lines is not acute; we may take this angle α to be at a vertex (h,k) in the first quadrant. Then $\alpha \geqslant \pi/2$ implies that $h+k \leqslant 2$, and this in turn implies that the area of C in the first quadrant does not exceed 1. Hence $A(C) \leqslant 4$.

B-6. (22, 4, 3, 0, 0, 0, 1, 0, 10, 4, 42, 119)

Let $X=(x_1,\ldots,x_n)$ and $Y=(y_1,\ldots,y_n)$. Also let $a+bi$ be either square root of $z_1^2+\cdots+z_n^2$. Then $ab=X\cdot Y=x_1y_1+\cdots+x_ny_n$ and

$$a^2-b^2=\|X\|^2-\|Y\|^2=(x_1^2+\cdots+x_n^2)-(y_1^2+\cdots+y_n^2).$$

The Cauchy–Schwarz inequality tells us that $|X\cdot Y| \leqslant \|X\|\cdot\|Y\|$ and hence $|a|\cdot|b| \leqslant \|X\|\cdot\|Y\|$. Therefore, the assumption that $|a|>\|X\|$ would imply that $|b|<\|Y\|$. This and $a^2=\|X\|^2-\|Y\|^2+b^2$ would yield $a^2<\|X\|$ and thus the contradiction $|a|<\|X\|$. Hence the assumption is false and $r=|a| \leqslant \|X\|$. Since $\|X\|^2 \leqslant (|x_1|+\cdots+|x_n|)^2$, this implies the desired $r \leqslant |x_1|+\cdots+|x_n|$.

MATHEMATICAL NOTES

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HOW TO CUT A CAKE FAIRLY

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In this note we prove that a cake can be divided fairly among n people, although each may have a different opinion as to which parts of the cake are most valuable. It can be done even if "fair" means that all people must receive their first choices!

In a simpler version of the problem, a division is regarded as "fair" if all people ("players") are satisfied that each has received at least $1/n$ of the cake. For this version, there is a simple and practical solution, attributed by Steinhaus [**1**] to Banach and Knaster. Martin Gardner describes the case $n=3$ in his newest book [**2**]:

> "One person moves a large knife slowly over a cake. The cake may be any shape, but the knife must move so that the amount of cake on one side continuously increases from zero to the maximum amount. As soon as any one of the three believes that the knife is in a position to cut a first slice equal to 1/3 of the cake, he/she shouts 'Cut!' The cut is made at that instant, and the person who shouted gets the piece. Since he/she is satisfied that he/she got 1/3, he/she drops out of the cutting ritual. In case two or all three shout 'Cut!' simultaneously, the piece is given to any one of them.
>
> "The remaining two persons are, of course, satisfied that at least 2/3 of the cake remain. The problem is thus reduced to the previous case . . .
>
> "This clearly generalizes to n persons."

Gardner then describes the more difficult version of the problem, in which a division is regarded as "fair" only if all players consider their own pieces to be at least as valuable as any of the others—essentially, all players get their first choices. The procedure described above doesn't always meet this test, because the player who claims the first piece may have a change of mind after seeing the remaining pieces. When $n=3$, we propose a new procedure to meet this objection:

A referee moves a sword from left to right over the cake, hypothetically dividing it into a small left piece and a large right piece. Each player holds a knife over what he considers to be the midpoint of the right piece. As the referee moves his sword, the players continually adjust their knives, always keeping them parallel to the sword (see Fig. 1). When any player shouts "cut," the cake is cut by the sword and by whichever of the players' knives happens to be the middle one of the three.

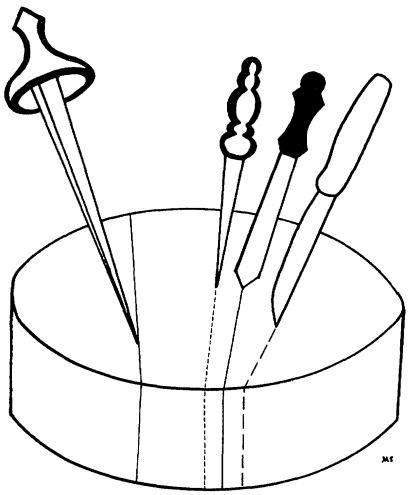

FIG. 1.

