

The Complexity of Grid Coloring

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1. $G_{n,m}$ is **c -colorable** if there is a c -coloring of $G_{n,m}$ such that no rectangle has all four corners the same color.
2. $\chi(G_{n,m})$ is the least c such that $G_{n,m}$ is c -colorable.

Examples

A FAILED 2-Coloring of $G_{4,4}$

R	B	B	R
B	R	R	B
B	B	R	R
R	R	R	B

A 2-Coloring of $G_{4,4}$

R	B	B	R
B	R	R	B
B	B	R	R
R	B	R	B

Example: a 3-Coloring of $G(10,10)$

Example: A 3-Coloring of $G_{10,10}$

R	R	R	R	B	B	G	G	B	G
R	B	B	G	R	R	R	G	G	B
G	R	B	G	R	B	B	R	R	G
G	B	R	B	B	R	G	R	G	R
R	B	G	G	G	B	G	B	R	R
G	R	B	B	G	G	R	B	B	R
B	G	R	B	G	B	R	G	R	B
B	B	G	R	R	G	B	G	B	R
G	G	G	R	B	R	B	B	R	B
B	G	B	R	B	G	R	R	G	G

It is known that **cannot** 2-color $G_{10,10}$. Hence $\chi(G_{10,10}) = 3$.

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$G_{n,m}$ is c -colorable iff no element of OBS_c is inside $G_{m,n}$.

2. We have a proof which shows $|OBS_c| \leq 2c^2$.
3. If OBS_c is known then the set of c -colorable grids is completely characterized.

OBS-2 and OBS-3 Known

We showed

$$OBS_2 = \{G_{3,7}, G_{5,5}, G_{7,3}\}$$

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$$OBS_3 = \{G_{4,19}, G_{5,16}, G_{7,13}, G_{10,11}, G_{11,10}, G_{13,7}, G_{16,5}, G_{19,4}\}$$

OBS-2 and OBS-3 Known

We showed

$$OBS_2 = \{G_{3,7}, G_{5,5}, G_{7,3}\}$$

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2-colorability table. *C* for Colorable, *U* for Uncolorability.

	2	3	4	5	6	7
2	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>
3	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>U</i>
4	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>U</i>
5	<i>C</i>	<i>C</i>	<i>C</i>	<i>U</i>	<i>U</i>	<i>U</i>
6	<i>C</i>	<i>C</i>	<i>C</i>	<i>U</i>	<i>U</i>	<i>U</i>
7	<i>C</i>	<i>U</i>	<i>U</i>	<i>U</i>	<i>U</i>	<i>U</i>

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3. 2009: Gasarch offered a prize of \$289.00 (17^2) to the first person to email him a 4-coloring of $G_{17,17}$.
4. Brian Hayes, Scientific American Math Editor, popularized the challenge.

Challenge Was Hard

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1. Lots of people worked on it.
2. For a while, no progress.
3. In 2012 Bernd Steinbach and Christian Posthoff [SP]. Clever SAT-solver designed for this purpose. Did not generalize.
4. They and others also found colorings that lead to $\text{OBS}_4 = \{$

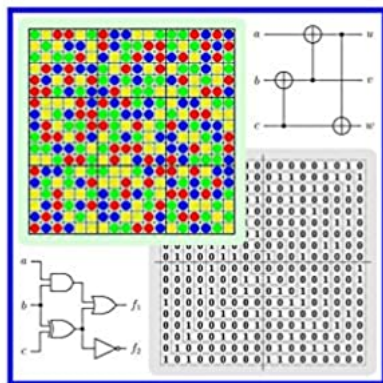
$G_{5,41}, G_{6,31}, G_{7,29}, G_{9,25}, G_{18,23}, G_{11,22}, G_{13,21}, G_{17,19},$
 $G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,18}, G_{22,11}, G_{21,13}, G_{19,17}$

$\}$

Coloring of $G_{18,18}$ on Book Cover!

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Is Grid Coloring Hard?

