SUCCINCTNESS OF THE COMPLEMENT AND INTERSECTION OF REGULAR EXPRESSIONS

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Abstract. We study the succinctness of the complement and intersection of regular expressions. In particular, we show that when constructing a regular expression defining the complement of a given regular expression, a double exponential size increase cannot be avoided. Similarly, when constructing a regular expression defining the intersection of a fixed and an arbitrary number of regular expressions, an exponential and double exponential size increase, respectively, can in worst-case not be avoided. All mentioned lower bounds improve the existing ones by one exponential and are tight in the sense that the target expression can be constructed in the corresponding time class, i.e., exponential or double exponential time. As a by-product, we generalize a theorem by Ehrenfeucht and Zeiger stating that there is a class of DFAs which are exponentially more succinct than regular expressions, to a fixed four-letter alphabet. When the given regular expressions are one-unambiguous, as for instance required by the XML Schema specification, the complement can be computed in polynomial time whereas the bounds concerning intersection continue to hold. For the subclass of single-occurrence regular expressions, we prove a tight exponential lower bound for intersection.

1. Introduction

The two central questions addressed in this paper are the following. Given regular expressions $r, r_1, \ldots, r_k$ over an alphabet Σ,

(1) what is the complexity of constructing a regular expression $r_\neg$ defining $\Sigma^* \setminus L(r)$, that is, the complement of r?
(2) what is the complexity of constructing a regular expression $r_\cap$ defining $L(r_1) \cap \cdots \cap L(r_k)$?

In both cases, the naive algorithm takes time double exponential in the size of the input. Indeed, for the complement, transform r to an NFA and determinize it (first exponential step), complement it and translate back to a regular expression (second exponential step). For the intersection there is a similar algorithm through a translation to NFAs, taking the crossproduct and a retranslation to a regular expression. Note that both algorithms do not only take double exponential time but also result in a regular expression of double

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exponential size. In this paper, we exhibit classes of regular expressions for which this double exponential size increase cannot be avoided. Furthermore, when the number k of regular expressions is fixed, $r_\cap$ can be constructed in exponential time and we prove a matching lower bound for the size increase. In addition, we consider the fragments of one-unambiguous and single-occurrence regular expressions relevant to XML schema languages [2, 3, 13, 23]. Our main results are summarized in Table 1.

The main technical part of the paper is centered around the generalization of a result by Ehrenfeucht and Zeiger [8]. They exhibit a class of languages $(Z_n)_{n\in\mathbb{N}}$ each of which can be accepted by a DFA of size $\mathcal{O}(n^2)$ but cannot be defined by a regular expression of size smaller than 2^{n-1}. The most direct way to define Z_n is by the DFA that accepts it: the DFA is a graph consisting of n states, labeled 0 to $n-1$, which are fully connected and the edge between state i and j carries the label $a_{i,j}$. It now accepts all paths in the graph, that is, all strings of the form $a_{i_0,i_1}a_{i_1,i_2}\cdots a_{i_k,i_{k+1}}$. Note that the alphabet over which Z_n is defined grows quadratically with n. We generalize their result to a four-letter alphabet. In particular, we define K_n as the binary encoding of Z_n using a suitable encoding for $a_{i,j}$ and prove that every regular expression defining K_n should be at least of size 2^n. As integers are encoded in binary the complement and intersection of regular expressions can now be used to separately encode K_{2^n} (and slight variations thereof) leading to the desired results. In [9] the same generalization as obtained here is attributed to Waizenegger [35]. Unfortunately, we believe that proof to be incorrect as we discuss in Remark A.1 in the Appendix.

Although the succinctness of various automata models have been investigated in depth [14] and more recently those of logics over (unary alphabet) strings [15], the succinctness of regular expressions has hardly been addressed. For the complement of a regular expression an exponential lower bound is given by Ellul et al [9]. For the intersection of an arbitrary number of regular expressions Petersen gave an exponential lower bound [28], while Ellul et al [9] mention a quadratic lower bound for the intersection of two regular expressions. In fact, in [9], it is explicitly asked what the maximum achievable blow-up is for the complement of one and the intersection of two regular expressions (Open Problems 4 and 5). Although we do not answer these questions in the most precise way, our lower bounds improve the existing ones by one exponential and are tight in the sense that the target expression can be constructed in the time class matching the space complexity of the lower bounds.

Succinctness of complement and intersection relate to the succinctness of semi-extended (RE($\cap$)) and extended regular expressions (RE($\cap$,$\neg$)). These are regular expressions augmented with intersection and both complement and intersection operators, respectively. Their membership problem has been extensively studied [18, 20, 26, 28, 30]. Furthermore, non-emptiness and equivalence of RE($\cap$,$\neg$) is non-elementary [33]. For RE($\cap$), inequivalence is EXPSPACE-complete [10, 16, 29], and non-emptiness is PSPACE-complete [10, 16] even when restricted to the intersection of a (non-constant) number of regular expressions [19]. Several of these papers hint upon the succinctness of the intersection operator and provide dedicated techniques in dealing with the new operator directly rather than through a translation to ordinary regular expressions [20, 28]. Our results present a double exponential lower bound in translating RE($\cap$) to RE and therefore justify even more the development for specialized techniques.

A final motivation for this research stems from its application in the emerging area of XML-theory [21, 27, 31, 34]. From a formal language viewpoint, XML documents can be seen as labeled unranked trees and collections of these documents are defined by schemas. A schema can take various forms, but the most common ones are Document Type Definitions

	complement	intersection (fixed)	intersection (arbitrary)
regular expression	2-exp	exp	2-exp
one-unambiguous	poly	exp	2-exp
single-occurrence	poly	exp	exp

Table 1: Overview of the size increase for the various operators and subclasses. All non-polynomial complexities are tight.

(DTDs) [4] and XML Schema Definitions (XSDs) [32] which are grammar based formalisms with regular expressions at right-hand sides of rules [23, 25]. Many questions concerning schemas reduce to corresponding questions on the classes of regular expressions used as right-hand sides of rules as is exemplified for the basic decision problems studied in [11] and [22]. Furthermore, the lower bounds presented here are utilized in [12] to prove, among other things, lower bounds on the succinctness of existential and universal pattern-based schemas on the one hand, and single-type EDTDs (a formalization of XSDs) and DTDs, on the other hand. As the DTD and XML Schema specification require regular expressions occurring in rules to be *deterministic*, formalized by Brüggemann-Klein and Wood in terms of one-unambiguous regular expressions [6], we also investigate the complement and intersection of those. In particular, we show that a one-unambiguous regular expressions can be complemented in polynomial time, whereas the lower bounds concerning intersection carry over from unrestricted regular expressions. A study in [2] reveals that most of the one-unambiguous regular expression used in practice take a very simple form: every alphabet symbol occurs at most once. We refer to those as single-occurrence regular expressions (SOREs) and show a tight exponential lower bound for intersection.

Outline. In Section 2, we introduce the necessary notions concerning (one-unambiguous) regular expressions and automata. In Section 3, we extend the result by Ehrenfeucht and Zeiger to a fixed alphabet using the family of languages $(K_n)_{n\in\mathbb{N}}$. In Section 4, we consider the succinctness of complement. In Section 5, we consider the succinctness of intersection of several classes of regular expressions. We conclude in Section 6.

