The Second Anti-Ramsey Theorem
An Exposition by
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1 Introduction

In the last writeup we proved the following:

Theorem 1.1 *Let* $n \in \mathsf{N}$*. Let* $\mathrm{COL}\colon [3n] \to [3]$*. Assume that every color appears in the image* n *times. Then there exists rainbow* $x, y, z \in [3n]$ *such* $x + y = z$

We give a definition for convenience and restate Theorem 1.1.

Definition 1.2 Let $m, c \in \mathsf{N}$. Let $\mathrm{COL}\colon [m] \to [c]$. A *rainbow solution* is x, y, z such that (1) they are all different colors, and (2) $x + y = z$.

We restate Theorem 1.1.

Theorem 1.3 *Let* $n \in \mathsf{N}$ *such that* $n \equiv 0 \pmod 3$*. Let* $\mathrm{COL}\colon [n] \to [3]$*. Assume that every color appears in the image* $\frac{n}{3}$ *times. Then there exists a rainbow solution.*

In Theorem 1.3 we demanded that R, B, G appear as much as possible. Can this be improved? What if we demanded that R, B, G each appear $\ge \frac{n}{4}$ times? Alas, then the theorem fails. The following is folklore. We include the proof for completeness.

Theorem 1.4 *Let* $n \in \mathsf{N}$ *such that* $n \equiv 0 \pmod 4$*. There exists* $\mathrm{COL}\colon [n] \to [3]$ *such that (1) every color appears* $\ge \frac{n}{4}$ *times, and (2) there is no rainbow solution.*

Proof:

$$\mathrm{COL}(w) = \begin{cases} \mathrm{R} & \text{if } 1 \le w \le \frac{n}{2} \text{ and } w \equiv 0 \pmod 2 \\ \mathrm{B} & \text{if } w \equiv 1 \pmod 2 \\ \mathrm{G} & \text{if } \frac{n}{2} + 1 \le w \le n \text{ and } w \equiv 0 \pmod 2 \end{cases} \tag{1}$$

There are $\frac{n}{4}$ red numbers, $\frac{n}{2}$ blue numbers, and $\frac{n}{4}$ green numbers.

If x, y, z is a rainbow solution then $x + y = z$ where they are different colors. Since all red numbers are even, all blue numbers are even, and all green numbers are odd, this is impossible. ∎

Is there some fraction $\frac{1}{4} < \alpha < \frac{1}{3}$ such that the following holds?

Let $n \in \mathsf{N}$*. Let* $\mathrm{COL}\colon [n] \to [3]$*. Assume that every color appears in the image* $\ge \alpha n$ *times. Then there exists rainbow solution.*

Actually any $\alpha > \frac{1}{4}$ works. In fact, the situation is better than that. Our main theorem states that if all colors appear $\ge \frac{n}{4} + 1$ times then there is a rainbow solution.

2 Main Theorem

Esther and George Szekeres [1] proved the following.

Theorem 2.1 *Let* COL: $[n] \to [3]$ *be such that R,B,G all appear* $> \frac{n}{4}$ *times. Then there exists a rainbow solution.*

Proof:

Let COL: $[n] \to [3]$ be such that R,B,G all appear $> n/4$ times.

Let

- $R = \{x_1 < \cdots < x_a\}$ be the set of RED elements,
- $B = \{y_1 < \cdots < y_b\}$ be the set of BLUE elements,
- $G = \{z_1 < \cdots < z_c\}$ be the set of GREEN elements,

We will say $w \in R$ instead of $\mathrm{COL}(w) = R$.

We can assume $x_1 = 1$ and $x_1 < y_1 < z_1$.

Note the following.

Note 2.2

1. If $w < y_1$ then $w \in R$.
2. If $w < z_1$ then $w \in R \cup B$.
3. $y_1 \geq 2$.
4. $z_1 \geq 4$ (If $z_1 = 2$ then $1 < y_1 < 2$. If $z_1 = 3$ then $y_1 = 2$ and x_1, y_1, z_1 is a rainbow solution.)