We view this two ways:

1. Is there an NP-complete problem lurking here somewhere?

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Part I of Talk—NP Completeness of GCE

**THERE IS AN NP-COMPLETE
PROBLEM LURKING!**

Grid Coloring Hard!-NP stuff

1. Let $c, N, M \in \mathbb{N}$. A partial mapping χ of $N \times M$ to $\{1, \dots, c\}$ is *extendable to a c -coloring* if there is an extension of χ to a total mapping which is a c -coloring of $N \times M$.

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1. Let $c, N, M \in \mathbb{N}$. A partial mapping χ of $N \times M$ to $\{1, \dots, c\}$ is *extendable to a c -coloring* if there is an extension of χ to a total mapping which is a c -coloring of $N \times M$.
- 2.

$$GCE = \{(N, M, c, \chi) \mid \chi \text{ is extendable}\}.$$

GCE is NP-complete!

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$\phi(x_1, \dots, x_n) = C_1 \wedge \dots \wedge C_m$ is a 3-CNF formula. We determine N, M, c and a partial c -coloring χ of $N \times M$ such that

$$\phi \in 3\text{-SAT} \text{ iff } (N, M, c, \chi) \in GCE$$

Forcing a Color to Only Appear Once in Main Grid

G								
G								
R		G						
G								
G								
G								
G	G	G	G	G	G	G	G	G

G can only appear once in the main grid, where it is, but what about **R**? (The double lines are not part of the construction. They are there to separate the main grid from the rest.)

Forcing a Color to Only Appear Once in Main Grid

R	G								
R	G								
R	R		G						
R	G								
R	G								
R	G								
R	G	G	G	G	G	G	G	G	G
R	G	R	R	R	R	R	R	R	R

G can only appear once in the main grid, where it is. **R** cannot appear anywhere in the main grid.

Using Variables

D means that the color is some *distinct*, unique color.

	D	D	D	D	D	D	D	D	D	D	D
\bar{x}_1		D	D	D	D	D	D	D	D	T	F
x_1		D	D	D	D	D	D	T	F	T	F
\bar{x}_1		D	D	D	D	T	F	T	F	D	D
x_1		D	D	T	F	T	F	D	D	D	D
\bar{x}_1		T	F	T	F	D	D	D	D	D	D
x_1		T	F	D	D	D	D	D	D	D	D

The labeled x_1, \bar{x}_1 are not part of the grid. They are visual aids.

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\bar{x}_1		D	D	D	D	D	D	D	D	T	F
x_1		D	D	D	D	D	D	T	F	T	F
\bar{x}_1		D	D	D	D	T	F	T	F	D	D
x_1		D	D	T	F	T	F	D	D	D	D
\bar{x}_1		T	F	T	F	D	D	D	D	D	D
x_1		T	F	D	D	D	D	D	D	D	D

The labeled x_1, \bar{x}_1 are not part of the grid. They are visual aids.
First col forced to be T-F-T-F-T-F or F-T-F-T-F-T

Coding a Clause

$C_1 = L_1 \vee L_2 \vee L_3$. Where L_1, L_2, L_3 are literals (vars or their negations).

	...	D	T	T
	⋮	⋮	⋮	⋮
L_1	...		D	F
	⋮	⋮	⋮	⋮
L_2	...			
	⋮	⋮	⋮	⋮
L_3	...		F	D
	⋮	⋮	⋮	⋮

The L_1, L_2, L_3 are not part of the grid. They are visual aids.

Coding a Clause—More Readable

$$C_1 = L_1 \vee L_2 \vee L_3.$$

	<i>D</i>	<i>T</i>	<i>T</i>
<i>L</i> ₁		<i>D</i>	<i>F</i>
<i>L</i> ₂			
<i>L</i> ₃		<i>F</i>	<i>D</i>

One can show that

- ▶ If put any of TTT, TTF, TFT, FTT, FFT, FTF, TFF in first column then can extend to full coloring.
- ▶ If put FFF in first column then cannot extend to a full coloring.

Example: (F,F,T)

$$C_1 = L_1 \vee L_2 \vee L_3.$$

	<i>D</i>	<i>T</i>	<i>T</i>
<i>L</i> ₁	<i>F</i>	<i>D</i>	<i>F</i>
<i>L</i> ₂	<i>F</i>		*
<i>L</i> ₃	<i>T</i>	<i>F</i>	<i>D</i>

The * is forced to be *T*.