2. Preliminaries

2.1. Regular expressions

By $\mathbb{N}$ we denote the natural numbers without zero. For the rest of the paper, Σ always denotes a finite alphabet. A Σ-*string* (or simply string) is a finite sequence $w = a_1 \cdots a_n$ of Σ-symbols. We define the length of w, denoted by $|w|$, to be n. We denote the empty string by ε. The set of *positions of* w is $\{1, \ldots, n\}$ and the *symbol of* w *at position* i is a_i. By $w_1 \cdot w_2$ we denote the *concatenation* of two strings w_1 and w_2. As usual, for readability, we denote the concatenation of w_1 and w_2 by $w_1 w_2$. The set of all strings is denoted by Σ^* and the set of all non-empty strings by Σ^+. A *string language* is a subset of Σ^*. For two string languages $L, L' \subseteq \Sigma^*$, we define their concatenation $L \cdot L'$ to be the set $\{w \cdot w' \mid w \in L, w' \in L'\}$. We abbreviate $L \cdot L \cdots L$ (i times) by L^i.

The set of *regular expressions* over Σ, denoted by RE, is defined in the usual way: $\emptyset$, ε, and every Σ-symbol is a regular expression; and when r_1 and r_2 are regular expressions, then $r_1 \cdot r_2$, $r_1 + r_2$, and r_1^* are also regular expressions.

By RE($\cap$,$\neg$) we denote the class of *extended regular expressions*, that is, RE extended with intersection and complementation operators. So, when r_1 and r_2 are RE($\cap$,$\neg$)-expressions then so are $r_1 \cap r_2$ and $\neg r_1$. By RE($\cap$) and RE($\neg$) we denote RE extended solely with the intersection and complement operator, respectively.

The language defined by an extended regular expression r, denoted by $L(r)$, is inductively defined as follows: $L(\emptyset) = \emptyset$; $L(\varepsilon) = \{\varepsilon\}$; $L(a) = \{a\}$; $L(r_1 r_2) = L(r_1) \cdot L(r_2)$; $L(r_1 + r_2) = L(r_1) \cup L(r_2)$; $L(r^*) = \{\varepsilon\} \cup \bigcup_{i=1}^{\infty} L(r)^i$; $L(r_1 \cap r_2) = L(r_1) \cap L(r_2)$; and $L(\neg r_1) = \Sigma^* \setminus L(r_1)$.

By $\bigcup_{i=1}^{k} r_i$, and r^k, with $k \in \mathbb{N}$, we abbreviate the expression $r_1 + \cdots + r_k$, and $rr \cdots r$ (k-times), respectively. For a set $S = \{a_1, \ldots, a_n\} \subseteq \Sigma$, we abbreviate by S the regular expression $a_1 + \cdots + a_n$.

We define the *size* of an extended regular expression r over Σ, denoted by $|r|$, as the number of Σ-symbols and operators occurring in r disregarding parentheses. This is equivalent to the length of its (parenthesis-free) reverse Polish form [37]. Formally, $|\emptyset| = |\varepsilon| = |a| = 1$, for $a \in \Sigma$, $|r_1 r_2| = |r_1 \cap r_2| = |r_1 + r_2| = |r_1| + |r_2| + 1$, and $|\neg r| = |r^*| = |r| + 1$.

Other possibilities considered in the literature for defining the size of a regular expression are: (1) counting all symbols, operators, and parentheses [1, 17]; or, (2) counting only the Σ-symbols. However, Ellul et al. [9] have shown that for regular expressions (so, without $\neg$ and $\cap$), provided they are preprocessed by syntactically eliminating superfluous $\emptyset$- and ε-symbols, and nested stars, the three length measures are identical up to a constant multiplicative factor. For extended regular expressions, counting only the Σ-symbols is not sufficient, since for instance the expression $(\neg\varepsilon)(\neg\varepsilon)(\neg\varepsilon)$ does not contain any Σ-symbols. Therefore, we define the size of an expression as the length of its reverse Polish form.

2.2. **One-unambiguous regular expressions and SOREs**

As mentioned in the introduction, several XML schema languages restrict regular expressions occurring in rules to be *deterministic*, formalized by Brüggemann-Klein and Wood [6] in terms of one-unambiguity. We introduce this notion next.

To indicate different occurrences of the same symbol in a regular expression, we mark symbols with subscripts. For instance, the *marking* of $(a + b)^* a + bc$ is $(a_1 + b_2)^* a_3 + b_4 c_5$. We denote by $r^\flat$ the marking of r and by $\mathrm{Sym}(r^\flat)$ the subscripted symbols occurring in $r^\flat$. When r is a marked expression, then $r^\natural$ over Σ is obtained from r by dropping all subscripts. This notion is extended to words and languages in the usual way.

Definition 1. A regular expression r is *one-unambiguous* iff for all strings $w, u, v \in \mathrm{Sym}(r^\flat)^*$, and all symbols $x, y \in \mathrm{Sym}(r^\flat)$, the conditions $uxv, uyw \in L(r^\flat)$ and $x \neq y$ imply $x^\natural \neq y^\natural$.

For instance, the regular expression $r = a^* a$, with marking $r^\flat = a_1^* a_2$, is not one-unambiguous. Indeed, the marked strings $a_1 a_2$ and $a_1 a_1 a_2$ both in $L(r^\flat)$ do not satisfy the conditions in the previous definition. The equivalent expression aa^*, however, is one-unambiguous. The intuition behind the definition is that positions in the input string can be matched in a deterministic way against a one-unambiguous regular expression without looking ahead. For instance, for the expression aa^*, the first a of an input string is always matched against the leading a in the expression, while every subsequent a is matched against the last a. Unfortunately, one-unambiguous regular languages do not form a very robust class as they are not even closed under the Boolean operations [6].

The following subclass captures the class of regular expressions occurring in XML schemas on the Web [2]:

Definition 2. A *single-occurrence regular expression (SORE)* is a regular expression where every alphabet symbol occurs at most once. In addition, we allow the operator r^+ which defines rr^*.

For instance, $(a+b)^+c$ is a SORE while $a^*(a+b)^+$ is not. Clearly, every SORE is one-unambiguous. Note that SOREs define local languages and that over a fixed alphabet there are only finitely many of them.

2.3. **Finite automata**

A non-deterministic finite automaton (NFA) A is a 4-tuple (Q, q_0, δ, F) where Q is the set of states, q_0 is the initial state, F is the set of final states and $\delta \subseteq Q \times \Sigma \times Q$ is the transition relation. We write $q \Rightarrow_{A,w} q'$ when w takes A from state q to q'. So, w is accepted by A if $q_0 \Rightarrow_{A,w} q'$ for some $q' \in F$. The set of strings accepted by A is denoted by $L(A)$. The size of an NFA is $|Q| + |\delta|$. An NFA is *deterministic* (or a DFA) if for all $a \in \Sigma, q \in Q$, $|\{(q, a, q') \in \delta \mid q' \in Q\}| \leq 1$.

We make use of the following known results.

Theorem 3. Let $A_1, \ldots, A_m$ be NFAs over Σ with $|A_i| = n_i$ for $i \leq m$, and $|\Sigma| = k$.

(1) A regular expression r, with $L(r) = L(A_1)$, can be constructed in time $\mathcal{O}(m_1 k 4^{m_1})$, where m_1 is the number of states of A_1 [24, 9].
(2) A DFA B with 2^{n_1} states, such that $L(B) = L(A_1)$, can be constructed in time $\mathcal{O}(2^{n_1})$ [36].
(3) A DFA B with 2^{n_1} states, such that $L(B) = \Sigma^* \setminus L(A_1)$, can be constructed in time $\mathcal{O}(2^{n_1})$ [36].
(4) Let $r \in$ RE. An NFA B with $|r| + 1$ states, such that $L(B) = L(r)$, can be constructed in time $\mathcal{O}(|r| \cdot |\Sigma|)$ [5].
(5) Let $r \in$ RE($\cap$). An NFA B with $2^{|r|}$ states, such that $L(B) = L(r)$, can be constructed in time exponential in the size of r [10].

