For any n points in the plane:
(a) There are $n^{0.66}$ distinct distances
(b) There are $n^{0.8}$ distinct distances
An Exposition by William Gasarch

1 Introduction

Let P be any set of n points in the plane. How may different distances are there between the points?

Example 1.1

1. If they are all in a line, one inch apart, then there are $n-1=\Theta(n)$ distinct distances.
2. If they are all in a line, but the distance keeps doubling, so $x_2-x_1=1$, $x_3-x_2=2$, $x_4-x_3=2^2$, $x_5-x_4=2^3$, etc, then there are $\binom{n}{2}=\Theta(n^2)$ different distances.
3. If they are in a uniform $\sqrt{n}\times\sqrt{n}$ grid then there are $O(n/\sqrt{\log n})$ different distances (this is not obvious).

How many different distances are guaranteed?

Def 1.2 Let

$$\begin{aligned} \text{diff-dist}(P) = & \ |\{d(p,q) : p,q \in P\}| \\ g(n) = & \ \min\{\text{diff-dist}(P) : P \text{ is a set of } n \text{ points in the plane}\} \end{aligned}$$

Székely [13] and Solymosi and Toth [6] state that Erdös conejctured $g(n)\geq\frac{n}{\sqrt{\log n}}$. Erdös conjectured that, for all ϵ, $g(n)=\Omega(n^{2-\epsilon})$. Chung [2] and Moser [9] state that Erdös conjectured $(\forall\epsilon)[g(n)\geq n^{2-\epsilon}]$ They all refer to [5]; however, that paper contains no such conjecture. It is possible that Erdös made some conjecture in talks he gave.

The following are known.

1. $O(n/\sqrt{\log n})\geq g(n)\geq\Omega(n^{0.5})$. Erdös [5].
2. $g(n)\geq\Omega(n^{0.66\cdots})$ (actually $\Omega(n^{2/3})$). Moser [9].
3. $g(n)\geq\Omega(n^{0.7143\cdots})$ (actually $\Omega(n^{5/7})$). Chung [2].
4. $g(n)\geq\Omega(n^{0.8}/\log n)$. Chung, Szemerédi, Trotter [3].
5. $g(n)\geq\Omega(n^{0.8})$. Székely [13].
6. $g(n)\geq\Omega(n^{0.8571})$ (actually $\Omega(n^{6/7})$). Solymosi and Toth [6].

7. $g(n) \geq \Omega(n^{0.8634\ldots})$ (actually $\Omega(n^{((4e/(5e-1))-\epsilon)})$). Tardos [15].

8. $g(n) \geq \Omega(n^{0.864\ldots})$ (actually $\Omega(n^{((48-14e)/(55-16e))-\epsilon})$). Katz and Tardos [7].

All of these papers are at
`http://www.cs.umd.edu/~gasarch/erdos_dist/erdos_dist.html`.
In this exposition we give a complete and motivated proof of the results

$$g(n) \geq \Omega(n^{0.66\ldots}).$$

and

$$g(n) \geq \Omega(n^{0.8}).$$

We present Székely's proof of the latter result which, early on, yields the former result. That is, the proof of $g(n) \geq \Omega(n^{0.66\ldots})$ is not the one originally given by Moser. It will fall out of the technology used to prove $g(n) \geq \Omega(n^{0.8})$.

We will actually prove something slightly stronger. We will show that if P is a set of points then there is some point $p \in P$ such that the set of distances from p is $\Omega(n^{0.66})$ and then $\Omega(n^{0.8})$.

Def 1.3 Let P be a set of points and let $p \in P$. Then

$$\begin{aligned} \text{dd-from-a-point}(P,p) &= |\{d(p,q) : q \in P\}| \\ \text{max-dd-from-a-point}(P) &= \max\{\text{dd-from-a-point}(P,p) : p \in P\} \\ g'(n) &= \min\{\text{max-dd-from-a-point}(P) : P \text{ is a set of } n \text{ points in the plane }\} \end{aligned}$$

Clearly $g(n) \geq g'(n)$. We show $g'(n) = \Omega(n^{0.66})$, and $g'(n) = \Omega(n^{0.8})$.

2 Motivation

We assume the following throughout.

1. P is a set of n points in the plane.

2. The maximum number of distances from any point is $t \leq n^{0.9}$ (if this does not hold we already have our theorems).

Picture the following: around every point place concentric circles that hit all of the other points. See Figure 1.

INSERT A FIGURE.

Fix a point p. The following are clear:

1. The number of concentric circles around p is $\leq t$.

2. The number of concentric circles that have only one point on them is $\leq t$.

We now form a multigraph.

Def 2.1 The multigraph G_P is defined as follows.

1. $V = P$.

2. Any two adjacent points on a circle form an edge.

Def 2.2 Let G be a graph. The *crossing number* of G is the minimal number of non-vertex crossings that the graph can be drawn with. Note that a planar graph has crossing number 0. We denote the crossing number of G by $c(G)$. We may use c if the graph is understood.

Lemma 2.3 *Let $G = G_P$.*

1. *The number of vertices is $v = n$.*

2. *The number of edges is $e = \Omega(n^2)$.*

3. *The maximum multiplicity of an edge is $m = O(t)$.*

4. *The crossing number is $c \leq O(n^2 t^2)$.*

Proof:
1) Clearly there are n vertices.

2) Let $p \in P$. How many edges does it create? For every vertex that is not alone on its circle centered at p, an edge is formed (say go clockwise). