CMSC858F Note Chapter 21: Polynomial Parity Arguments for Directed Graphs

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1 Summary

This chapter explores problems whose solutions are known to exist but may be difficult to find. This usually results from the solutions' existence being proven in a non-constructive way. The chapter starts by defining the *End of the Line* (EOL) problem as a motivational example. Moreover, it is worth remarking that it is unlikely for EOL to be NP-complete; it turns out that the same can be said for most of the relevant problems presented in this chapter.

The text next introduces Nash equilibriums. It gives examples through prisoner's dilemma (deterministic strategies) and penalty shots (randomized strategies). To wrap up the subsection, the known results are presented:

1. In general, Nash equilibriums can be irrational.
2. 2-player games with integer specifications have rational Nash equilibriums with poly-sized numerators and denominators.
3. All games have Nash equilibriums, though the difficulty of finding them range from impossible to most likely NP-intermediate depending on the specific setting.

The chapter then presents Brouwer's fixed point theorem. It is another problem that fits the theme of a solution that's known to exist but is difficult to find. Moreover, it is used to analyze Nash equilibriums described above. Finally, another problem with similar properties, Sperner's lemma is introduced in its 2-dimension special case.

The chapter then makes the connection between all the problems defined above. It defines the FNP and PPAD classes, and states that all four problems are PPAD-complete. The envy-free cake cutting problem is also briefly defined and stated to be PPAD-complete with certain parameter choices.

To prove some of the claims above, the textbook then gives the following reductions:

$$\mathsf{EOL} \leq \mathsf{3D-Sperner} \leq \mathsf{ARITHCIRCSAT} \leq \mathsf{POLYMATRIXNASH} \leq \mathsf{2-NASH}$$

I will omit the details of the proofs. They are in the textbook, and I do not see the point of reproducing them here.

The chapter next presents other related complexity classes in FNP:

- PPA (Polynomial Parity Argument): ODG (Odd Degree Node) and HAM − 3reg are complete problems. CHEVALLEY is in PPA; it is open whether it is PPA-complete.
- PLS (Polynomial Local Search): This is defined so that FS (Find Sink) and LMC (Local Max Cut) are PLS complete.
- PPP (Polynomial Pigeonhole Principle): COLLISION is complete. DS (Distinct Subsets) is in PPP; open problem if complete.

The textbook finally states the following inclusions:

$$\mathsf{FP} \subseteq \mathsf{PPAD} \subseteq \mathsf{PPA} \subseteq \mathsf{FNP}$$

$$\mathsf{FP} \subseteq \mathsf{PPAD} \subseteq \mathsf{PPP} \subseteq \mathsf{FNP}$$

Proving these inclusions are given as exerices in the textbook.

2 Comments and Suggestions

The chapter describes way more than 10 complexity problems already, and includes almost everything I can find. The only thing I can find that's not included yet is that *Arrow-Debreu equilibrium* is in PPA, according to Complexity Zoo.

I've taken undergrad-level economics and also have heard of Brouwer's fixed point theorem before, so I might not be the best person to say whether these subsections are accessible to beginners. I will give my thoughts regardless, as is required for the class assignment.

Presenting all 4 problems (EOL, Nash, Brouwser, Sperner's) before FNP and PPAD is a good presentation to me. This is however because I already knew what Nash equilibriums are, and I also have heard about Brouwser's fixed point theorem. For readers without background, I'm wondering if this creases a few unresolved threads before they all get tied together in FNP and PPAD. This strategy does motivate PPAD very well though.

I did not check the details of the proofs related to 2 − NASH, though the result is quite interesting. The related classes PPA, PLS, and PPP and the list of problems in them are very informative as well.

3 Typos, Potential Errors, and Points of Confusion

- On page 445, there is an awkward gap between the section headers 21.2 Game Theory and 21.2.1 Prisoner's dilemma. Maybe you wanted to give a smooth transition and forgot to write it?

- The last paragraph on page 452 is confusing. I have difficulty inferring the dimensions of x from the context, and nothing seems to line up with the special case n=2 of Sperner's presented. "Color each node according to the direction of $f(x) - x$ in one of three colors" does not seem to be a well-defined operation either.
- On page 465, Therem 21.8.6 talks about the "solution" of a system. We are however given a list of polynomials. Are we defining the system by setting all the polynomials to zero? Also, is "degree" well-defined here? Let's say we have a term $x_1^2 x_2^3 x_3^4$, is the degree here 4 or $2 + 3 + 4 = 9$?
- On page 466: Polynomial Local [s]earch does not have "search" capitalized. Is that intended?