3. **A generalization of a Theorem by Ehrenfeucht and Zeiger to a fixed alphabet**

We first introduce the family $(Z_n)_{n \in \mathbb{N}}$ defined by Ehrenfeucht and Zeiger over an alphabet whose size grows quadratically with the parameter n [8]:

Definition 4. Let $n \in \mathbb{N}$ and $\Sigma_n = \{a_{i,j} \mid 0 \leq i, j \leq n-1\}$. Then, Z_n contains exactly all strings of the form $a_{i_0,i_1} a_{i_1,i_2} \cdots a_{i_{k-1},i_k}$ where $k \in \mathbb{N}$.

A way to interpret Z_n is to consider the DFA with states $\{0, \ldots, n-1\}$ which is fully connected and where the edge between state i and j is labeled with $a_{i,j}$. The language Z_n then consists of all paths in the DFA. [1]

Ehrenfeucht and Zeiger obtained the succinctness of DFAs with respect to regular expressions through the following theorem:

[1] Actually, in [8], only paths from state 0 to state $n-1$ are considered. We use our slightly modified definition as it will be easier to generalize to a fixed arity alphabet suited for our purpose in the sequel.

Theorem 5 ([8])**.** For $n \in \mathbb{N}$, any regular expression defining Z_n must be of size at least 2^{n-1}. Furthermore, there is a DFA of size $\mathcal{O}(n^2)$ accepting Z_n.

Our language K_n is then the straightforward binary encoding of Z_n that additionally swaps the pair of indices in every symbol $a_{i,j}$. Thereto, for $a_{i,j} \in \Sigma_n$, define the function ρ_n as

$$\rho_n(a_{i,j}) = \text{enc}(j)\$\text{enc}(i)\#,$$

where enc(i) and enc(j) denote the $\lceil \log(n) \rceil$-bit binary encodings of i and j, respectively. Note that since $i, j < n$, i and j can be encoded using only $\lceil \log(n) \rceil$-bits. We extend the definition of ρ_n to strings in the usual way: $\rho_n(a_{i_0,i_1} \cdots a_{i_{k-1},i_k}) = \rho_n(a_{i_0,i_1}) \cdots \rho_n(a_{i_{k-1},i_k})$.

We are now ready to define K_n.

Definition 6. Let $\Sigma_K = \{0, 1, \$, \#\}$. For $n \in \mathbb{N}$, let $K_n = \{\rho_n(w) \mid w \in Z_n\}$.

For instance, for $n = 5$, $w = a_{3,2}a_{2,1}a_{1,4}a_{4,2} \in Z_5$ and thus

$$\rho_n(w) = 010\$011\#001\$010\#100\$001\#010\$100\# \in K_5.$$

We generalize the previous theorem as follows:

Theorem 7. For any $n \in \mathbb{N}$, with $n \geq 2$,

(1) any regular expression defining K_n is of size at least 2^n; and,
(2) there is a DFA A_n of size $\mathcal{O}(n^2 \log n)$ defining K_n.

The construction of A_n is given in the Appendix. The rest of this section is devoted to the proof of Theorem 7(1). It follows the structure of the proof of Ehrenfeucht and Zeiger but is technically more involved as it deals with binary encodings of integers.

We start by introducing some terminology. Let $w = a_{i_0,i_1}a_{i_1,i_2} \cdots a_{i_{k-1},i_k} \in Z_n$. We say that i_0 is the *start-point* of w and i_k is its *end-point*. Furthermore, we say that w *contains* i or i *occurs in* w if i occurs as an index of some symbol in w. That is, $a_{i,j}$ or $a_{j,i}$ occurs in w for some j. For instance, $a_{0,2}a_{2,2}a_{2,1} \in Z_5$, has start-point 0, end-point 1, and contains 0, 1 and 2. The notions of contains, occurs, start- and end-point of a string w are also extended to K_n. So, the start and end-points of $\rho_n(w)$ are the start and end-points of w, and w contains the same integers as $\rho_n(w)$.

For a regular expression r, we say that i is a *sidekick* of r when it occurs in every non-empty string defined by r. A regular expression s is a starred subexpression of a regular expression r when s is a subexpression of r and is of the form t^*.

In the Appendix, we prove the following lemma:

Lemma 8. Any starred subexpression s of a regular expression r defining K_n has a sidekick.

We now say that a regular expression r is *normal* if every starred subexpression of r has a sidekick. In particular, any expression defining K_n is normal. We say that a regular expression r *covers* a string w if there exist strings $u, u' \in \Sigma^*$ such that $uwu' \in L(r)$. If there is a greatest integer m for which r covers w^m, we call m the *index* of w in r and denote it by $I_w(r)$. In this case we say that r is *w-finite*. Otherwise, we say that r is *w-infinite*. The index of a regular expression can be used to give a lowerbound on its size according to the following lemma.

Lemma 9 ([8])**.** For any regular expression r and string w, if r is w-finite, then $I_w(r) < 2|r|$.[2]

[2]In fact, in [8] the length of an expression is defined as the number of Σ-symbols occurring in it. However, since our length measure also contains these Σ-symbols, this lemma still holds in our setting.

Now, we can state the most important property of K_n.

Lemma 10. Let $n \geq 2$. For any $C \subseteq \{0, \ldots, n-1\}$ of cardinality k and $i \in C$, there exists a string $w \in K_n$ with start- and end-point i only containing integers in C, such that any normal regular expression r which covers w is of size at least 2^k.

Proof. The proof is by induction on the value of k. For $k = 1$, $C = \{i\}$. Then, define $w = \text{enc}(i)\$\text{enc}(i)\#$, which satisfies all conditions and any expression covering w must definitely have a size of at least 2.

For the inductive step, let $C = \{j_1, \ldots, j_k\}$. Define $C_\ell = C \setminus \{j_{(\ell \mod k)+1}\}$ and let w_ℓ be the string given by the induction hypothesis with respect to C_ℓ (of size $k-1$) and j_ℓ. Note that $j_\ell \in C_\ell$. Further, define $m = 2^{k+1}$ and set

$$w = \text{enc}(j_1)\$\text{enc}(i)\#w_1^m\text{enc}(j_2)\$\text{enc}(j_1)\#w_2^m\text{enc}(j_3)\$\text{enc}(j_2)\#\cdots w_k^m\text{enc}(i)\$\text{enc}(j_k)\#.$$

Then, $w \in K_n$, has i as start and end-point and only contains integers in C. It only remains to show that any expression r which is normal and covers w is of size at least 2^k.

Fix such a regular expression r. If r is w_ℓ-finite for some $\ell \leq k$. Then, $I_{w_\ell}(r_k) \geq m = 2^{k+1}$ by construction of w. By Lemma 9, $|r| \geq 2^k$ and we are done.

