Chapter 16: 3SUM-Hardness: A Method for Obtaining Quadratic Lower Bounds

CMSC 858M: Algorithmic Lower Bounds: Fun with Hardness
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Textbook: Fun with Hardness: Algorithmic Lower bounds

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1 Introduction

In contrast to parts of the book which show some problems are probably not solvable in polynomial time or space or variants thereof. But even within polynomial time there are distinctions. We will show (1) certain problems are probably not in subquadratic time, (2) certain problems are probably not in subcubic time.

2 The 3SUM problem

Gajentaan and Overmars [GO12] used the following problem to show that other problems are probably not in subquadratic time.

Definition 1 (3SUM)**.** *Instance: n integers.*
Question: Do three of the integers sum to 0?

The following thereom gives better and better algorithms for 3SUM.

Theorem 1. *1. 3SUM can be solved in $O(n^3)$ time.*

2. 3SUM can be solved in $O(n^2 \log n)$ time.

3. 3SUM can be solved in randomized $O(n^2)$ time.

4. 3SUM can be solved in deterministic $O(n^2)$ time.

Definition 2 (subquadratic)**.** *An algothim is subquadratic if there exists an $\varepsilon > 0$ such that it runs in time $O(n^{2-\varepsilon})$.*

Despite enormous effort nobody has obtained a subquadratic algorithm for 3SUM. In the next section we make this a conjecture and, from the conjecture, obtain quadratic lower bounds on other problems.

3 Definition of 3SUM-Hardness

Gajentaan and Overmars [GO12] (essentially) defined 3SUM-hardness and showed many problems are 3SUM-hard. Many of the problems are in computational geoemtry.

Conjecture 1 (3SUM conjecture)**.** *There is no subquadratic algorithm for* 3SUM*.*

Definition 3. *Let A and B be sets or functions (though usually sets).*

1. *$A \leq_{sq} B$ means that if there is a subquadratic algorithm for B then it can be used to obtain a subquadratic algorithm for A. Usually this will be just like $\leq_p$ where you solve an instance of A by quickly creating an instance of B. However, there are times when you need to make more queries to B. The sq stands for sub-quadratic.*

2. *$A \equiv_{sq} B$ if $A \leq_{sq} B$ and $B \leq_{sq} A$.*

Definition 4. *A problem A is* 3SUM*-hard if* 3SUM $\leq_{sq} A$*.*

Both NP-hardness and 3SUM-hardness state that there are no polynomial time or subcubic algorithms contingent on $\mathsf{P} \neq \mathsf{NP}$ and the 3SUM conjecture, respectively. Do note that 3SUM is *not* a natural class hence there is also no notion of 3SUM-complete.

4 Some 3SUM-hard problems

4.1 Three variants of 3SUM

Recall that if the integers of an instance of 3SUM are in $[-u, u]$ then 3SUM can be solved in time $O(n + u \log n)$. For $u = O(n^{2-\varepsilon})$ this yields a subquadratic algorthim for 3SUM. For the case when u is bigger, we have the following theorem.

Theorem 2. *[BDP08] The* 3SUM *problem restricted to $[-n^3, n^3]$ is* 3SUM*-hard.*

Definition 5 (Convolution 3SUM)**.** *Instance: A set of n integers $a_1, \ldots, a_n$. Question: Is there $i \neq j$, such that $a_{i+j} = a_i a_j$. Note that proving Convolution 3SUm is in $O(n^2)$ is much easier than showing 3SUm is in $O(n^2)$.*

Theorem 3. *Convolution* 3SUM *is* 3SUM*-hard.*

Definition 6 (3SUM')**.** *Instance: Three sets A, B, and C of n integers. Question: Is there $a \in A$, $b \in B$, $c \in C$ such that $a + b = c$.*

Theorem 4. $\text{3SUM} \equiv_{sq} \text{3SUM}'$.

