The Finite Canonical Ramsey Theorem:Intro and Erdos-Rado's Proof

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Ramsey's Theorem For Graphs

Theorem: $(\forall k)(\exists n)$ for every $COL : \binom{[n]}{2} \rightarrow [c]$ there is a homog set of size k.

What if the number of colors was unbounded?

Do not necc get a homog set since could color EVERY edge differently. But then get infinite *rainbow set*.

Attempt

Theorem: $(\forall k)(\exists n)$ for every $COL : \binom{[n]}{2} \rightarrow \omega$ there is either a homog or rainbow set of size k.
FALSE:

- $COL(i,j) = \min\{i,j\}$.
- $COL(i,j) = \max\{i,j\}$.

Min-Homog, Max-Homog, Rainbow

Definition: Let $COL : \binom{[n]}{2} \rightarrow \omega$. Let $V \subseteq [n]$.

- V is *homogenous* if $COL(a,b) = COL(c,d)$ iff $TRUE$.
- V is *min-homogenous* if $COL(a,b) = COL(c,d)$ iff $a = c$.
- V is *max-homogenous* if $COL(a,b) = COL(c,d)$ iff $b = d$.
- V is *rainbow* if $COL(a,b) = COL(c,d)$ iff $a = c$ and $b = d$.

Definition: Let $COL : \binom{[n]}{1} \rightarrow \omega$. Let $V \subseteq [n]$.

- V is *homogenous* if $COL(a) = COL(c)$ iff $TRUE$.
- V is *rainbow* if $COL(a) = COL(c)$ iff $a = c$.

We write the next Theorem in an odd way to make it conform to the *a*-ary Can Ramsey Theorem.

Theorem: Let $COL : \binom{[k^2]}{1} \rightarrow \omega$. Then there exists either a homog set or a rainbow set of size k.

Canonical Ramsey Theorem for Graphs

Theorem: $(\forall k)(\exists n)$ for all $COL : \binom{[n]}{2} \rightarrow \omega$ there is either

- an homog set of size k,
- an min-homog set of size k,
- an max-homog set of size k,
- a rainbow set of size k.

I-homog for a-hypergraphs

Definition: Let $COL : \binom{[n]}{a} \rightarrow \omega$. Let $V \subseteq [n]$. Let $I \subseteq [a]$. The set V is *I-homog* if for all $x_1 < \cdots < x_a \in \binom{[n]}{a}$ and $y_1 < \cdots < y_a \in \binom{[n]}{a}$,

$$COL(x_1, \ldots, x_a) = COL(y_1, \ldots, y_a) \text{ iff } (\forall i \in I)[x_i = y_i].$$

Canonical Ramsey Theorem for a-hypergraphs

Theorem: $(\forall a)(\forall k)(\exists n)$ for all $COL : \binom{[n]}{a} \rightarrow \omega$ there exists $I \subseteq [a]$ and $V \subseteq [n]$, $|V| = k$ and V is I-homog.

Definition: $ER_a(k)$ is the least n that works.

Note: $ER_1(k) \leq k^2$.

Definition:

$\Gamma_0(k) = k$, $\Gamma_{a+1}(k) = 2^{\Gamma_a(k)}$.

Recall:

- $R_1(k) = 2k - 1 \le \Gamma_0(O(k))$
- $R_a(k) \le \Gamma_{a-1}(O(k))$. (Constant depends on a.)
- $R_a^c(k) \le \Gamma_{a-1}(O(k))$. (Constant depends on a, c.)

GOAL

We give MANY proofs of:

- Can Ramsey for graphs.
- Can Ramsey for a-hypergraphs.

We note

- Ease of proof.
- Bound on $ER_a(k)$ in terms of Γ.

This is original proof due to Erdos-Rado (1950)'
This proof:

- Bounds $ER_2(k)$ using ER_1 and R_4
- Bounds $ER_a(k)$ using ER_{a-1} and R_{2a}.
- Shows $ER_a(k) \le \Gamma_{a^2-1}(O(k^2))$.

Proof of Can Ramsey Theorem for Graphs

Given $COL : \binom{[n]}{2} \rightarrow \omega$ define $COL' : \binom{[n]}{4} \rightarrow [16]$