Therefore, assume that r is w_ℓ-infinite for every $\ell \leq k$. For every $\ell \leq k$, consider all subexpressions of r which are w_ℓ-infinite. It is easy to see that all minimal elements in this set of subexpressions must be starred subexpressions. Here and in the following, we say that an expression is minimal with respect to a set simply when no other expression in the set is a subexpression. Indeed, a subexpression of the form a or ε can never be w_ℓ-infinite and a subexpression of the form r_1r_2 or $r_1 + r_2$ can only be w_ℓ-infinite if r_1 and/or r_2 are w_ℓ-infinite and is thus not minimal with respect to w_ℓ-infinity. Among these minimal starred subexpressions for w_ℓ, choose one and denote it by s_ℓ. Let $E = \{s_1, \ldots, s_k\}$. Note that since r is normal, all its subexpressions are also normal. As in addition each s_ℓ covers w_ℓ, by the induction hypothesis the size of each s_ℓ is at least 2^{k-1}. Now, choose from E some expression s_ℓ such that s_ℓ is minimal with respect to the other elements in E.

As r is normal and s_ℓ is a starred subexpression of r, there is an integer j such that every non-empty string in $L(s_\ell)$ contains j. By definition of the strings $w_1, \ldots, w_k$, there is some w_p, $p \leq k$, such that w_p does not contain j. Denote by s_p the starred subexpression from E which is w_p-infinite. In particular, s_ℓ and s_p cannot be the same subexpression of r.

Now, there are three possibilities:

- s_ℓ and s_p are completely disjoint subexpressions of r. That is, they are both not a subexpression of one another. By induction they must both be of size 2^{k-1} and thus $|r| \geq 2^{k-1} + 2^{k-1} = 2^k$.
- s_p is a strict subexpression of s_ℓ. This is not possible since s_ℓ is chosen to be a minimum element from E.
- s_ℓ is a strict subexpression of s_p. We show that if we replace s_ℓ by ε in s_p, then s_p is still w_p-infinite. It then follows that s_p still covers w_p, and thus s_p without s_ℓ is of size at least 2^{k-1}. As $|s_\ell| \geq 2^{k-1}$ as well it follows that $|r| \geq 2^k$.

  To see that s_p without s_ℓ is still w_p-infinite, recall that any non-empty string defined by s_ℓ contains j and j does not occur in w_p. Therefore, a full iteration of s_ℓ can never contribute to the matching of any number of repetitions of w_p. So, s_p can only lose its w_p-infinity by this replacement if s_ℓ contains a subexpression which is itself w_p-infinite. However, this then also is a subexpression of s_p and s_p is chosen

to be minimal with respect to w_p-infinity, a contradiction. We can only conclude that s_p without s_ℓ is still w_p-infinite.

■

Since by Lemma 8 any expression defining K_n is normal, Theorem 7(1) directly follows from Lemma 10 by choosing $i = 0$, $k = n$. This concludes the proof of Theorem 7(1).

In Remark A.1 in the Appendix, we relate our proof to the one by Waizenegger [35].

4. Complementing regular expressions

It is known that extended regular expressions are non-elementary more succinct than classical ones [7, 33]. Intuitively, each exponent in the tower requires nesting of an additional complement. In this section, we show that in defining the complement of a single regular expression, a double-exponential size increase cannot be avoided in general. In contrast, when the expression is one-unambiguous its complement can be computed in polynomial time.

Theorem 11. (1) For every regular expression r over Σ, a regular expression s with $L(s) = \Sigma^* \setminus L(r)$ can be constructed in time $\mathcal{O}(2^{|r|+1} \cdot |\Sigma| \cdot 4^{2^{|r|+1}})$.

(2) Let Σ be a four-letter alphabet. For every $n \in \mathbb{N}$, there is a regular expressions r_n of size $\mathcal{O}(n)$ such that any regular expression r defining $\Sigma^* \setminus L(r_n)$ is of size at least 2^{2^n}.

Proof. (2) Take Σ as Σ_K, that is, $\{0, 1, \$, \#\}$. Let $n \in \mathbb{N}$. We define an expression r_n of size $\mathcal{O}(n)$, such that $\Sigma^* \setminus L(r_n) = K_{2^n}$. By Theorem 7, any regular expression defining K_{2^n} is of size exponential in 2^n, that is, of size 2^{2^n}. By $r^{[0,n-1]}$ we abbreviate the expression $(\varepsilon + r(\varepsilon + r(\varepsilon \cdots (\varepsilon + r))))$, with a *nesting depth* of $n-1$. We then define r_n as the disjunction of the following expressions:

- all strings that do not start with a prefix in $(0+1)^n\$$:
$$\Sigma^{[0,n]} + (0+1)^{[0,n-1]}(\$ + \#)\Sigma^* + (0+1)^n(0+1+\#)\Sigma^*$$
- all strings where a \$ is not followed by a string in $(0+1)^n\#$:
$$\Sigma^*\$\big(\Sigma^{[0,n-1]}(\# + \$) + \Sigma^n(0+1+\$)\big)\Sigma^*$$
- all strings where a non-final # is not followed by a string in $(0+1)^n\$$:
$$\Sigma^*\#\big(\Sigma^{[0,n-1]}(\# + \$) + \Sigma^n(0+1+\#)\big)\Sigma^*$$
- all strings that do not end in #:
$$\Sigma^*(0+1+\$)$$
- all strings where the corresponding bits of corresponding blocks are different:
$$((0+1)^* + \Sigma^*\#(0+1)^*)0\Sigma^{3n+2}1\Sigma^* + ((0+1)^* + \Sigma^*\#(0+1)^*)1\Sigma^{3n+2}0\Sigma^*.$$

It should be clear that a string over $\{0, 1, \$, \#\}$ is matched by none of the above expressions if and only if it belongs to K_{2^n}. So, the complement of r_n defines exactly K_{2^n}. ■

The previous theorem essentially shows that in complementing a regular expression, there is no better algorithm than translating to a DFA, computing the complement and translating back to a regular expression which includes two exponential steps. However, when the given regular expression is one-unambiguous, a corresponding DFA can be computed in quadratic time through the Glushkov construction [6] eliminating already one exponential step. The proof of the next theorem shows that the complement of that DFA can be directly defined by a regular expression of polynomial size.

Theorem 12. For any one-unambiguous regular expression r over an alphabet Σ, a regular expression s defining $\Sigma^* \setminus L(r)$ can be constructed in time $\mathcal{O}(n^3)$, where n is the size of r.

Proof. Let r be a one-unambiguous expression over Σ. We introduce some notation.

- The set Not-First(r) contains all Σ-symbols which are not the first symbol in any word defined by r, that is, Not-First$(r) = \Sigma \setminus \{a \mid a \in \Sigma \wedge \exists w \in \Sigma^*, aw \in L(r)\}$.
- For any symbol $x \in \mathrm{Sym}(r^\flat)$, the set Not-Follow$(r, x)$ contains all Σ-symbols of which no marked version can follow x in any word defined by $r^\flat$. That is, Not-Follow$(r, x) = \Sigma \setminus \{y^\natural \mid y \in \mathrm{Sym}(r^\flat) \wedge \exists w, w' \in \mathrm{Sym}(r^\flat)^*, wxyw' \in L(r^\flat)\}$.
- The set Last(r) contains all *marked* symbols which are the last symbol of some word defined by $r^\flat$. Formally, Last$(r) = \{x \mid x \in \mathrm{Sym}(r^\flat) \wedge \exists w \in \Sigma^*, wx \in L(r^\flat)\}$.

We define the following regular expressions:

- $\mathrm{init}(r) = \begin{cases} \text{Not-First}(r)\Sigma^* & \text{if } \varepsilon \in L(r); \text{ and} \\ \varepsilon + \text{Not-First}(r)\Sigma^* & \text{if } \varepsilon \notin L(r). \end{cases}$
- For every $x \in \mathrm{Sym}(r^\flat)$, let $r^\flat_x$ be the expression defining $\{wx \mid w \in \mathrm{Sym}(r^\flat)^* \wedge \exists u \in \mathrm{Sym}(r^\flat)^*, wxu \in L(r^\flat)\}$. That is, all prefixes of strings in $r^\flat$ ending in x. Then, let r_x define $L(r^\flat_x)^\natural$.