Example: (F,F,T)

$$C_1 = L_1 \vee L_2 \vee L_3.$$

	<i>D</i>	<i>T</i>	<i>T</i>
<i>L</i> ₁	<i>F</i>	<i>D</i>	<i>F</i>
<i>L</i> ₂	<i>F</i>	*	<i>T</i>
<i>L</i> ₃	<i>T</i>	<i>F</i>	<i>D</i>

The * is forced to be *F*.

Example: (F,F,T)

$$C_1 = L_1 \vee L_2 \vee L_3.$$

	<i>D</i>	<i>T</i>	<i>T</i>
<i>L</i> ₁	<i>F</i>	<i>D</i>	<i>F</i>
<i>L</i> ₂	<i>F</i>	<i>F</i>	<i>T</i>
<i>L</i> ₃	<i>T</i>	<i>F</i>	<i>D</i>

Other Assignments

1. We did (F, F, T) .
2. (F, T, F) , (T, F, F) are similar.
3. (F, T, T) , (T, F, T) , (T, T, F) , (T, T, T) are easier.

Cannot Use (F,F,F)

$C_1 = L_1 \vee L_2 \vee L_3$. Want that (F, F, F) **cannot** be extended to a coloring.

	D	T	T
L_1	F	D	F
L_2	F	*	*
L_3	F	F	D

The *'s are forced to be T .

Cannot Use (F,F,F)

	<i>D</i>	T	T
<i>L</i> ₁	<i>F</i>	<i>D</i>	<i>F</i>
<i>L</i> ₂	<i>F</i>	T	T
<i>L</i> ₃	<i>F</i>	<i>F</i>	<i>D</i>

There is a mono rectangle of *T*'s. **Not** a valid coloring!

Put it all Together

Do the above for all variables and all clauses to obtain the result that GRID EXT is NP-complete!

Big Example

$$(x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee x_3 \vee x_4) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4)$$

												C_1	C_1	C_2	C_2	C_3	C_3
	D	D	D	D	D	D	D	D	D	D	D	T	T	T	T	T	T
\bar{x}_4		D	D	D	D	D	D	D	D	T	F	D	D	D	D	D	F
x_4		D	D	D	D	D	D	D	D	T	F	D	D	D	F	D	D
\bar{x}_3		D	D	D	D	D	D	T	F	D	D	D	D	D	D	D	D
x_3		D	D	D	D	T	F	T	F	D	D	D	D			D	D
\bar{x}_3		D	D	D	D	T	F	D	D	D	D	D	F	D	D		
\bar{x}_2		D	D	T	F	D	D	D	D	D	D	D	D	F	D	D	D
x_2		D	D	T	F	D	D	D	D	D	D			D	D	D	D
\bar{x}_1		T	F	D	D	D	D	D	D	D	D	D	D	D	D	F	D
x_1		T	F	D	D	D	D	D	D	D	D	F	D	D	D	D	D

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1. **Maybe Not** GCE is Fixed Parameter Tractable. For fixed c GCE_c is in time $O(N^2M^2 + 2^{O(c^4)})$. But for $c = 4$ this is huge!
2. **Maybe Yes** Gives a sense that brute force search is needed—not shortcuts.

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3. **Maybe Not** Our result says nothing about the case where the grid is originally all blank.

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2. **Maybe Yes** Gives a sense that brute force search is needed—not shortcuts.
3. **Maybe Not** Our result says nothing about the case where the grid is originally all blank.
4. **Maybe Yes** *GCE* problem should be easier than starting with all blanks.

Key to $O(N^2M^2 + 2^{O(c^4)})$ Result

Lemma Let χ be a partial c -coloring of $G_{n,m}$. Let U be the uncolored grid points. Let $|U| = u$. There is an algorithm that will determine if χ can be extended to a full c -coloring that runs in time $O(cnm2^{2u}) = 2^{O(nm)}$.