Notation 2.3

1. A pair of elements are *adjacent elements of* G if they are of the form (z_i, z_{i+1}).
2. Let r be the minimum difference between adjacent elements of G. Formally

$$r = \min_{1 \leq k \leq c-1} z_{k+1} - z_k.$$

3. Let k be the least k such that $z_{k+1} - z_k = r$.

Informally, we are interested in r since, if r is large, then G is small, perhaps $\leq n/4$.

There are five cases. Each case assumes the negation of the prior cases. The cases will have subcases. Every case and subcase concludes either (1) there is a rainbow solution, or (2) the case cannot occur.

Begin The Five Cases

Case 1 $r \geq 4$. By Note 2.2.4, $z_1 \geq 4$. Since $r \geq 4$, inductively, $(\forall 1 \leq i \leq c)[z_i \geq 4i]$. Hence $4c \leq z_c \leq n$, so $c \leq \frac{n}{4}$. This is contrary to hypothesis so cannot occur.

Case 2 $r < y_1$. By Note 2.2.1, $r \in R$ and $y_1 - r \in R$.

We look at the color of $z_k + r - y_1$ (since $r < y_1$, $z_k + r - y_1 < z_k$). First we need that $z_k + r - y_1 \geq 1$.

If $z_k + r - y_1 \leq 0$ then $z_k \leq y_1 - r \leq y_1$, so by Note 2.2.1 $z_k \in R$, but we know that $z_k \in G$. So $z_k + r - y_1 \geq 1$.

Color of $z_k + r - y_1$

1. $z_k + r - y_1 \in R$. Then

$$\begin{array}{ccc} (z_k + r - y_1)+ & y_1 = & z_k + r = z_{k+1} \\ R & B & G \end{array}$$

is a rainbow solution.

2. $z_k + r - y_1 \in B$. Then

$$\begin{array}{ccc} (z_k + r - y_1)+ & (y_1 - r) = & z_k \\ B & R & G \end{array}$$

is a rainbow solution.

3. $z_k + r - y_1 \in G$. Then there exists k' such that $z_k + r - y_1 = z_{k'}$. Since $z_k + r - y_1 < z_k$, $k' < k$. We will use this later.

   For now, we look at $z_k - y_1$.

   (a) $z_k - y_1 \in R$. Then

$$\begin{array}{ccc} (z_k - y_1)+ & y_1 = & z_k \\ R & B & G \end{array}$$

   is a rainbow solution.

   (b) $z_k - y_1 \in B$. Then

$$\begin{array}{ccc} (z_k - y_1)+ & r = & z_k + r - y_1 \\ B & R & G \end{array}$$

   is a rainbow solution.

   (c) $z_k - y_1 \in G$. We are not going to get a rainbow solution. We will contradict the definition of k.

   If $z_k - y_1 \in G$ then there exists k'' such that $z_k - y_1 = z_{k''}$. Since $z_k - y_1 < z_k + r - y_1$, $k'' < k'$.

   $z_{k'} - z_{k''} = z_k + r - y_1 - (z_k - y_1) = r$

   Recall that k is the *least* number such that $z_{k+1} - z_k = r$. But we have just seen that $z_{k'} - z_{k''} = r$ and $k'' < k$. Hence this subcase cannot occur.

Recap We assume the negation of Case 1 and Case 2 which we denote $\neg C1$ and $\neg C2$.

1. $\neg C1$ implies $r \leq 3$.

2. $\neg C2$ and Note 2.2.3 implies $2 \leq y_1 \leq r$.

By the two points above $r \in \{2, 3\}$. By $\neg C2$, $y_1 \leq r$. Hence

$$(r, y_1) \in \{(3,3), (3,2), (2,2)\}.$$

Therefore there are three more cases.

Case 3 $(r, y_1) = (3, 3)$. Since $y_1 = 3$, by Note 2.2.1 we have $1, 2 \in R$.

We consider the colors of $z_t - 1$ and $z_t - 2$ for all $1 \le t \le c$.