We are now ready to define s:

$$\mathrm{init}(r) \quad + \bigcup_{x \notin \mathrm{Last}(r)} r_x(\varepsilon \quad + \quad \text{Not-Follow}(r, x)\Sigma^*) \quad + \bigcup_{x \in \mathrm{Last}(r)} r_x \text{Not-Follow}(r, x)\Sigma^*.$$

In the Appendix we show that s can be constructed in time cubic in the size of r and that s defines the complement of r. The latter is proved by exhibiting a direct correspondence between s and the complement of the Glushkov automaton of r. ∎

We conclude this section by remarking that one-unambiguous regular expressions are not closed under complement and that the constructed s is therefore not necessarily one-unambiguous.

5. Intersecting regular expressions

In this section, we study the succinctness of intersection. In particular, we show that the intersection of two (or any fixed number) and an arbitrary number of regular expressions are exponentially and double exponentially more succinct than regular expressions, respectively. Actually, the exponential bound for a fixed number of expressions already holds for single-occurrence regular expressions, whereas the double exponential bound for an arbitrary number of expressions only carries over to one-unambiguous expressions. For single-occurrence expressions this can again be done in exponential time.

In this respect, we introduce a slightly altered version of K_n.

Definition 13. Let $\Sigma_L = \{0, 1, \$, \#, \triangle\}$. For all $n \in \mathbb{N}$, $L_n = \{\rho_n(w)\triangle \mid w \in Z_n \wedge |w| \text{ is even}\}$.

We also define a variant of Z_n which only slightly alters the $a_{i,j}$ symbols in Z_n. Thereto, let $\Sigma_n^\circ = \{a_{i^\circ,j}, a_{i,j^\circ} \mid 0 \leq i, j < n\}$ and set $\hat{\rho}(a_{i,j}a_{j,k}) = \triangleright_i a_{i,j^\circ}a_{j^\circ,k}$ and $\hat{\rho}(a_{i_0,i_1}a_{i_1,i_2} \cdots a_{i_{k-2},i_{k-1}}a_{i_{k-1},i_k}) = \hat{\rho}(a_{i_0,i_1}a_{i_1,i_2}) \cdots \hat{\rho}(a_{i_{k-2},i_{k-1}}a_{i_{k-1},i_k})$, where k is even.

Definition 14. Let $n \in \mathbb{N}$ and $\Sigma_M^n = \Sigma_n^\circ \cup \{\triangleright_0, \triangle_0, \ldots, \triangleright_{n-1}, \triangle_{n-2}\}$. Then, $M_n = \{\hat{\rho}(w)\triangle_i \mid w \in Z_n \wedge |w| \text{ is even} \wedge i \text{ is the end-point of } w\}$.

Note that paths in M_n are those in Z_n where every odd position is promoted to a circled one ($^\circ$), and triangles labeled with the non-circled positions are added. For instance, the string $a_{2,4}a_{4,3}a_{3,3}a_{3,0} \in Z_5$ is mapped to the string $\triangleright_2 a_{2,4^\circ}a_{4^\circ,3} \triangleright_3 a_{3,3^\circ}a_{3^\circ,0}\triangle_0 \in M_5$.

We make use of the following property, which is proved in the Appendix:

Lemma 15. Let $n \in \mathbb{N}$.

(1) Any regular expression defining L_n is of size at least 2^n.
(2) Any regular expression defining M_n is of size at least 2^{n-1}.

The next theorem shows the succinctness of the intersection operator.

Theorem 16.
(1) For any $k \in \mathbb{N}$ and regular expressions $r_1, \ldots, r_k$, a regular expression defining $\bigcap_{i \leq k} L(r_k)$ can be constructed in time $\mathcal{O}((m+1)^k \cdot |\Sigma| \cdot 4^{(m+1)^k})$, where $m = \max\{|r_i| \mid 1 \leq i \leq k\}$.
(2) For every $n \in \mathbb{N}$, there are SOREs r_n and s_n of size $\mathcal{O}(n^2)$ such that any regular expression defining $L(r_n) \cap L(s_n)$ is of size at least 2^{n-1}.
(3) For each $r \in \text{RE}(\cap)$ an equivalent regular expression can be constructed in time $\mathcal{O}(2^{|r|} \cdot |\Sigma| \cdot 4^{2^{|r|}})$.
(4) For every $n \in \mathbb{N}$, there are one-unambiguous regular expressions $r_1, \ldots, r_m$, with $m = 2n + 1$, of size $\mathcal{O}(n)$ such that any regular expression defining $\bigcap_{i \leq m} L(r_i)$ is of size at least 2^{2^n}.
(5) Let $r_1, \ldots, r_n$ be SOREs. A regular expression defining $\bigcap_{i \leq n} L(r_n)$ can be constructed in time $\mathcal{O}(m \cdot |\Sigma| \cdot 4^m)$, where $m = \sum_{i \leq n} |r_i|$.

Proof. The proofs of (1), (3), and (5) have been moved to the Appendix.
(2) Let $n \in \mathbb{N}$. By Lemma 15(2), any regular expression defining M_n is of size at least 2^{n-1}. We define SOREs r_n and s_n of size quadratic in n, such that $L(r_n) \cap L(s_n) = M_n$. We start by partitioning Σ_M^n in two different ways. To this end, for every $i < n$, define $\text{Out}_i = \{a_{i,j^\circ} \mid 0 \leq j < n\}$, $\text{In}_i = \{a_{j^\circ,i} \mid 0 \leq j < n\}$, $\text{Out}_{i^\circ} = \{a_{i^\circ,j} \mid 0 \leq j < n\}$, and, $\text{In}_{i^\circ} = \{a_{j,i^\circ} \mid 0 \leq j < n\}$. Then,

$$\Sigma_M^n = \bigcup_i \text{In}_i \cup \text{Out}_i \cup \{\triangleright_i, \triangle_i\} = \bigcup_{i^\circ} \text{In}_{i^\circ} \cup \text{Out}_{i^\circ} \cup \{\triangleright_i, \triangle_i\}.$$

Further, define

$$r_n = ((\triangleright_0 + \cdots + \triangleright_{n-1}) \bigcup_{i^\circ} \text{In}_{i^\circ}\text{Out}_{i^\circ})^+(\triangle_0 + \cdots + \triangle_{n-1})$$

and

$$s_n = \Big(\bigcup_i (\text{In}_i + \varepsilon)(\triangleright_i + \triangle_i)(\text{Out}_i + \varepsilon)\Big)^*.$$

Now, r_n checks that every string consists of a sequence of blocks of the form $\triangleright_i a_{j,k^\circ} a_{k^\circ,\ell}$, for $i, j, k, \ell < n$, ending with a $\triangle_i$, for $i < n$. It thus sets the format of the strings and checks whether the circled indices are equal. Further, s_n checks whether the non-circled indices are equal and whether the triangles have the correct indices. Since the alphabet of M_n is of size $\mathcal{O}(n^2)$, also r_n and s_n are of size $\mathcal{O}(n^2)$.
(4) Let $n \in \mathbb{N}$. We define $m = 2n + 1$ one-unambiguous regular expressions of size $\mathcal{O}(n)$, such that their intersection defines L_{2^n}. By Lemma 15(1), any regular expression defining L_{2^n} is of size at least 2^{2^n} and the theorem follows. For ease of readability, we denote Σ_L simply by Σ. The expressions are as follows. There should be an even length sequence of blocks:

$$\big((0+1)^n\$(0+1)^n\#(0+1)^n\$(0+1)^n\#\big)^*\triangle.$$

For all $i \in \{0, \ldots, n-1\}$, the $(i+1)$th bit of the two numbers surrounding an odd # should be equal:

$$\big(\Sigma^i(0\Sigma^{3n+2}0 + 1\Sigma^{3n+2}1)\Sigma^{n-i-1}\#\big)^*\triangle.$$

