CMSC 858M: Algorithmic Lower Bounds: Fun with Hardness Proofs
Fall 2020

Instructor: Mohammad T. Hajiaghayi
Scribe: Jacob Prinz

TBD

1 Overview

In complexity theory, one often deals with large classes of time complexity, like P, the set of all polynomial time functions. However, in some applications, one may want to deal with more specific time complexity classes. In this section, we present the All Pairs Shortest Path problem. This problem is known to be able to be solved in cubic time. It has been shown to be equivalent to a number of other problems. None of these problems has a known solution in sub-cubic time (which we will define in this section). Therefore, we conjecture that APSP, and therefore many other problems, can not be solved in sub-cubic time. One may then use this assumption to demonstrate the hardness of other problems by reducing them to APSP.

2 All Pairs Shortest Path

Definition 1 (Graph Notation) *A Graph is defined as a triple* (V, E, w)*, where* $w : V \times E \to \mathbb{Z}$ *is the weight function.* $n = |V|$*, and* $m = |E|$*.*

Definition 2 (Shortest Path) *Given a graph* $G = (V, E, w)$*, with weights in* $\mathbb{N}$*, if* $x, y \in G$*, then* $\mathtt{dist}_G(x, y)$ *is the length of the shortest path between* x *and* y*.*

Definition 3 (All Pairs Shortest Path (APSP)) *In a graph* $G = (V, E, w)$*, compute* $\mathtt{dist}_G(x, y)$ *for all* $x, y \in V$*.*

Note that for the purpose of finding the time complexity of algorithms in this section, we consider arithmetic operations to have unit cost.

There is a well-known algorithm due to Floyd and Warshall which runs in $O(n^3)$ time.

- for all $x \in V$, $y \in V$, $z \in V$, do
  - if $\mathtt{dist}(y, z) > \mathtt{dist}(x, y) + \mathtt{dist}(x, z)$, then
    * $\mathtt{dist}(y, z) \leftarrow \mathtt{dist}(x, y) + \mathtt{dist}(x, z)$

One can also use Dijkstra's algorithm on each vertex, for the same worst case time complexity.

A natural question is if there is an algorithm for APSP which is better than cubic. The following are known:

- Due to Fredman [?], we have an algorithm that runs in $O(\frac{n^3 (\log \log n)^{1/3}}{(\log n)^{1/3}})$
- Due to Williams [?] we have an algorithm that runs in $O(\frac{n^3}{2^{\Omega(\sqrt{\log n})}})$
- Due to Zwick [?] we have a good approximation. For all $\epsilon > 0$, there exists a $(1 + \epsilon)$ approximation to APSP that runs in $O(\frac{n^\omega}{\epsilon})$, where ω is the exponent of the best known matrix multiplication algorithm, which is currently 2.3728659.

None of the above algorithms, however, meet the following criteria:

Definition 4 (subcubic time) *An algorithm runs in subcubic time if there exists $\epsilon > 0$ such that it runs in time $O(n^{3-\epsilon})$*

Nobody has ever been able to find a subcubic algorithm for ASPS.

3 APSS Hardness

Conjecture 1 *There is no subcubic time algorithm for APSP.*

Definition 5 *If A and B are decision problems, then $A \leq_{sc} B$ means that if there is a subcubic time algorithm for B then there is a subcubic time algorithm for A. Typically, this is just a direct reduction of A to B¿*

Definition 6 $A \equiv_{sc} B$ *iff* $A \leq_{sc} B$ *and* $B \leq_{sc} A$. *We say that A and B are subcubic equivalent.*

Definition 7 (ASPS-hard)
- *A is ASPS-hard if* $\mathsf{ASPS} \leq_{sc} A$
- *A is ASPS-complete if A is ASPS-hard and* $A \leq_{sc} \mathsf{APSP}$

Because of the conjecture, we believe that problems which are APSP- hard have no subcubic algorithm.

Scribe: Jacob Prinz
Lecture 1b **Date:** 09/01/2010

4 Centrality Measures

There is a collection of measurements on graphs called *Centrality Measures*. These measurements have real world uses, like measusring properties of social or transportation networks. It is an important open problem to show that they are APSP-hard. However, this is an open problem. Instead, we will reduce some problems to some other problems.

Definition 8 *Let* G *be a weighted directed graph. Let* v *be a vertex in* G. *Let* $\alpha_v = \max_{v'} \mathrm{dist}(v, v')$.