Color of $z_t - 1$

- If $z_t - 1 \in B$ then
$$\begin{array}{ccc} (z_t - 1)+ & 1 & = z_t \\ B & R & G \end{array}$$
is a rainbow solution.

- If $z_t - 1 \in G$ then $z_t - 1 = z_{t-1}$ and z_t are adjacent elements of G with difference 1. This contradicts the minimum difference between two adjacent elements of G being $r = 3$. Hence this subcase cannot occur.

- We can assume $(\forall t)[z_t - 1 \in R]$.

Color of $z_t - 2$

- If $z_t - 2 \in B$ then
$$\begin{array}{ccc} (z_t - 2)+ & 2 & = z_t \\ B & R & G \end{array}$$
is a rainbow solution.

- If $z_t - 2 \in G$ then $z_t - 2 = z_{t-1}$ and z_t are adjacent elements of G with difference 2. This contradicts the minimum difference between two adjacent elements of G being $r = 3$. Hence this subcase cannot occur.

- We can assume $(\forall t)[z_t - 2 \in R]$.

To recap: for all $z \in G$, $z - 1, z - 2 \in R$. Hence
$a \ge 2c$
Since $c > \frac{n}{4}$ we get $a > \frac{n}{2}$.
Since $a + b + c = n$

$$b = n - a - c < n - \frac{n}{2} - \frac{n}{4} = \frac{n}{4}$$

This is contrary to hypothesis.

Case 4 $(r, y_1) = (3, 2)$. So $1 \in R$ and $2 \in B$.

Note that $z_{k+1} = z_k + 3$.

We look at the color of $z_k + 1$.

Color of $z_k + 1$

- If $z_k + 1 \in R$ then

$$\begin{array}{cccc} (z_k+1) & +2 & = z_k + 3 = z_{k+1} \\ R & B & G \end{array}$$

  is a rainbow solution.

- If $z_k + 1 \in B$ then

$$\begin{array}{cccc} z_k & +1 & = z_k + 1 \\ G & R & B \end{array}$$

  is a rainbow solution.

- If $z_k + 1 \in G$ then $z_k + 1$ and z_k are adjacent elements of G with difference 1. This contradicts that minimum different between adjacent elements of G is $r = 3$. Hence this subcase cannot occur.

Case 5 $(r, y_1) = (2, 2)$. So $1 \in R$ and $2 \in B$.

We will look at the colors of $z_k - 1$, $z_k + 1$, $z_k - 2$, and 3.

Color of $z_k - 1$

- If $z_k - 1 \in B$ then

$$\begin{array}{ccc} (z_k - 1) & +1 & = z_k \\ B & R & G \end{array}$$

  is a rainbow solution.

- If $z_k - 1 \in G$ then $z_{k-1} = z_k - 1$ and z_k are adjacent elements of G that are 1 apart, which contradicts $r = 2$. Hence this subcase cannot occur.

- We can assume $z_k - 1 \in R$.

Color of $z_k + 1$

- If $z_k + 1 \in B$ then

$$\begin{array}{ccc} z_k & +1 & = (z_k + 1) \\ G & R & B \end{array}$$

  is a rainbow solution.

- If $z_k + 1 \in G$ then $z_{k+1} = z_k + 1$ and z_k are two adjacent elements of G that are 1 apart, which contradicts $r = 2$. Hence this cannot occur.

- We can assume $z_k + 1 \in R$.

Color of $z_k - 2$

- If $z_k - 2 \in R$ then

$$\begin{array}{ccc} (z_k - 2) & +2 & = z_k \\ R & B & G \end{array}$$

  is a rainbow solution.

- If $z_k - 2 \in G$ then note that $z_k - 2, z_k - 1, z_k$ are three consecutive numbers and that (from the case Color of $z_k - 1$) we know that $z_k - 1 \in R$. Hence within G we have $z_k - 2, z_k$ are adjacent, so $z_k - 2 = z_{k-1}$. This contradicts that k is the least index with $z_{k+1} - z_k = r = 2$. Hence this subcase cannot occur.