For all $i \in \{0, \ldots, n-1\}$, the $(i+1)$th bit of the two numbers surrounding an even # should be equal:

$$\Sigma^{2n+2}\Big(\Sigma^i(0\Sigma^{2n-i+1}(\triangle + \Sigma^{n+i+1}0\Sigma^{n-i-1}\#) + (1\Sigma^{2n-i+1}(\triangle + \Sigma^{n+i+1}1\Sigma^{n-i-1}\#)))\Big)^*.$$

Clearly, the intersection of the above expressions defines L_{2^n}. Furthermore, every expression is of size $\mathcal{O}(n)$ and is one-unambiguous as the Glushkov construction translates them into a DFA [6]. ∎

6. Conclusion

In this paper we showed that the complement and intersection of regular expressions are double exponentially more succinct than ordinary regular expressions. For complement, complexity can be reduced to polynomial for the class of one-unambiguous regular expressions although the obtained expressions could fall outside that class. For intersection, restriction to SOREs reduces complexity to exponential. It remains open whether there are natural classes of regular expressions for which both the complement and intersection can be computed in polynomial time.

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Appendix

Remark A.1 It remains to discuss Waizenegger's proof of Theorem 7. He takes an approach similar to ours and defines a binary encoding of Z_n as follows. For $n \in \mathbb{N}$, Σ_n consists of n^2 symbols. If we list these, in some unspecified order, and associate a natural number from 0 to $n^2 - 1$ to each of them, we can encode Z_n using $\log(n^2)$ bits. For instance, if $n = 2$, then $\Sigma_n = \{a_{0,0}, a_{0,1}, a_{1,0}, a_{1,1}\}$ and we can encode these by the numbers $0, 1, 2, 3$, respectively. Now, a string in Z_n is encoded by replacing every symbol by its binary encoding and additionally adding an a and b to the start and end of each symbol. For instance, $a_{0,0}a_{0,1}$ becomes $a00ba01b$, which gives a language, say W_n, over the four letter alphabet $\{0, 1, a, b\}$.

Then, Waizenegger shows that W_n can be described by an NFA of size $\mathcal{O}(k^2 \cdot \log k^2)$, but every regular expression defining it must be of size at least exponential in n. The latter is done by using the same proof techniques as in [8]. However, in this proof it is claimed that if r_n is an expression defining W_n, then every string defined by a starred subexpression of r_n must start with an a. However, this statement is incorrect. For instance, if r_2 is an expression defining W_2, and we still use the same encoding as above, then $((a0(0ba0)^*0b) + \varepsilon)r_2$ still defines W_2 but does not have this property.[3] Albeit small, the mistake has serious consequences. It follows from this claim that every starred subexpression has a sidekick (using our terminology), and the rest of the proof builds upon this fact. To see the importance, consider the language K'_n obtained from Z_n like K_n but without swapping the indices of the symbols in Z_n. This language can be defined by a regular expressions of size $\mathcal{O}(n \log(n))$:

$$\bigcup_{i<n} \mathrm{enc}(i)\$\Big(\bigcup_{i<n} \mathrm{enc}(i)\#\mathrm{enc}(i)\$\Big)^* \bigcup_{i<n} \mathrm{enc}(i)\#.$$

However, if we assume that every string defined by a starred subexpression starts with a $\#$, we can reuse our proof for K_n and show that any regular expression defining K'_n must be of at least exponential size which is clearly false.

To summarize, Waizenegger's claim that the specific encoding of the n^2 symbols in Σ does not matter in the proof for Theorem 7 is false. Our encoding used in defining K_n allows to prove the sidekick lemma (Lemma 8) in a correct way.

Proof of Theorem 7(2) *For every n, with $n \geq 2$, there is a DFA A_n of size $\mathcal{O}(n^2 \log n)$ defining K_n.*

Proof. Let $n \in \mathbb{N}$ and $n \geq 2$. We compose A_n from a number of subautomata which each will be able to read one block of a string. That is, strings defined by the regular expression $(0+1)^{\lceil \log n \rceil}\$(0+1)^{\lceil \log n \rceil}$. These DFAs are:

- $B = (Q_B, q_B, \delta_B, \{f_0, \ldots, f_{n-1}\})$, with $L(B) = \{w\$w' \mid w, w' \in (0+1)^{\lceil \log n \rceil} \wedge 0 \leq \mathrm{enc}(w') < n\}$, such that for any $j < n$, $q_B \Rightarrow_{B, w\$w'} f_j$ iff $\mathrm{enc}(w) = j$. That is, B reads strings of the form $w\$w'$, checks whether w' encodes a number between 0 and $n-1$, and remembers the value of w in its accepting state.

[3] For convenience, we assume here that all strings in W_n must have start point 0, an additional constraint which is actually present in [8, 35]

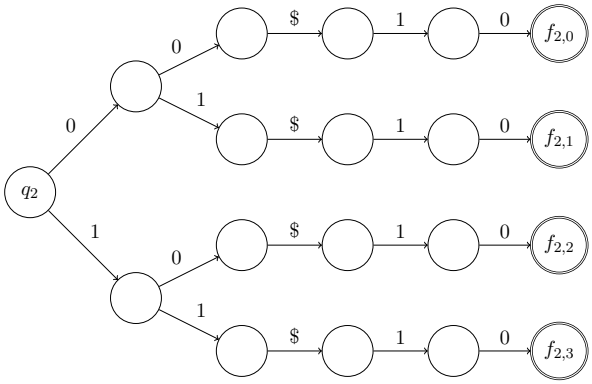

Figure 1: The automaton B_2, for $n = 4$.

- For any $i < n$, $B_i = (Q_i, q_i, \delta_i, \{f_{i,0}, \ldots, f_{i,n-1}\})$, with $L(B_i) = \{w\$w' \mid w, w' \in (0+1)^{\lceil \log n \rceil} \wedge \mathrm{enc}(w') = i\}$, such that for any $j < n$, $q \Rightarrow_{B_i, w\$w'} f_{i,j}$ iff $\mathrm{enc}(w) = j$. That is, B_i reads strings of the form $w\$w'$, checks whether w' encodes i and remembers the value of w in its accepting state.

The construction of B and each of the B_i is very similar. We illustrate the construction of B_2, for $n = 4$, in Figure 1.

Now, let $A_n = (Q, q_B, \delta, \{q_0, \ldots, q_{n-1}\})$ where $Q = Q_B \cup \bigcup_{i<n} Q_i$ and δ contains $\delta_B \cup \bigcup_{i<n} \delta_i$ plus for every $i, j < n$, $(f_j, \#, q_j)$, $(f_{i,j}, \#, q_j) \in \delta$. So, A_n works as follows: first B reads the first block of the string and remembers the first integer therein, say j. Then it passes control to B_j which reads the next block in which it remembers the first integer, say k, and checks whether the second integer is j. If so, it passes control to B_k, and so on. Whenever A_n reaches an initial state of one of the B_i, a valid string has been read and A_n can accept.