The player who shouted "cut" receives the left piece. He must be satisfied, because he knew what all three pieces would be when he said the word. Then the player whose knife ended nearest to the sword, if he didn't shout "cut," takes the center piece; and the player whose knife was farthest from the sword, if he didn't shout "cut," takes the right piece. The player whose knife was used to cut the cake, if he hasn't already taken the left piece, will be satisfied with whichever piece is left over. If ties must be broken—either because two or three players shout simultaneously or because two or three knives coincide—they may be broken arbitrarily.

This procedure does not generalize to larger n. John Selfridge, John Conway, and Richard Guy, in their research on the fair division of wine, have discovered a more elegant algorithm for $n=3$, but it, too, fails to generalize. In this note we shall be content with a nonconstructive existence theorem valid for all n.

One existence theorem, operating on quite different principles, has already appeared in this MONTHLY [3]. Dubins and Spanier (in an article with the same title as this note) assumed that each player's preferences are defined by a nonatomic measure over the cake. They proved that, given any finite number of measures (including those of the players and those of the kibitzers as well), there is a partition of the cake into n parts that are equal according to all of the measures. This was one of several results illustrating the power of Lyapunov's Theorem and other measure-theoretic techniques. Unfortunately, their result depends on a liberal definition of a "piece" of cake, in which the possible pieces form an entire σ-algebra of subsets. A player who hopes only for a modest interval of cake may be presented instead with a countable union of crumbs.

In this note we shall adhere more closely to the original model by imposing a rigid structure on the ways in which the cake may be cut. In particular, we shall insist that it be cut by $(n-1)$ planes, each parallel to a given plane. The possible cuts can then be represented by numbers in the interval $[0,1]$; and the possible divisions of the cake, by vectors $x=(x_1,x_2,\ldots,x_{n-1})$ such that $0\leqslant x_1\leqslant x_2\leqslant\cdots\leqslant x_{n-1}\leqslant 1$. By convention let $x_0=0$ and $x_n=1$, so that the ith piece is the interval $[x_{i-1},x_i]$. The possible divisions form a compact set in $\mathbb{R}^{n-1}$, which we call the *division simplex*,

$$S=\{(x_1,\ldots,x_{n-1})|0\leqslant x_1\leqslant\cdots\leqslant x_{n-1}\leqslant 1\}.$$

See Fig. 2. S has the shape of an $(n-1)$-simplex with vertices at $v_1=(1,1,\ldots,1), v_2=(0,1,\ldots,1),\ldots,v_n=(0,0,\ldots,0)$. The vertex v_i represents the division in which the ith piece is the whole cake, and the face opposite v_i, which we shall call S_i, consists of divisions for which the ith piece is empty.

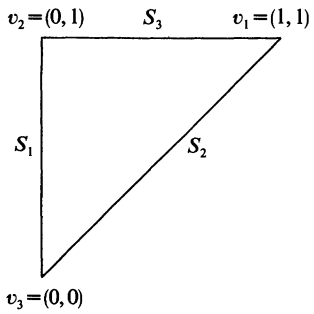

FIG. 2. When $n=3, S \subseteq R^2$.

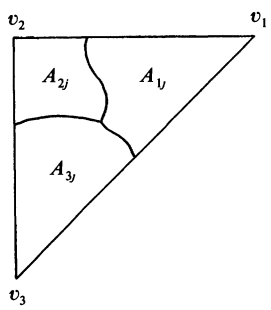

FIG. 3. Preferences of the jth player.

We allow great generality in the player's preferences. We assume that the choice for the jth player is based on a real-valued evaluation function f_j, which gives the value of the ith piece in terms of $x_1,\ldots,x_{n-1}$ (and i). Thus the value of the ith piece to the jth player is denoted by $f_j(x,i)$. Intuitively, one expects $f_j(x,i)$ to depend only on x_{i-1} and x_i, but the added generality comes at no extra cost.

For a given x, we say that player j *prefers* the ith piece if $f_j(x,i) \geq f_j(x,k)$ for all k. For some divisions a player may be indifferent to two or more "preferred" pieces. The division is fair if each player can be given a preferred piece.

We assume that each f_j is a continuous function of x. We must also assume that no player ever prefers an empty piece of cake.