Set Up the Algorithm For $S \subseteq U$ and $1 \leq i \leq c$ let

$$f(S, i) = \begin{cases} \text{Yes} & \text{if } \chi \text{ can be extended to } S \text{ using colors } \{1, \dots, i\}; \\ \text{No} & \text{if not.} \end{cases}$$

For $S \subseteq U$ and $1 \leq i \leq c$ use Dynamic Programming to compute $f(S, i)$. $f(U, c)$ is your answer.

End of Set Up of Algorithm

Computing $f(S, i)$

Assume that $(\forall S', |S'| < |S|)(\forall 1 \leq i \leq c)[f(S', i)$ is known].

1. For all 1-colorable $T \subseteq S$ do the following
 - 1.1 If $f(S - T, i) = NO$ then $f(S, i) = NO$ and STOP.
 - 1.2 If $f(S - T, i - 1) = YES$ then determine if coloring T with i works. If yes then $f(S, i) = YES$ and STOP. Note that this takes $O(nm)$.
2. We know that for all 1-colorable $T \subseteq S$ $f(S - T, i) = YES$ and either
 - (1) $f(S - T, i - 1) = NO$ or
 - (2) $f(S - T, i - 1) = YES$ and coloring T with i bad.In all cases $f(S, i) = NO$.

Open Questions

1. Improve Fixed Parameter Tractable algorithm.
2. NPC results for mono squares? Other shapes?
3. Show that

$$\{(n, m, c) : G_{n,m} \text{ is } c\text{-colorable}\}$$

is hard.

- ▶ If n, m in unary then sparse set, not NPC—New framework for hardness needed.
- ▶ If n, m binary then not in NP. Could try to prove NEXP-complete. But the difficulty of the problem is not with the grid being large, but with the number-of-possibilities being large.

Part II of Talk—Lower Bounds on Tree Resolution

**YOU SAY YOU WANT A
RESOLUTION!**

Part II of Talk—Lower Bounds on Tree Resolution

YOU SAY YOU WANT A RESOLUTION!

If you write a good parody of the Beatles **You say you want a revolution** about resolution theorem proving, I will treat you to lunch at **The Food Factory**, if they open up again.

Resolution

Def Let $\varphi = C_1 \wedge \cdots \wedge C_L$ be a CNF formula.

A **Resolution Proof** of $\varphi \notin \text{SAT}$ is a sequence of clauses such that on each line you have either

1. One of the C 's in φ (called an **Axiom**).
2. $A \vee B$ if $A \vee x$ and $B \vee \neg x$ were on prior lines. Variable that is **resolved on** is x .
3. The last line has the empty clause.

Example

$$\varphi = x_1 \wedge x_2 \wedge (\neg x_1 \vee \neg x_2)$$

1. x_1 **Axiom**
2. $\neg x_1 \vee \neg x_2$ **AXIOM**
3. $\neg x_2$ (From lines 1,2, resolve on x_1 .)
4. x_2 **Axiom**
5. \emptyset (From lines 3,4, resolve on x_2 .)

Resolution is Complete

Def Let $\varphi = C_1 \wedge \dots \wedge C_L$ be a CNF formula on n variables.

1. If exists a Res Proof of $\varphi \notin SAT$ then $\varphi \notin SAT$.

Proof Any assignment that satisfies φ satisfies any node of the Res Proof including the last node \emptyset .

2. If $\varphi \notin SAT$ then exists a Res Proof of $\varphi \notin SAT$ of size $2^{O(n)}$.

Proof Form a Decision Tree that has at every node on level i the variable x_i . Right= T and Left= F . A leaf is the first clause that is false with that assignment. **Turn Decision Tree upside down! View nodes as which var to resolve on! This will be Res Proof!** (It will even be Tree Res Proof.)

Another Example

The **and** of the following:

1. For $i, j \in \{1, \dots, 5\}$

$$x_{ij1} \vee x_{ij2}.$$

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2. For $i, j, i', j' \in \{1, \dots, 5\}$

$$\neg x_{ij1} \vee \neg x_{i'j'1} \vee \neg x_{ij'1} \vee \neg x_{i'j1}$$

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3. For $i, j, i', j' \in \{1, \dots, 5\}$

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Interpretation There is no mono 2-rectangle.

We interpret this statement as saying

There is a 2-coloring of $G_{5,5}$.