Finally, for the size of A_n, consider the subautomata B_i as illustrated in Figure 1. The first, tree-like, part consists of at most $n + n/2 + n/4 + \cdots + 1$ nodes and transitions, which is bounded by $2n$. Further, the *linear* automata following this part are each of size $\lceil \log n \rceil$ and there are n of these. So, the total size of any B_i is $\mathcal{O}(n \log n)$. Similarly, the size of B is $\mathcal{O}(n^2)$. Since A_n consists of one copy of B and a linear number of B_i subautomata, A_n is of size $\mathcal{O}(n^2 \log n)$.

■

Proof of Lemma 8: *Any starred subexpression s of a regular expression r defining K_n has a sidekick.*

Proof. We prove this lemma by a detailed examination of the structure of strings defined by s. We start by making the following observation. For any string $w \in L(s)$, there exist strings $u, u' \in \Sigma_K^*$ such that $uwu' \in L(r)$. Furthermore, w can be pumped in uwu' and still form a string defined by r. That is, for every $j \in \mathbb{N}$, $uw^j u' \in L(r)$. In addition, for every other $w' \in L(s)$, $uw'u' \in L(r)$.

Let w be a non-empty string in $L(s)$ and let u, u' be such that $uwu' \in L(r)$. Then w must contain at least one \$-symbol. Towards a contradiction, suppose it does not. When w contains a # then $uwwu' \in L(r)$ but $uwwu' \notin K_n$ which leads to the desired contradiction. When w contains no # and therefore only consists of 0's and 1's, then $uw^n u' \in L(r)$ but $uw^n u' \notin K_n$ which again is a contradiction. In a similar way, one can show that (i) w

contains at least one #-symbol; (*ii*) w contains an equal number of \$ and #-symbols; (*iii*) the \$ and #-symbols must alternate; and (*iv*) between any consecutive \$ and #-symbol there is a string of length $\lceil \log(n) \rceil$ containing only 0's and 1's.

From the above it follows that w matches one of the following expressions:

(1) $\alpha_1 = (0+1)^*\$(0+1)^{\lceil \log n \rceil}\#\Sigma_K^*$

(2) $\alpha_2 = \Sigma_K^*\#(0+1)^{\lceil \log n \rceil}\$(0+1)^*$.

We refer to the strings defined by α_1 and α_2 as strings of type-1 and type-2. We next show that all strings defined by s are either all of type-1 or all of type-2. Towards a contradiction assume there is a type-1 string w_1 and a type-2 string w_2 in $L(s)$. Then, $w_2 w_1 \in L(s)$ and thus there exist $u, u' \in \Sigma_K^*$ such that $uw_2w_1u' \in L(r)$. However, because of the concatenation of w_2 and w_1, there are two \$-symbols without an intermediate #-symbol and therefore $uw_2w_1u' \notin K_n$.

Assume that all strings defined by s are of type-1. We next argue that the substring of length $\lceil \log n \rceil$, that is, the integer i, between the first \$ and #-symbol is the same for every $w \in L(s)$ which gives us our sidekick. For a type-1 string, we refer to this integer as the start block. Towards a contradiction, suppose that w, w_1 and w_2 are non-empty strings in $L(s)$ such that w_1 and w_2 have different start blocks. Let u, u' be such that $uww_1u' \in L(r)$ and therefore $uww_1u' \in K_n$. Now, uw contains at least one \$ and #-symbol. Therefore, by definition of K_n, the value of the start block of w_1 is uniquely determined by uw. Now also $uww_2u' \in L(r)$ as s is a starred subexpression, but $uww_2u' \notin K_n$ as w_2 has a different start block, which yields the desired contradiction.

The same kind of reasoning can be used to show that s has a sidekick when all defined strings are of type-2. ∎

Proof of Theorem 11: (continued)

Proof. (1) Let $r \in$ RE. We first construct a DFA A, with $L(A) = \Sigma^* \setminus L(r)$ and then construct the regular expression s equivalent to A. According to Theorem 3(3) and (4) A contains at most $2^{|r|+1}$ states and can be constructed in time exponential in the size of A. Then, by Theorem 3(1), the total algorithm is in time $\mathcal{O}(2^{(|r|+1)} \cdot |\Sigma| \cdot 4^{2^{(|r|+1)}})$. ∎

Proof of Theorem 12: (continued)

Proof. We prove that $L(s) = \Sigma^* \setminus L(r)$ by highlighting the correspondence between s and the complement of the Glushkov automaton G_r of r. The Glushkov automaton G_r is the DFA (Q, q_0, δ, F), where

- $Q = \{q_0\} \cup \mathrm{Sym}(r^\flat)$;
- $F = \mathrm{Last}(r^\flat)$ (plus q_0 if $\varepsilon \in L(r)$); and
- for $x, y \in \mathrm{Sym}(r^\flat)$, there is
  - a transition $(q_0, x^\natural, x) \in \delta$ if x is the first symbol in some word defined by $r^\flat$; and,
  - a transition $(x, y^\natural, y) \in \delta$ if y follows x in some word defined by $r^\flat$.

It is known that $L(r) = L(G_r)$ and that G_r is deterministic whenever r is one-unambiguous [6].

The complement automaton $\overline{G_r} = (\overline{Q}, q_0, \overline{\delta}, \overline{F})$ is obtained from G_r by making it complete and interchanging final and non-final states. Formally, $\overline{Q} = Q \cup \{q_-\}$ (with $q_- \notin Q$) and

$\overline{\delta}$ contains δ plus the triples $(q_{\_}, a, q_{\_})$, for every $a \in \Sigma$, and $(q, a, q_{\_})$ for every state $q \in Q$ and symbol $a \in \Sigma$ for which there is no $q' \in Q$ with $(q, a, q') \in \delta$. Finally, $\overline{F} = \{q_{\_}\} \cup (Q \setminus F)$. Clearly, $L(\overline{G_r}) = \Sigma^* \setminus L(G_r)$.

Now, we show that $L(s) = L(\overline{G_r})$. First, by definition of s, $\varepsilon \in L(s)$ iff $\varepsilon \notin L(r)$ iff $\varepsilon \in L(\overline{G_r})$. We prove that for any non-empty word w, $w \in L(s)$ iff $w \in L(\overline{G_r})$, from which the lemma follows. Thereto, we show that the non-empty words defined by the different disjuncts of s correspond exactly to subsets of the words defined by $\overline{G_r}$.

- init(r) defines exactly the non-empty words for which $\overline{G_r}$ immediately goes from q_0 to $q_{\_}$ and reads the rest of the word in $q_{\_}$.
- For any $x \in \text{Last}(r)$, $r_x(\text{Not-Follow}(r, x)\Sigma^*)$ defines exactly all strings w for which $\overline{G_r}$ arrives in x after reading a part of w, goes to $q_{\_}$ and reads the rest of w there.
- For any $x \notin \text{Last}(r)$, $r_x(\varepsilon + \text{Not-Follow}(r, x)\Sigma^*)$ defines exactly all strings w for which $\overline{G_r}$ either (1) arrives in x after reading w and accepts because x is an accepting state; or (2) arrives in x after reading a part of w, goes to $q_{\_}$ and reads the rest of w there.

Note that because there are no incoming transitions in q_0, and $q_{\_}$ only has transitions to itself, we have described exactly all accepting runs of $\overline{G_r}$. Therefore, any non-empty string $w \in L(s)$ iff $w \in \overline{G_r}$.