Proof. $\text{3SUM} \leq_{sq} \text{3SUM}'$. This reduction follows easily by restricting 3SUM$'$ with $A = S, B = S$, and $C = -S$ where S is the set of integers of 3SUM. ■

4.2 3SUM-hard problems in computational geometry

All of the following results are by [GO12].

Definition 7 (GEOMBASE)**.** *Instance: n points in $\mathbb{Z}^2$ with y-coordinate in $\{0, 1, 2\}$.*
Question: Does there exist a non-horizontal line hitting 3 points of this set.

Theorem 5. $\text{3SUM}' \equiv_{sq}$ GEOMBASE*. Hence,* GEOMBASE *is* 3SUM*-hard.*

Definition 8 (GEOMBASE')**.** *Instance: n points in $\mathbb{Z}^2$ with y-coordinate in $\{0, 1, 2\}$, and ε. View the points as holes in the $y = 0, 1, 2$ lines and enlarge them to be ϵ-long. In addition, we view the $y = 0, 1, 2$ lines as finite segments.*
Question: Does there exist a non-horizontal line going through 3 of the holes i.e., hitting 3 points of this set.

Theorem 6. $\text{GEOMBASE} \leq_{sq} \text{GEOMBASE}'$.

Definition 9 (COLLINEAR)**.** *Instance: n points in $\mathbb{Z} \times \mathbb{Z}$.*
Question: Are three of the points collinear.

Theorem 7. $\text{3SUM} \leq_{sq}$ COLLINEAR*, hence* COLLINEAR *is* 3SUM*-hard.*

5 Further results

1. [AM17] Role-matchmaking is a problem where we have players with skills levels as well as preferences over specific roles they would like to play (e.g., in soccer we have roles of goalkeeper, defender, midfield,

or forward). This problem has immediate applications to many popular online videogames such as *League of Legends* and *Dota 2*. Then, assuming the 3SUM conjecture, role-matchmaking is intractable. Intractibility follows from the fact that these games have many players queuing up at the same time so even quadratic time is too much.

2. [KPP16] Chiba and Nishizeki's $O(m\alpha)$-time algorithm (SICOMP 1985) for enumerating all triangles in a graph with arboricity/degeneracy α is essentially optimal, for any α.

3. [KPP16] Bjørklund, Pagh, Williams, and Zwick's algorithm (ICALP 2014) for listing t triangles is essentially optimal (assuming the matrix multiplication exponent is $\omega = 2$)

4. [KPP16] Any static data structure for SetDisjointness that answers queries in constant time must spend $\Omega(N^{2-o(1)})$ time in preprocessing, where N is the size of the set system.

5. [GRT97] 3-POINTS-ON-LINE is 3SUM-hard.

6. [GO12] MINIMUM-AREA-TRIANGLE is 3SUM-hard. Given a set S of n points on the plane, what is the area of the smallest triangle formed by any three of these points? the problem of

7. [GO12] SEPARATOR is 3SUM-hard. Given a set S of line segments, is there a line that divides S into nonempty subsets?

8. [GO12] STRIPS-COVER-BOX is 3SUM-hard. Given a rectangle and n infinite strips (the space between two parallel lines), does the union of the strips cover the rectangle?

9. [GO12] TRIANGLES-COVER-TRIANGLE is 3SUM-hard. Given a set S of n triangles and a triangle t, does the union of the triangles in S cover t?

6 Chapter suggestions

1. Definition 16.3.3 "of" should read "if".

2. Statement of the Convolution 3SUM problem has a two typos: (1) Convlution instead of Convolution and (2) a_J instead of a_j.

3. Description of GEOMBASE says "I addition". Should that be "in addition"?

4. Some more exercises for this section would be helpful.

References

[AM17] Josh Alman and Dylan McKay. Theoretical foundations of team matchmaking. In *Proceedings of the 16th Conference on Autonomous Agents and MultiAgent Systems*, pages 1073–1081, 2017.

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