This statement is known to be false.

GRID(n, m, c)

Def Let $n, m, c \in \mathbb{N}$. GRID(n, m, c) is the **and** of the following:

1. For $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$,

$$x_{ij1} \vee x_{ij2} \vee \dots \vee x_{ijc}$$

Interpretation (i, j) is colored either 1 or \dots or c .

2. For $i, i' \in \{1, \dots, n\}$, $j, j' \in \{1, \dots, m\}$, $k \in \{1, \dots, c\}$,

$$\neg x_{ijk} \vee \neg x_{i'jk} \vee \neg x_{ij'k} \vee \neg x_{i'j'k}$$

Interpretation There is no mono rectangle.

We interpret this statement as saying

There is a c -coloring of $G_{n,m}$.

Note GRID(n, m, c) has nmc **VARS** and $O(cn^2m^2)$ **CLAUSES**.

GRID(n, m, c)—How to View Assignments

Given an assignment:

1. For all $i \in [n]$ and $j \in [m]$ let k be the **least** number such that $x_{ijk} = T$. View this as saying that (i, j) is colored k .
2. If there is NO such number then (i, j) is not colored and this assignment makes GRID(n, m, c) **false**.

Hence we view assignments as attempted colorings of the grid where some points are not colored.

Two Ways to Invalidate $\text{GRID}(n, m, c)$

1. There is a mono rectangle.
2. There is some point that is not colored: there is some i, j such that all x_{ijk} are **false**.

Tree Resolution Proofs

Def A **Tree Res Proof** is a Res Proof where the underlying graph is a tree. Note that if you remove the bottom node that is labeled \emptyset then the Tree Res Proof is cut into two **disjoint** parts.

Known If $\varphi \notin \text{SAT}$ and φ has v variables then there is a Tree Res Proof of φ of size $2^{O(v)}$.

Our Goal

Assume that there is no c -coloring of $G_{n,m}$.

1. GRID(n, m, c) has a size $2^{O(cnm)}$ Tree Res Proof.
2. We show $2^{\Omega(c)}$ size is **required**. This is our point!
3. The lower bound is **independent** of n, m .

Interesting Examples

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1. We showed that $G_{2c^2-c, 2c}$ is not c -colorable. Hence

$$\text{GRID}(2c^2 - c, 2c)$$

has $O(c^3)$ vars, $O(c^6)$ clauses but $2^{\Omega(c)}$ Tree Res proof.

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1. We showed that $G_{2c^2-c, 2c}$ is not c -colorable. Hence

$$\text{GRID}(2c^2 - c, 2c)$$

has $O(c^3)$ vars, $O(c^6)$ clauses but $2^{\Omega(c)}$ Tree Res proof.

2. Easy to show G_{c^3, c^3} is not c -colorable.

$$\text{GRID}(c^3, c^3, c)$$

has $O(c^7)$ vars, $O(c^{13})$ clauses and $2^{\Omega(c)}$ Tree Res proof.

These are poly-in- c formulas that **require** $2^{\Omega(c)}$ Tree Res proofs.

The Prover-Delayer Game

(Due to Pudlak and Impagliazzo [PI].) Parameters of the game:

$$p \in \mathbb{R}^+,$$

$$\varphi = C_1 \wedge \cdots \wedge C_L \notin \text{SAT}.$$

Do the following until a clause is proven false:

1. **PROVER** picks a variable x that was not already picked.
2. **DEL** either
 - 2.1 Sets x to F or T , OR
 - 2.2 Defers to **PROVER** who then sets x to T or F while **DEL** gets a point.

At end if **DEL** has $\geq p$ pts then he **wins**; else **PROVER wins**.

Convention

We assume that **PROVER** and **DEL** play perfectly.

1. **PROVER** *wins* means **PROVER** *has a winning strategy*.
2. **DEL** *wins* means **DEL** *has a winning strategy*.

Prover-Delayer Game and Tree Res Proofs

Lemma Let $p \in \mathbb{R}^+$, $\varphi \notin \text{SAT}$. If φ has a Tree Res Proof of size $< 2^p$ then **PROVER** wins.

