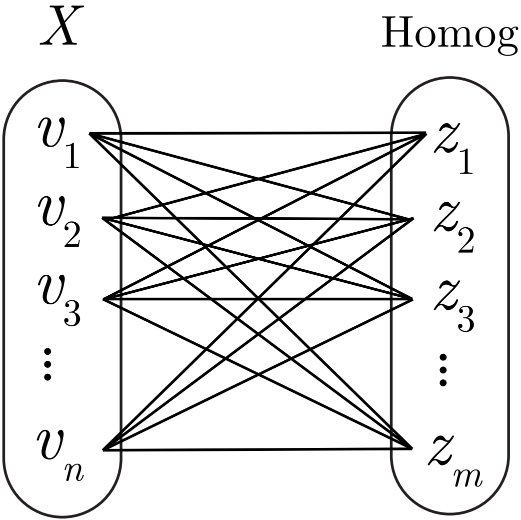
$$\{w_1, \ldots, w_{R(m)}\}.$$

to obtain homog set

$$z_1, \ldots, z_m.$$

We will assume the $z_1, \ldots, z_m$ form an ind set.
(The case where the form a clique is similar.)
We call the set **Super Homog** since it looks the same to U and to each other.
Picture on the next slide.

The Super Homog Set



Proof of Claim 2 Continued

$$(\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)]$$

- We will assume the z_i's form a clique (the other case is similar).
- All of the z_i's map to the same vector. Hence they all look the same to $v_1, \ldots, v_n$.

Example All z_i have edge to $\{v_1, v_3, v_{17}\}$ but no other v_j.

Let H_0 be G restricted to $X \cup \{z_1, \ldots, z_m\}$. By Claim 1 $H_0 \models \phi$.
For every $p \geq 1$ we form a graph $H_p = (V_p, E_p)$ on $n + m + p$ vertices such that $H_p \models \phi$:
Informally add $m + p$ vertices that are **just like the z_i's**.
Formally Next Slide.

Proof of Claim 2 Continued, Formal $H_p = (V_p, E_p)$

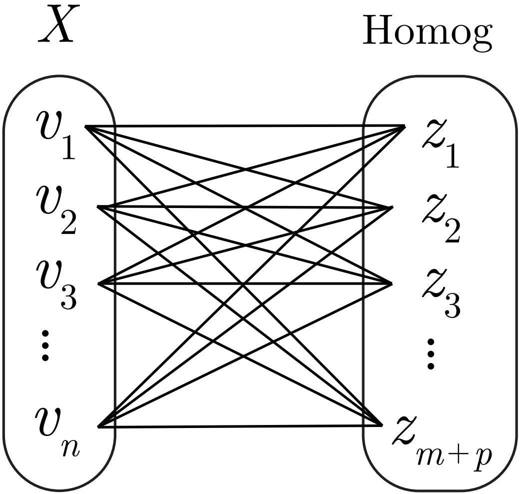
$$(\exists x_1)\cdots(\exists x_n)(\forall y_1)\cdots(\forall y_m)[\psi(x_1,\ldots,x_n,y_1,\ldots,y_m)]$$

- $V_p = X \cup \{z_1,\ldots,z_m,z_{m+1},\ldots,z_{m+p}\}$ where $z_{m+1},\ldots,z_{m+p}$ are new vertices.
- E_p is the union of the following edges.
  - The edges in H_0,
  - Make $\{z_1,\ldots,z_{m+p}\}$ form a clique.
  - Let $(b_1,\ldots,b_n)$ be the vector that all of the elements of $\{z_1,\ldots,z_m\}$ mapped to. For $m+1 \le j \le m+p$, for $1 \le i \le m$ such that $b_i = 1$, put an edge between z_j and v_i. **Example** All of the z_j's have a edge to $\{v_1, v_3, v_{17}\}$ but nothing else.

X sees all of the $z_1,\ldots,z_{m+p}$ as the same. Hence any subset of the $\{z_1,\ldots,z_{m+p}\}$ of size m looks the same to X and to the other z_i's. Hence $H_p \models \phi$, so $n+m+p \in \mathrm{spec}(\phi)$.

End of Proof of Claim 2

Can Add Vertices



Claim 3

$\phi = (\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)]$.
$N_0 = n + 2^n R(m)$.
$N_0 \notin \mathrm{spec}(\phi) \implies \mathrm{spec}(\phi) \subseteq \{0, \ldots, N_0 - 1\}$.