Definition 9 (Center and Radius of graph) *The center of a graph is the vertex with minimum* v_α. *This value is the radius of the graph.*

Definition 10 (Diameter (DIAM)) *The diameter of a graph* G *is the maximum distance between any two vertices.*

Definition 11 (Median) *The median of a graph is* $\min_x \sum_y \mathrm{dist}(x, y)$

The following measurement will be helpful to determine how useful a vetex is to a shortest path in a graph.

Definition 12 (Betweenness Centrality (BC)) $BC_{s,t}(x)$ *is the fraction of shortest paths between* s *and* t *which pass through* x.

Definition 13 $BC(x) = \sum_{s,t} BC_{s,t}(x)$

Definition 14 (Positive Betweenness Centrality) $PBC(x) =$ *Is* $BC(x) > 0$*?*

Definition 15 (Negative Triangle (NEGTRI)) *Given a weighted directed graph* $G = (V, E, w)$ *with weights in* $\{-M, ..., M\}$*, is there a triangle with negative sum of weights?*

5 Subcubic Equivalence

Clearly, all of the problems in the previous section are $\leq_{sc}$ APSP and therefore in $O(n^3)$. Can one make a subcubic algorithm?

Theorem 1 *RADIUS, MEDIAN, BC, and NEGTRI are all APSP-complete [**?**].*

There is good evidence for the following conjecture:

Conjecture 2 *DIAM is APSP-complete*

Due to Abboud et al. [**?**] and Williams and Williams [**?**], we knowo that all of the problems mentioned in the above theorem are APSP-complete.

6 DIAM and PBC

Theorem 2 *DIAM* $\leq_{sc}$ *PBC*

Given a graph G, create a new graph G_D which is $G \cup \{x\}$, with an edge of weight $\frac{D}{2}$ between x and each other vertex.

If the betweeness centrality of x is positive, then that means that D is smaller than the shortest path between some two points in G. If it is zero, then D is larger than the longest shortest path between some two points.

So, do I binary search on D to find the diameter of the graph.

Theorem 3 *PBC* $\leq_{sc}$ *DIAM*

Given a graph $G = (V, E)$ and $x \in V$ create a new graph $G' = (V', E')$ with $V' = V \cup V_a \cup V_b$. Let M be the largest weight, and let $W = 3M|V|$ Let $E' = E \cup$ the following edges:

- For each $v \in V - \{x\}$, an edge $(v_a, v, W - \mathtt{dist}_G(v, x))$ and $(v, v_b, W - \mathtt{dist}_G(x, v))$
- For each $v \in V$, edges $(v, v_a, 0)$ and $(v_b, v, 0)$.

Note that we can compute all of the $\mathtt{dists}$ in subcubic time with Dijkstra's algorithm.

Then, $\mathsf{PBC}(x) = 1$ iff $\mathtt{dist}_{G'}(s_a, t_b) = 2W$.

Theorem 4 $\mathsf{NEGTRI} \leq_{sc} \mathsf{RADIUS}$

Input a weighted directed graph $G = (V, E < w)$ with weights in $\{-M, ..., M\}$. Let $Q = 3M$. Then, construct G' weighted undirected for input into RADIUS.

- Let $V' = \{x\} \cup V_a \cup V_b \cup V_c \cup V_d$ four copies of V.
- Make an edge of weight $2Q + M$ between (x, v_a) for each $v_a \in V_a$.
- for each edge $(u, v) \in E$, put an edge of weight $2Q$ between (u_a, v_d).
- for each $(u, v, w) \in E$, put three edges in G': $(u_a, v_b, Q + w)$, $(u_b, v_c, Q + w)$, $(u_c, v_d, Q + w)$.

Now, if there is a negative triangle, then the radius will be $9M$. If not, then the radius will be $\geq 9M$.

7 Connection to the Strong Exponential Time Hypothesis

The Strong Exponential Time Hypothesis (SETH) says that for any $\delta < 1$, SAT can't be solved in $O(2^{\delta n})$ time.

Roditty and Williams [?] found a lower bound of the complexity of DIAMETER given SETH.

Theorem 5
- *There is a probabalistic algorithm with expected runtime* $O(m\sqrt{n})$ *for a* 1.5*-approximation of DIAMETER.*
- *Assuming SETH, there is no* ϵ *such that there is an* $O(m^{2-\epsilon})$ *time* 1.5*-approximation algorithm for DIAMETER.*

8 Open Problems

Here are some open problems in this area:

- Are DIAM and APSP subcubic equivalent?
- Is it possible to get a lower bound for approximating RADIUS and MEDIAN similar to the one for DIAMETER?
- Does there exist a sub-cubic time algorithm for APSP?