THEOREM. *Under these assumptions, there is a division x and a way to assign the pieces to the players such that all players prefer their assigned pieces.*

Proof. For each i,j, let A_{ij} be the set of divisions $x \in S$ for which the jth player prefers the ith piece. From the continuity of the functions f_j we know that each A_{ij} is closed. For each j, the sets A_{ij} cover S. The assumption that no player prefers an empty piece implies that A_{ij} has empty intersection with the face S_i for each i,j. The sets A_{ij} provide all the information we need about the players' preferences, so we shall not refer again to the functions f_j. See Fig. 3.

Define $B_{ij}=\cap_{k\neq i}(S-A_{kj})$. Thus B_{ij} is the set of divisions for which the jth player prefers *only* the ith piece. Typically B_{ij} is the interior of A_{ij}, but that is not necessary. Each B_{ij} is open (relative to S). Note that, for a given j, the sets B_{ij} do *not* cover S; the uncovered part $(S-\cup_i B_{ij})$ consists of divisions for which the jth player is indifferent to two or more acceptable pieces.

Now define $U_i=\cup_j B_{ij}$. Thus U_i is the set of divisions for which *some* player prefers *only* the ith piece. Note that each U_i is open (as always, relative to S) and that U_i does not intersect S_i. We now divide the proof into two cases.

Usual Case: *The sets U_i cover S.* In this case we rely on a topological lemma.

LEMMA. *Suppose an $(n-1)$-simplex S is covered by n open sets $U_1,\ldots,U_n$, such that no U_i intersects the corresponding face S_i. Then the common intersection of $U_1,\ldots,U_n$ is nonempty.*

To see how this lemma proves the theorem (in the usual case), choose a division x in the common intersection of the U_i's. Since $x \in U_i$ for each i, every piece will be the unique acceptable piece for some player. Since there are exactly enough pieces to go around, all players can take their own first-choice pieces.

Proof of Lemma. We continue to regard S as a subset of the vector space $\mathbb{R}^{n-1}$, and use v_i and S_i as before. Write ∂S for the boundary of S. For each i, and for each $x \in S$, let $d_i(x)$ be the distance from x to the closed set $(S - U_i)$, and define $D(x) = \Sigma_i d_i(x)$. Since $x \in U_i$ for some i, some $d_i(x)$ is positive and so is $D(x)$. We may therefore define $f: S \to S$ by

$$f(x) = \sum_i \frac{d_i(x)}{D(x)} v_i.$$

The restriction of f to ∂S is a function $f_0: \partial S \to \partial S$ that takes each face S_i to itself. Hence we may define maps $f_t: \partial S \to \partial S$ by $f_t(x) = tx + (1-t)f(x)$. These maps define a homotopy between f_0 and the identity on ∂S. Therefore f_0 cannot be extended to a map from S to ∂S. In particular, since f is an extension of f_0, its image must intersect the interior of S.

Finally, if x is any point in f^{-1} (int S), then x is in each U_i.

Unusual Case: *The sets U_i do not cover S.* This case is unusual because it depends on a coincidence: if x is not in any U_i, then it is not in any B_{ij} for any j, so it must leave *every* player indifferent to two or more acceptable pieces. But this is not impossible. For example, if all players have identical preferences, the "coincidence" is certain to occur.

Our strategy in this case is to modify the players' preferences. We will approximate the sets A_{ij} by sets A'_{ij}, which are more orderly and for which the "coincidence" does not occur. By applying the lemma we shall find a division that would be fair, if the players' preferences were described by the sets A'_{ij}. As the approximations improve, these approximate solutions will converge to a division that is fair according to the actual preferences.

We start by choosing irrational numbers $\alpha_1, \ldots, \alpha_n$, one for each player, that are linearly independent over the rationals. We say that a number is *related* to α_j if its difference from α_j is rational. Numbers related to α_j are dense in $\mathbb{R}$, but no number can be related to both α_j and α_k if $j \neq k$.

Let M be a (large) integer.

For each j, construct A'_{ij} as follows. Divide S into cells by all planes of the form $\{x | x_k = (L/M) + \alpha_j\}$ for $k = 1, \ldots, n$ and for all integers L. A cell, together with its boundary, is part of A'_{ij} if i is the smallest subscript for which A_{ij} intersects the cell. See Fig. 4.