- We can assume $z_k - 2 \in B$.

Color of 3

- If $3 \in B$ then
$$\underset{R}{(z_k - 1)} \quad \underset{B}{+3} \quad \underset{G}{= z_k + 2 = z_{k+1}}$$
is a rainbow solution.
- If $3 \in G$ then
$$\underset{R}{1} \quad \underset{B}{+2} \quad \underset{G}{= 3}$$
is a rainbow solution.
- We can assume $3 \in R$.

Claim There is a rainbow solution or every odd number $\le n$ is in R.
Proof

We try to prove that every odd number $\le n$ is in R. We will either succeed or show there is a rainbow solution. The proof is by induction.

Base Case Since we are in Case 5, $1 \in R$. We have also shown $3 \in R$.
Induction Hypothesis $i \ge 1$ and $2i-1, 2i+1 \in R$.
Induction Step We show $2i+3 \in R$.

We consider two cases: $2i+1 < z_k$ and $z_k < 2i+1$. Note that $2i+1 = z_k$ is impossible since $2i+1 \in R$ and $z_k \in G$.

Case 5.1 $2i+1 < z_k$.
Color of $z_k - (2i+1)$ (This is ≥ 1 by the Case we are in.)

- If $z_k - (2i+1) \in B$ then

$$\begin{array}{ccc} (2i+1) & +(z_k-(2i+1)) & = z_k \\ R & B & G \end{array}$$

  which is a rainbow solution.

- If $z_k - (2i+1) \in G$ then

$$\begin{array}{ccc} (z_k-(2i+1)) & +(2i-1) = & z_k - 2 \\ G & R & B \end{array}$$

  which is a rainbow solution

- We can assume $z_k - (2i+1) \in R$

Color of $2i+3$

- If $2i+3 \in B$ then

$$\begin{array}{ccc} (z_k-(2i+1)) & +(2i+3) & = z_k+2 = z_{k+1} \\ R & B & G \end{array}$$

  which is a rainbow solution

- If $2i+3 \in G$ then

$$\begin{array}{ccc} (2i+1) & +2 & = 2i+3 \\ R & B & G \end{array}$$

  which is a rainbow solution

- We have $2i+3 \in$ R which was our goal.

Case 5.2 $z_k < 2i+1$.
Color of $\mathbf{2i+1-z_k}$ (This is ≥ 1 by the Case we are in.)

- If $2i+1-z_k \in B$ then
$$\begin{array}{ccc} z_k & +(2i+1-z_k) & =2i+1 \\ G & B & R \end{array}$$
is a rainbow solution.
- If $2i+1-z_k \in G$ then
$$\begin{array}{ccc} (z_k-2) & +(2i+1-z_k) & =2i-1 \\ B & G & R \end{array}$$
is a rainbow solution.
- We can assume $2i+1-z_k \in R$.

Color of $\mathbf{2i+3}$

- If $2i+3 \in B$ then
$$\begin{array}{ccc} (z_k+2) & +(2i+1-z_k) & =(2i+3) \\ G & R & B \end{array}$$
which is a rainbow solution
- If $2i+3 \in G$ then
$$\begin{array}{ccc} (2i+1) & +2 & =2i+3 \\ R & B & G \end{array}$$
- We have $2i+3 \in R$ which was or goal.

End of Proof
Since every odd number $\leq n$ is in R, $|R| \geq n/2$. Hence $|B|+|G| \leq n/2$, so either $|B| \leq n/4$ or $|G| \leq n/4$. Either one contradicts the premise of the theorem. Hence this subcase cannot occur.
End of the Five Cases

Recap: There were 5 cases and several subcases. Every single case either (1) yields a rainbow solution, or (2) cannot occur. Hence there is a rainbow solution. ▮

References

[1] E. Szekeres and G. Szekeres. Adding numbers. *James Cook Math Notes*, 35:4073–4075, 1984. `https://webhomes.maths.ed.ac.uk/cook/iss_no35.PDF`.