Prover-Delayer Game and Tree Res Proofs

Lemma Let $p \in \mathbb{R}^+$, $\varphi \notin \text{SAT}$. If φ has a Tree Res Proof of size $< 2^p$ then **PROVER** wins. **Pf PROVER** Strategy:

1. Initially T is res tree of size $< 2^p$ and **DEL** has 0 pts.
2. **PROVER** picks x , the **last** var **resolved on**.
3. If **DEL** sets x then **DEL** gets no pts.
4. If **DEL** defers then **PROVER** sets T or F —**whichever yields a smaller tree**. **Note** One of the trees will be of size $< 2^{p-1}$. **DEL** gets 1 point.
5. Repeat: after i th stage will always have T of size $< 2^{p-i}$, and **DEL** has $\leq i$ pts.

Contrapositive is Awesome!

Recall:

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Contrapositive

Lemma Let $p \in \mathbb{R}^+$, $\varphi \notin \text{SAT}$. If **DEL** wins then **EVERY** Tree Res Proof for φ has size $\geq 2^p$. **Plan** Get **awesome** strategy for **DEL** when $\varphi = \text{GRID}(n, m, c)$.

GRID(n, m, c) Requires Exp Tree Res Proofs

Thm Let n, m, c be such that $G_{n,m}$ is not c -colorable. Let $c \geq 2$. Any tree resolution proof of $\text{GRID}(n, m, c) \notin \text{SAT}$ requires size $2^{0.5c}$.

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Pf Parameters: $p = 0.5c, \varphi = \text{GRID}(n, m, c)$.

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1. If setting $x_{ijk} = T$ creates a mono rect (of color k) then **DEL** **does not** let this happen— he sets x_{ijk} to F .

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Del Strategy

Assume x_{ijk} was chosen by the **PROVER**.

1. If setting $x_{ijk} = T$ creates a mono rect (of color k) then **DEL** **does not** let this happen— he sets x_{ijk} to F .
2. If none of the x_{ij*} are T and $\geq \frac{c}{2}$ of the x_{ij*} are F via **PROVER** then **DEL** sets x_{ijk} to T .
3. In all other cases the **DEL** defers to the **PROVER**.

Case 1: Prover Set $c/2$ Vars to F

Game ends when there is some i, j such that

$$x_{ij1} = x_{ij2} = \dots = x_{ijc} = F.$$

Who set those variables to F ?

Case 1 At least $\frac{c}{2}$ set F by Prover. Then **DEL** gets at least

0.5c pts.

Case 2: Del Set $c/2$ Vars to F

$x_{ij1} = x_{ij2} = \dots = x_{ijc} = F$. Who set those vars to F ?

Case 2 At least $\frac{c}{2}$ set F by **DEL**. Assume they are $x_{ij1}, x_{ij2}, \dots, x_{ijc/2}$.

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$x_{ij1}, x_{ij2}, \dots, x_{ijc/2}$.

- ▶ x_{ij1} set to F by **DEL**. Why? There exists i', j' such that $x_{i'j'1}, x_{ij'1}, x_{i'j'1}$ all set to T . (Do not know by who.)

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- ▶ x_{ij1} set to F by **DEL**. Why? There exists i', j' such that $x_{i'j'1}, x_{ij'1}, x_{i'j'1}$ all set to T . (Do not know by who.)
- ▶ x_{ij2} set to F by **DEL**. Why? There exists i'', j'' such that $x_{i''j''2}, x_{ij''2}, x_{i''j''2}$ all set to T . (Do not know by who.)

Case 2: Del Set $c/2$ Vars to F

$x_{ij1} = x_{ij2} = \dots = x_{ijc} = F$. Who set those vars to F ?

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- ▶ x_{ij1} set to F by **DEL**. Why? There exists i', j' such that $x_{i'j'1}, x_{ij'1}, x_{i'j'1}$ all set to T . (Do not know by who.)
- ▶ x_{ij2} set to F by **DEL**. Why? There exists i'', j'' such that $x_{i''j''2}, x_{ij''2}, x_{i''j''2}$ all set to T . (Do not know by who.)
- ▶ etc.

