CMSC 858M: Algorithmic Lower Bounds
Spring 2021
Chapter 2: SAT and its Variants

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December 12, 2021

1 General Overview

Overall, I really enjoyed this chapter. The variants of SAT and Schaefer's Dichotomy Theorem were interesting. Moreover, the chapter provided knowledge and helpful examples about gadgets, which seem like a powerful tool for such problems, and the examples of video games were fun and novel. Some of the following critiques lie in that I think there are too many examples, but depending on the authors' goal of the text, I may want to retract such statements. My background and interests are primarily in combinatorics, where conventional presentation is the introduction of a technique, followed by a few examples to see how to apply it. If this book has a similar goal, then I do think the number of examples seem a bit overkill; after a certain point I do not think any particularly unique ideas are being presented in the examples, and they get somewhat repetitive. If your goal however is to provide a larger perspective on the state of the area in a survey-type fashion, ignore some of my comments.

2 Section 2.1: Introduction

Good – no comments.

3 Section 2.2: Variants of SAT

This section is very notation / definition heavy, and as such took me a few passes to fully understand and remember all the definitions. However, I doubt there's anyway around this, simply by the nature of the section. I also think Schaefer's Dichotomy Theorem is a really cool result, so I'm glad you mention it.

4 Section 2.3: Easy Example: 2-Colorable Perfect Matching

I think this section is a crucial example to get the reader more comfortable with what gadgets are and how to use them before moving to more advanced examples. As a very small critique, the gadget images in Figure 2.3 don't look as nice as the graphs in Figure 2.2. It would be nice if there was a consistent graph format throughout the text, although I realize that probably entails a lot of extra work.

5 Section 2.4: Complex Example: Cryptarithms

Interesting application and a good increase in difficulty from the previous section – no real critiques.

6 Section 2.5: Pushing 1×1 Blocks

This is a cool section, and it's clear that such an application would be useful in proving many games NP-hard. My main criticism is with the presentation: there are tons of images, which while possibly helpful for the reader / necessary for the proofs ultimately dominate the section and make it difficult to follow the paragraphs sometimes. For example, the start sentence at the end of page 43 is separated from its end by two full pages of large images. This critique of possibly too many / too large images carries over throughout the rest of the chapter, although this section is definitely the worst offender.

7 Section 2.6: Video Games that are NP-Hard

I find it fascinating that you can prove such things are NP-Hard. However, I think the interest in the chapter is far more for the novelty of being able to show such things; the proofs are not terribly enlightening by this point in the chapter, so I'm glad each subsection is kept quite brief. I'm not entirely sure why "Conway's Phutball" is in this section, as it does not appear to be a video game. I'm in favor of deleting it, as I think there are enough applications already.

As a quick note, "the" on page 54 in "Here are th gadgets you need" is missing an e. Additionally, "individual" right before section 2.7 is missing an l, and the abrupt ending of section 2.6 feels weird (I get that it is setting up for the next section, but I'm not a fan).

8 Section 2.7: A Classic Game: Checkers

I would be in favor of deleting this section for conciseness. I think the chapter already has many (more interesting) applications, and at this point (in my opinion) no noticeably new / distinct information is occurring.

9 Section 2.8: The Complexity of Origami

I'm not particularly interested in origami, but I think this is distinct enough of an application from the games of the prior section to be worth mentioning, and I'm sure if I was into origami I would find this chapter particularly fascinating. In particular, I think this section does the best job of incorporating images without disrupting the flow of the text.

10 Important Related Problems

The chapter seemingly covers every variant of SAT known to man that could be argued is "natural" in any way, so I don't think I can suggest any more variants. However, there are several generalizations of Schaefer's Dichotomy Theorem that I think are interesting. I noticed later chapters in the book also contain some variants of that theorem, but from what I was able to gather, these are different.

- Allender et al. [2] refined Schaefer's theorem to show it can be determined in polynomial time whether the problem is in co-NLOGTIME, L-complete, NL-complete, $\oplus$L-complete, P-complete, or NP-complete.
- Bodirsky and Pinsker [4] provide a version of Schaefer's theorem for propositional logic of graphs instead of Boolean logic. The primary theorem is as follows.

  Theorem 1 *For all ϕ, the problem Graph-SAT(ϕ) is either NP-complete or in P. Moreover, the problem of deciding for given ϕ whether Graph-SAT(ϕ) is NP-complete or in P is decidable.*

- Creignou and Hermann [5] found a similar result for counting the number of solutions. In particular, they showed the following theorem.

  Theorem 2 *Let S be a finite set of logical relations. If every relation in S is affine then #SAT(S) is in FP, otherwise #SAT(S) is #P-complete.*

If you're interested in adding more examples of NP-hard video/computer games, the following are some interesting examples. (I tried to avoid examples like Candy Crush that showed up in later chapters, which admittedly is most interesting examples I found online.)

- Starcraft is a strategy game, where players build buildings and armies to attack other players by collecting resources from the environment. There are multiple races a player can play as, each with its own advantages and disadvantages. The game is over when only one player's buildings remain standing. Viglietta [9] proved that based on a particular configuration of game with two different races being used, the game is NP-hard.

- Tron is an arcade game based on a movie by the same name. The goal of the game is to control the character, Tron, through 12 levels of four sub-games. One of the sub-games is a type of race, and Viglietta [9] showed it, and thus Tron in general, is NP-hard.

- Hanano is a 2D puzzle game based on colored stones and colored flowers. One swaps stones, which causes flowers to spread to adjacent stones of the same color. The puzzle is solved when all stones are covered by flowers. Liu and Yang showed that (even in certain restricted cases) deciding whether a level admits a solution in NP-hard [6].

- LaserTank is a computer puzzle game, where a player must control a tank through movement or shooting to reach a flag, completing the puzzle. Alexandersson and Restadh [1] showed solving such puzzles is NPC.

If you're interested in adding more examples of NP-hard classical games or activities, the following are some interesting examples.

- Hanabi is a multi-player card game with imperfect information, somewhat similar to solitaire. There is a deck of 50 cards, each card with one of five colors and one of five values (there are repeated combinations). The goal is to create five piles (for each color) of numerically increasing cards by taking turns providing either a hint to another player about color or number, discarding a card to obtain another hint, or playing a card into the piles. The difficulty lies in the fact that a particular player cannot see their own hand, only everyone else's.d Baffier et al. [3] showed Hanabi is NPC, even if players are allowed to cheat in certain ways.

- Battleships puzzles (not to be confused with battleship, the popular game with pegs) are a type of packing puzzles, where a variety of different shaped ships must be put onto a grid with several conditions, such as no two ships being in adjacent squares, and the number of ship segments in a particular column is equal to the value of that column tally. Sevenster [7] showed these puzzles are NPC.

- Kakuro, or Cross Sums, is a type of logic puzzle, similar to a crossword puzzle. There is a grid of black and white cells, some of which contain a diagonal line and contain numerical entries as clues. The goal is to fill the cells with numerical values such that the sum of the numbers in each entry matches the corresponding clue(s). These puzzles were proven NPC by Seta [8].

References

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