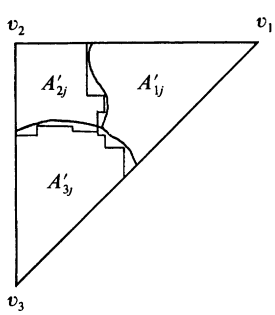

FIG. 4. Approximating the preferences.

The important properties of A'_{ij} are (1) every point on the boundary of A'_{ij} has some coordinate related to α_j, and (2) A'_{ij} approximates A_{ij} in the sense that every point A'_{ij} is within $\sqrt{n}/M$ of some point of A_{ij}.

Now for each i,j, define $B'_{ij} = \cap_{k \neq i}(S - A'_{kj})$—this *is* equal to the interior of A'_{ij}—and define $U'_i = \cup_j B'_{ij}$. As before, the sets U'_i are open and (if M has been chosen large enough) U'_i does not intersect S_i. To prove that the sets U'_i cover S, note that if x is not in any U'_i, it must not be in any B'_{ij}, so it must be on the boundary of some A'_{ij} for every j. That means that x must have a coordinate related to each of the α_j's. But this is impossible, because x has only $(n-1)$ coordinates, and each may be related to only one α_j.

Therefore, we may apply the lemma to find a point in the common intersection of the sets U_i'. We call it x_M. If the cake is divided according to x_M, the pieces can be assigned to the players in such a way that if the jth player receives the ith piece, then x_M is contained in A_{ij}' and is within $\sqrt{n}/M$ of A_{ij}.

As M increases, we can generate a sequence of divisions $\{x_M\}$. Since S is compact, we can find a subsequence that converges to some division $x \in S$; and by reducing to another subsequence if necessary, we can guarantee that the assignment of pieces to players is the same for each division x_M in the subsequence. Cut the cake according to x and assign the pieces to the players as for these x_M. Then if the jth player receives the ith piece, x must be arbitrarily close to A_{ij}. Since A_{ij} is closed, this implies that $x \in A_{ij}$, and the jth player prefers the assigned piece.

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NAPOLEON'S THEOREM AND THE PARALLELOGRAM INEQUALITY FOR AFFINE-REGULAR POLYGONS

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A well-known theorem, first proved in [2] but credited to Napoleon, reads as follows:

Construct equilateral triangles outwardly on the sides of any triangle. Their centers form the vertices of an equilateral triangle.

A lesser-known theorem of Thébault [11] (which is easily obtained as a corollary of Van Aubel's theorem [6],[1]) states:

Construct squares outwardly on the sides of any parallelogram. Their centers form the vertices of a square.

Clearly these theorems are related, and we may conjecture that they are part of a sequence of theorems leading from some m-gons to regular m-gons. In what sense, however, is a parallelogram, rather than the general quadrilateral, the successor of an arbitrary triangle? An answer to this question is provided by the following observations:

(a) Any triangle is the image of an equilateral triangle under an affine transformation.

(b) A quadrilateral is a parallelogram if and only if it is the affine image of a square.

These suggest the following result, which, in spite of its simplicity, appears to be new for $m \geq 5$, since even Thébault's result is not mentioned in survey articles [1], [7], [8], and [9]. Throughout the paper all subscripts will be taken modulo m.

THEOREM 1. *Let $\mathcal{P} = P_0P_1 \cdots P_{m-1}$ be a simple plane m-gon and construct regular m-gons on the sides of $\mathcal{P}$, one set outwardly and one set inwardly. Their centers form the vertices of m-gons $\mathcal{Q}$ and $\mathcal{Q}'$, and the centroids of $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{Q}'$ coincide. If $\mathcal{P}$ is the affine image of a regular m-gon, then*:

1. *The m-gons $\mathcal{Q}$ and $\mathcal{Q}'$ are regular.*
2. *The difference of the areas of $\mathcal{Q}$ and $\mathcal{Q}'$ is $4\cos^2(\pi/m)$ times the area of $\mathcal{P}$.*
3. *The sum of the squares of the edges of $\mathcal{Q}$ and $\mathcal{Q}'$ is $4\cos^2(\pi/m)$ times the sum of the squares of the edges of $\mathcal{P}$.*

Conversely, if $\mathcal{Q}$ (or $\mathcal{Q}'$) is regular, then $\mathcal{P}$ is affine-regular.