For every k such that x_{ijk} is set to F by **DEL** there exists **three** vars of form x_{**k} set to T .

Key All these 3-sets are **disjoint**, so at least $3c/2$ vars set T (by who?).

Case 2a: Prover Set $3c/2$ Vars to T

Key At least $3c/2$ vars set T (by who?).

Case 2a PROVER set $\geq \frac{3c}{4}$ to T . **DEL** gets at least

$$0.75c = 0.75c \text{ pts.}$$

Case 2b: Del Set $3c/2$ Vars To T

Case 2b DEL set $\geq \frac{3c}{4}$ to T .

DEL set x_{ijk} to T :

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Case 2b DEL set $\geq \frac{3c}{4}$ to T .

DEL set x_{ijk} to T :

- ▶ At time there are $c/2$ k' such that **PROVER** set $x_{ijk'}$ to F .

Case 2b: Del Set $3c/2$ Vars To T

Case 2b DEL set $\geq \frac{3c}{4}$ to T .

DEL set x_{ijk} to T :

- ▶ At time there are $c/2$ k' such that **PROVER** set $x_{ijk'}$ to F .
- ▶ **DEL** will **never** set an x_{ij*} to T again! **never!!**

Every x_{ijk} set T by **DEL** implies that $c/2$ vars set F by **PROVER**, and these sets of $c/2$ vars are disjoint.

Upshot PROVER had set $\frac{3c}{4} \times \frac{c}{2}$ to F . **DEL** gets at least

$$0.375c^2 = 0.375c^2 \text{ pts.}$$

Final Analysis

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- ▶ **Case 1 DEL** gets at least $0.5c$ pts.

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- ▶ **Case 1 DEL** gets at least $0.5c$ pts.
- ▶ **Case 2a DEL** gets at least $0.75c$ pts.
- ▶ **Case 2b DEL** gets at least $0.375c^2$ pts.

Upshot For $c \geq 2$ **DEL** gets at least $0.5c$ pts.

Punchline By **Lemma** any Tree Res Proof has size $\geq 2^{0.5c}$.

Optimize

1. In construction use cutoff of $c/2$ for when **DEL** sets x_{ijk} to T . Choose fraction **carefully**.
2. In analysis we twice do a half-half cutoff. Choose fractions **carefully**!
3. Use asymmetric **PROVER-DEL** game (next slide) and choose a, b **carefully**!

Thm Let n, m, c be such that $G_{n,m}$ is not c -colorable. Let $c \geq 9288$. Any tree resolution proof of $\text{GRID}(n, m, c) \notin \text{SAT}$ requires size $2^{0.836c}$.

Asymmetric Prover-Delayer Game

(Due to Beyersdorr, Galesi, Lauria [BGL].) Parameters of the game: $a, b \in (1, \infty)$, with $\frac{1}{a} + \frac{1}{b} = 1$, $p \in \mathbb{R}^+$,

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 - 2.2.2 If **PROVER** sets $x = T$ then **DEL** gets $\lg b$ pts.

At end if **DEL** has $\geq p$ pts then he **wins**; else **PROVER wins**.

Other Shapes

What is special about rectangles? **Nothing!** **Def** (Informally) Let S be a set of at least 2 grid points. Let $\text{GRID}(n, m, c; S)$ be the prop statement that there is a c -coloring of $G_{n,m}$ with no mono configuration that is “like S ”.

Thm (Informally) Let S be a set of at least 2 grid points. Let n, m, c be such that $\text{GRID}(n, m, c; S) \notin \text{SAT}$. Any tree resolution proof of $\text{GRID}(n, m, c; S) \notin \text{SAT}$ requires size $2^{\Omega(c)}$.

Open Questions

1. Want matching upper bounds for Tree Res Proofs of $\text{GRID}(n, m, c) \notin \text{SAT}$.
2. Want lower bounds on Gen Res Proofs of $\text{GRID}(n, m, c) \notin \text{SAT}$.
3. Want lower bounds on in other proof systems $\text{GRID}(n, m, c) \notin \text{SAT}$.
4. Upper and lower bounds for $\text{GRID}(n, m, c; S)$ for various S in various proof systems.

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