9 Extra Related Problems

- Bringmann, Gawrychowski, Mozes, and Weimann [?] show the APSP-hardness of the problem of tree-edit distance. Tree edit distace is the problem of given two trees, how many changes must be made to get from one tree to the other in the shortest way?
- Boroujeni, Dehghani, Ehsani, HajiAghayi, and Seddighin showed that Radius, Median, and Betweeness Centrality are all subcubic-equivalent.
- Boroujeni et. al. Also showed that Reach Centrality is subcubic equivalent to Diameter.
- Chan and Williams [?] found an algorithm for APSP in deterministic $n^3/2^{\Omega(EQUATION)}$ time, matching the previous known randomized algorithm.
- Bringmann, Kunnemann, Wegrzycki found an efficient algorithm for approximating APSP [?]. They designed an approximation scheme which runs in
$$O(n\omega/\epsilon polylog(n/\epsilon))$$

References

[1] Bringmann, K., Gawrychowski, P., Mozes, S., & Weimann, O. (2020). Tree Edit Distance Cannot be Computed in Strongly Subcubic Time (Unless APSP Can). ACM Transactions on Algorithms, 16(4), 1–22. https://doi.org/10.1145/3381878

[2] Mahdi Boroujeni, Sina Dehghani, Soheil Ehsani, MohammadTaghi Haji-Aghayi, Saeed Seddighin Subcubic Equivalences Between Graph Centrality Measures and Complementary Problems SODA '15: Proceedings of the twenty-sixth annual ACM-SIAM symposium on Discrete algorithms

[3] Chan, T. M., & Williams, R. R. (2021). Deterministic APSP, Orthogonal Vectors, and More. ACM Transactions on Algorithms, 17(1), 1–14. https://doi.org/10.1145/3402926

[4] Bringmann, K., Kunnemann, M., & Wegrzycki, K. (2019, June 23). Approximating APSP without scaling: equivalence of approximate min-plus and exact min-max. Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing. STOC '19: 51st Annual ACM SIGACT Symposium on the Theory of Computing. https://doi.org/10.1145/3313276.3316373

[5] Erik Demaine, William Gasarch, Mohammad Hajiaghayi *Fun with Hardness: Algorithmmic Lower Bounds*. Not yet Published.

[6] Amir Abboud, Fabrizio Grandoni, and Virginia Vassilevska Williams. Subcubic equivalences between graph centrality problems, APSP and diameter. In Piotr Indyk, editor, *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2015, San Diego, CA, USA, January 4-6, 2015,* pages 1681–1697. SIAM, 2015. https://doi.org/10.1137/1.9781611973730.112.

[7] Mahdi Boroujeni, Sina Dehghani, Soheil Ehsani, Mohammad Taghi Hajiaghayi, and Saeed Seddighin. Subcubic equivalences between graph centrality measures and complementary problems, 2019. http://arxiv.org/abs/1905.08127.

[8] Michael Fredmann. New bounds on the complexity of the shortest path problem. *SIAM Journal on Computing*, 5(1):83–89, 1976. https://epubs.siam.org/doi/pdf/10.1137/0205006.

[9] Liam Roditty and Virginia Vassilevska Williams. Fast approximation algorithms for the diameter and radius of sparse graphs. In Dan Boneh, Tim Rough- garden, and Joan Feigenbaum, editors, *Symposium on Theory of Computing Conference, STOC'13, Palo Alto, CA, USA, June 1-4, 2013, pages 515–524. ACM, 2013.* https://doi.org/10.1145/2488608.2488673.

[10] R. Ryan Williams. Faster all-pairs shortest paths via circuit complexity. *SIAM Journal on Computing*, 47(5):1965–1985, 2018. https://doi.org/10.1137/15M1024524.

[11] Virginia Vassilevska Williams and R. Ryan Williams. Subcubic equivalences between path, matrix, and triangle problems. *Journal of the Association of Computing Machinery (JACM)*, 65(5):27:1–27:38, 2018. https://doi.org/10.1145/3186893.

[12] UUri Zwick. All pairs shortest paths in weighted directed graph 34 exact and almost exact algorithms. In *39th Annual Symposium on Foundations of Com- puter Science, FOCS '98, November 8-11, 1998, Palo Alto, California, USA,* pages 310–319. IEEE Computer Society, 1998. https://doi.org/10.1109/SFCS.1998.743464.

[13] Liam Roditty and Virginia Vassilevska Williams. Fast approximationalgorithms for the diameter and radius of sparse graphs. In Dan Boneh,Tim Roughgarden, and Joan Feigenbaum, editors, *Symposium on Theoryof Computing Conference, STOC'13, Palo Alto, CA, USA, June 1-4,2013*, pages 515–524. ACM, 2013.https://doi.org/10.1145/2488608.2488673.