We now show that s can be computed in time cubic in the size of r. By a result of Brüggemann-Klein [5] the Glushkov automaton G_r corresponding to r as defined above can be computed in time quadratic in the size of r. Using G_r, the sets Not-First(r), Not-Follow(r, x) and Last(r) can be computed in time linear in the size of r. So, all three sets can be computed in time quadratic in the size of r. The expression init(r) can be constructed in linear time. We next show that for any $x \in \text{Sym}(r^\flat)$, the expression $r^\flat_x$ can be constructed in time quadratic in the size of r. As $r_x = (r^\flat_x)^\natural$, it follows that s can be constructed in cubic time. The expression $r^\flat_x$ is inductively defined as follows:

- For $r^\flat = \varepsilon$ or $r^\flat = \emptyset$, $r^\flat_x = \emptyset$.
- For $r^\flat = y \in \text{Sym}(r^\flat)$,
$$r^\flat_x = \begin{cases} x & \text{if } y = x \\ \emptyset & \text{otherwise.} \end{cases}$$
- For $r^\flat = \alpha\beta$,
$$r^\flat_x = \begin{cases} \alpha_x & \text{if } x \text{ occurs in } \alpha \\ \alpha\beta_x & \text{otherwise.} \end{cases}$$
- For $r^\flat = \alpha + \beta$,
$$r^\flat_x = \begin{cases} \alpha_x & \text{if } x \text{ occurs in } \alpha \\ \beta_x & \text{otherwise.} \end{cases}$$
- For $r^\flat = \alpha^*$, $r_x = \alpha^*\alpha_x$

The correctness is easily proved by induction on the structure of $r^\flat$. Note that there are no ambiguities in the inductive definition of concatenation and disjunction since the expressions are marked and therefore every marked symbol occurs only once in the expression. All steps in the above inductive definition are linear apart from the last one which is quadratic. Hence, the result follows.

■

We illustrate the construction in the previous proof by means of an example. Let $r = a(ab^*c)^*$, and $r^\flat = a_1(a_2b_3^*c_4)^*$. Then,

- $r^\flat_{a_1} = a_1$,
- $r^\flat_{a_2} = a_1(a_2b_3^*c_4)^*a_2$,
- $r^\flat_{b_3} = a_1(a_2b_3^*c_4)^*a_2b_3^*b_3$,
- $r^\flat_{c_4} = a_1(a_2b_3^*c_4)^*a_2b_3^*c_4$, and
- $\text{init}(r) = \varepsilon + (b+c)^*\Sigma^*$.

Then the complement of r is defined by

$$\begin{aligned}\varepsilon + (b+c)\Sigma^* &\\ &+ a(ab^*c)^*a(\varepsilon + a\Sigma^*) + a(ab^*c)^*ab^*b(\varepsilon + a\Sigma^*)\\ &+ a((b+c)\Sigma^*) + a(ab^*c)^*ab^*c((b+c)\Sigma^*).\end{aligned}$$

Proof of Lemma 15:

Proof. (1) Analogous to Lemma 8, any regular expression r defining L_n must also be normal and must furthermore cover any string $w \in K_n$. By choosing $i = 0$ and $k = n$ in Lemma 10, we see that r must be of size at least 2^n.

(2) Let $n \in \mathbb{N}$ and r_M be a regular expression defining M_n. Let r_Z^2 be the regular expression obtained from r_M by replacing $\triangleright_i$ and $\triangle_i$, with $i < n$, by ε and any $a_{i^\circ,j}$ or a_{i,j°, with $0 \leq i, j < n$, by $a_{i,j}$. Then, $|r_Z^2| \leq |r_M|$ and r_Z^2 defines exactly all strings of even length in Z_n, and thus also covers every string in Z_n. Since the proof in [8] also constructs a string $w \in Z_n$ such that any normal expression covering w must be of size at least 2^{n-1}, it immediately follows that r_Z^2 and thus r_M must be of size at least 2^{n-1}.

■

Proof of Theorem 16: (continued)

Proof. (1) First, construct NFAs $A_1, \ldots, A_k$ such that $L(A_i) = L(r_i)$, for any $i \leq k$. If $m = \max\{|r_i| \mid 1 \leq i \leq k\}$, then by Theorem 3(4) any A_i has at most $n+1$ states and can be constructed in time $\mathcal{O}(m \cdot |\Sigma|)$. Then, an NFA A with $(m+1)^k$ states, such that $L(A) = \bigcap_{i \leq k} L(A_i)$, can be constructed in time $\mathcal{O}((m+1)^k)$ by means of a product construction. By Theorem 3(1), a regular expression defining $L(A)$ can then be constructed in time $\mathcal{O}((m+1)^k \cdot |\Sigma| \cdot 4^{(m+1)^k})$.

(3) For an expression r in RE($\cap$), an NFA A with $2^{|r|}$ states defining $L(r)$ can be constructed in time exponential in the size of r (Theorem 3(5)). Then, by Theorem 3(1), a regular expression (so, without $\neg$ and $\cap$) defining $L(A)$ can be constructed in time $\mathcal{O}(2^{|r|} \cdot |\Sigma| \cdot 4^{2^{|r|}})$.

(5) We show that given SOREs $r_1, \ldots, r_n$, we can construct an NFA A with $|\Sigma| + 1$ states defining $\bigcap_{i \leq n} L(r_i)$ in time cubic in the sizes of $r_1, \ldots, r_n$. It then follows from Theorem 3(1) that an expression defining $\bigcap_{i \leq n} L(r_i)$ can be constructed in time $\mathcal{O}((m+1) \cdot |\Sigma| \cdot 4^{(m+1)})$, where $m = \sum_{i \leq n} |r_i|$ since $m \geq |\Sigma|$.

We first construct NFAs $A_1, \ldots, A_n$ such that $L(A_i) = L(r_i)$, for any $i \leq n$, by using the Glushkov construction [5]. This construction creates an automaton which has an initial state, with only outgoing transitions, and additionally one state for each symbol in the regular expressions. Furthermore, all incoming transitions for that state are labeled with

that symbol. So, we could also say that each symbol is labeled with a symbol, and that all incoming transitions carry the label of that state. Since $r_1, \ldots, r_k$ are SOREs, for every symbol there exists at most one state labeled with that symbol in any A_i. Now, let $A_i = (Q_i, q_0^i, \delta_i, F_i)$ then we say that $Q_i = \{q_0^i\} \cup \{q_a^i \mid a \in \Sigma\}$, where q_a^i is the state labeled with a. For ease of exposition, if a symbol a does not occur in an expression r_i, we add a state q_a^i to Q_i which does not have any incoming or outgoing transitions.

Now, we are ready to construct the NFA $A = (Q, q_0, \delta, F)$ defining the intersection of $A_1, \ldots, A_n$. First, Q has again an initial state and one state for each symbol: $Q = \{q_0\} \cup \{q_a \mid a \in \Sigma\}$. A state is accepting if all its corresponding states are accepting: $F = \{q_a \mid \forall i \leq n, q_a^i \in F_i\}$. Here, a can denote 0 or an alphabet symbol. Finally, there is a transition between q_a and q_b if there is a transition between q_a^i and q_b^i, in every A_i: $\delta = \{(q_a, b, q_b) \mid \forall i \leq n, (q_a^i, b, q_b^i) \in \delta^i\}$. Now, $L(A) = \bigcap_{i \leq n} L(A_i)$. Since the Glushkov construction takes quadratic time [5], and we have to construct n automata, the total construction can be done in cubic time.

■