Proof of Claim 3

By Claim 2
$\{N_0, \ldots\} \cap \mathrm{spec}(\phi) \neq \emptyset \implies \{n + m, \ldots, N_0, \ldots\} \subseteq \mathrm{spec}(\phi)$.
We take the contrapositive with a weaker premise.

$$N_0 \notin \mathrm{spec}(\phi) \implies \{N_0, \ldots\} \cap \mathrm{spec}(\phi) = \emptyset$$

$$\implies \mathrm{spec}(\phi) \subseteq \{0, \ldots, N_0 - 1\}.$$

End of Proof of Claim 3

Recap Both Claims

We put a subcase of Claim 2, and Claim 3, next to each other to recap what we know.
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We put a subcase of Claim 2, and Claim 3, next to each other to recap what we know.
Let $N_0 = n + 2^n R(m)$.

Claim 2

$$\text{If } N_0 \in \mathrm{spec}(\phi) \text{ then } \{n+m, \ldots, \} \subseteq \mathrm{spec}(\phi).$$

Claim 3

$$\text{If } N_0 \notin \mathrm{spec}(\phi) \text{ then } \mathrm{spec}(\phi) \subseteq \{0, \ldots, N_0 - 1\}.$$

Algorithm for Finding $\mathrm{spec}(\phi)$

1. Input

$$\phi = (\exists x_1)\cdots(\exists x_n)(\forall y_1)\cdots(\forall y_m)[\psi(x_1,\ldots,x_n,y_1,\ldots,y_m)].$$

2. Let $N_0 = n + 2^n R(m)$. Determine if $N_0 \in \mathrm{spec}(\phi)$.

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    2.2 If NO then, by Claim 3 $\mathrm{spec}(\phi) \subseteq \{0,\ldots,N_0-1\}$. For $0 \le i \le N_0 - 1$ test if $i \in \mathrm{spec}(\phi)$. You now know $\mathrm{spec}(\phi)$ which is finite set. Output it.

End of Proof of Main Theorem

Other Sentences. Part I

What other Sentences could we look at?
$\exists^*\forall^*$ sentences with more complicated objects than graphs.

1. **Colored Graphs** c kinds of edges.
2. **a-ary Hypergraphs** a-ary Hyperedges.
3. **Colored a-ary Hypergraphs** c kinds of a-ary Hyperedges.
4. **$\leq a$-ary Hypergraphs** all arities $\leq a$ allowed.
5. **Colored $\leq a$-ary Hypergraphs** c_i colors for the i-arity sets.

Discuss for which of these is spec decidable.

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YES.
Key ingredient Ramsey theory on $\le a$-hypergraphs.

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$(\exists^*\forall^*)^*$-sentences, only predicate $E(x,y)$. **Morgan** sentences.

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YES, NO, Unknown to Science.

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Is spec for Morgan Sentences decidable? **Vote**
YES, NO, Unknown to Science. YES
Known If ϕ is a Morgan sentence then $\mathrm{spec}(\phi)$ is a union of arithmetic progressions OR the complement of such (proof is hard). So for example

$$\{4, 7, 10, \ldots\} \cup \{11, 22, 33, \ldots\}.$$

Known If A is a Union of AP's then $(\exists\phi)[\mathrm{spec}(\phi) = A]$.

Other Sentences. Part III

$(\exists^*\forall^*)^*$-sentences, predicates of arity $\leq a$-ary. **McKenzie** sentences (that name is real).

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Is spec for McKenzie Sentences decidable? **Vote**.

YES, NO, Unknown to Science.YES.

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Is spec for McKenzie Sentences decidable? **Vote**.

YES, NO, Unknown to Science.YES.
Known If ϕ is a Mackenzie sentence then $\mathrm{spec}(\phi) \in \mathit{EXPTIME}$.

Other Sentences. Part III

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Is spec for McKenzie Sentences decidable? **Vote**.

YES, NO, Unknown to Science.YES.
Known If ϕ is a Mackenzie sentence then $\mathrm{spec}(\phi) \in \mathit{EXPTIME}$.
Also Known If $A \in \mathit{EXPTIME}$ then there exists Mackenzie ϕ such that $\mathrm{spec}(\phi) = A$.

App, “App”, or ““App””

Vote App OR “App” OR ““App””

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App This was **not** a problem people came up with to find an app of Ramsey’s Theorem. Ramsey was working on this problem in logic and proved Ramsey’s Theorem to help him solve it. So the question in Logic is legit.

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““App”” This would be unfair. I reserve the 4-quotes if either NOBODY cares or ONLY I care. (When I prove primes are infinite FROM van Der Waerden’s Theorem, feel free to use 4 quotes. I am not kidding.)

Vote App OR “App” OR ““App””