1. If $COL(x_1, x_2) = COL(x_1, x_3)$ then $COL'(x_1, x_2, x_3, x_4) = 1$.
2. If $COL(x_1, x_2) = COL(x_1, x_4)$ then $COL'(x_1, x_2, x_3, x_4) = 2$.
3. If $COL(x_1, x_2) = COL(x_2, x_3)$ then $COL'(x_1, x_2, x_3, x_4) = 3$.
4. If $COL(x_1, x_2) = COL(x_2, x_4)$ then $COL'(x_1, x_2, x_3, x_4) = 4$.
5. If $COL(x_1, x_2) = COL(x_3, x_4)$ then $COL'(x_1, x_2, x_3, x_4) = 5$.
6. If $COL(x_1, x_3) = COL(x_1, x_4)$ then $COL'(x_1, x_2, x_3, x_4) = 6$.
7. If $COL(x_1, x_3) = COL(x_2, x_3)$ then $COL'(x_1, x_2, x_3, x_4) = 7$.
8. If $COL(x_1, x_3) = COL(x_2, x_4)$ then $COL'(x_1, x_2, x_3, x_4) = 8$.
9. If $COL(x_1, x_3) = COL(x_3, x_4)$ then $COL'(x_1, x_2, x_3, x_4) = 9$.
10. If $COL(x_1, x_4) = COL(x_2, x_3)$ then $COL'(x_1, x_2, x_3, x_4) = 10$.
11. If $COL(x_1, x_4) = COL(x_2, x_4)$ then $COL'(x_1, x_2, x_3, x_4) = 11$.
12. If $COL(x_1, x_4) = COL(x_3, x_4)$ then $COL'(x_1, x_2, x_3, x_4) = 12$.
13. If $COL(x_2, x_3) = COL(x_2, x_4)$ then $COL'(x_1, x_2, x_3, x_4) = 13$.
14. If $COL(x_2, x_3) = COL(x_3, x_4)$ then $COL'(x_1, x_2, x_3, x_4) = 14$.
15. If $COL(x_2, x_4) = COL(x_3, x_4)$ then $COL'(x_1, x_2, x_3, x_4) = 15$.

Finish up the proof

If NONE of the above then $COL'(x_1, x_2, x_3, x_4) = 16$.

CLASS DO IN GROUPS: Cases 1-15. Some use One-Dim Can Ramsey.

If color is 16 get Rainbow EASILY.

$ER_2(k) \le R_4(ER_1(k)) \le R_4(k^2) \le \Gamma_3(O(k^2))$.

- GOOD- All cases EASY.
- GOOD- Rainbow case trivial.
- BAD- number of cases is large.
- BAD- Proof yields $ER_2(k) \le \Gamma_3(O(k^2))$ LARGE!

PROOF ONE: The a-ary Case

List all unordered pairs of elements of $\binom{2a}{a}$.

$COL'(x_1, \ldots, x_{2a})$ is the least i such that the ith pair is equal.

Else color it $\binom{\binom{2a}{a}}{2} + 1$. (Get rainbow EASILY.)

Need to prove it this works.

When get Homog set $\{h_1, h_2, h_3, \ldots h_r\}$ actually take $\{h_a, h_{2a}, h_{3a}, \ldots\}$. We ignore this in the analysis.

Proof by Example

I need a number a

I need two subsets of $[2a]$

$(i_1, i_2, i_3, i_4, i_5)$ and $(j_1, j_2, j_3, j_4, j_5)$
such that some coordinates are the same.

Proof by Example

$a = 5$.
$(1, 5, 7, 9, 10)$ and $(2, 5, 6, 8, 10)$

$$COL(x_1, x_5, x_7, x_9, x_{10}) = COL(x_2, x_5, x_6, x_8, x_{10})$$

Define $COL'(x, y) = COL(-, x, -, -, y)$ (Here is where we use $\{h_a, h_{2a}, \ldots\}$.)
Easy: COL' is well defined. Apply ER_2. Say Max-homog.

$$COL(y_1, y_2, y_3, y_4, y_5) = COL(z_1, z_2, z_3, z_4, z_5) \text{ iff}$$

$$COL'(y_2, y_5) = COL'(z_2, z_5)(\text{Def of } COL') \text{ iff}$$

$$y_5 = z_5 \ (COL' \text{ is Max-homog}).$$

SO get $\{5\}$-homog

Proof by Harder Example

$a = 7$.
$(1, 2, 7, 8, 10, 11, 13)$ and $(2, 3, 7, 8, 9, 11, 14)$

$$COL(x_1, x_2, x_7, x_8, x_{10}, x_{11}, x_{13}) = COL(x_2, x_3, x_7, x_8, x_9, x_{11}, x_{14})$$

Define $COL'(x, y, z) = COL(-, -x, y, -, z, -)$
COL' is well defined (HW). If get $\{1, 3\}$-homog.

$$COL(y_1, y_2, y_3, y_4, y_5, y_6, y_7) = COL(z_1, z_2, z_3, z_4, z_5, z_6, z_7) \text{ iff}$$

$$COL'(y_3, y_4, y_6) = COL'(z_3, z_4, z_6) \text{ (Def of } COL'\text{) iff}$$

$$y_3 = z_3 \text{ AND } y_6 = z_6 \text{ (} COL' \text{ is } \{1, 3\}\text{-homog).}$$

SO get $\{3, 6\}$-homog.

Upshot and PROS/CONS

Arity: $2a$

Number of colors: $c = \binom{\binom{2a}{a}}{2} + 1$.

Get $ER_a(k) \le R^c_{2a}(ER_{a-1}(k))$ Can show

$$ER_a(k) \le \Gamma_{a^2-1}(O(k^2))$$

- GOOD- All cases EASY.
- GOOD- Rainbow case trivial.
- BAD- number of cases is large.
- BAD- $ER_a(k) \le \Gamma_{a^2-1}(O(k^2))$. LARGE!