Van Der Warden's (VDW) Theorem

Exposition by William Gasarch

April 21, 2022

VDW's Theorem

Definition Let $W, k, c \in \mathbb{N}$. Let $\mathrm{COL}\colon [W] \to [c]$. A **mono k-AP** is an arithmetic progression of length k where every elements has the same color. We often say

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We will determine W later.
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Worst Case Scenario B_1 and B_{33} same color. So need B_{65} to exist.

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How many colorings of a block already have a mono 3-AP.

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RRRXY with $X, Y \in \{R, B\}$. 4 colorings.
BBBXY with $X, Y \in \{R, B\}$. 4 colorings.
RBRRR
RBRBR
BRBBB
BRBRB
RBBBX with $X \in \{R, B\}$. 2 colorings.
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I really do not care.

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Go to White Board to finish proof.

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One can work out by hand that

$$W(3,2) = 9.$$

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Note that we **do not** do
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How large? The bounds are not primitive recursive.

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- Proof is elementary. Can be in a this class but won't.
- Bounds still large. Not able to write down.

Deep Math From Search for Better Upper Bounds on VDW Numbers

Exposition by William Gasarch

April 21, 2022

A Man, A Plan, A Canal: Panama!

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Well, a plan anyway.
We outline a plan for getting better upper bounds on $W(k, c)$.

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It DID succeed! (Oh! Thats a good thing!)

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Examples

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2. $\{x^2 : x \in \mathbb{N}\}$ has upper den 0.

A Conjecture, 1936

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Conjecture If $A \subseteq \mathbb{N}$ has positive upper density then, for all k, A has a k-AP.

Theorem Conj implies VDW's Theorem. HW or Final.

The hope was that the proof of Conj would require a new proof of VDW's Theorem that would lead to better bounds.

Roth's Theorem, 1952

Theorem If $A \subseteq \mathbb{N}$ has positive upper density then A has a 3-AP.

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- Roth won the Fields Medal in 1958 for his work on Diophantine approximation (so not for this work).

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- Even so, it introduced very deep methods.

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  - Causes of change: (1) combinatorics using deep math, (2) CS inspired new problems in combinatorics.

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None of these results used mathematics of interest.

Known Lower Bounds

1. Easy Use of Prob Method (was on HW) $W(k,2) \geq \sqrt{k}2^{k/2}$ (Easy extension to 3 colors)
2. Very sophisticated use yields $W(k,2) \geq \frac{2^k}{k^{\epsilon}}$ (Does not extend to 3 colors.)
3. If p is prime then $W(p,2) \geq p(2^p - 1)$. Constructive! (Does not extend to 3 colors.)

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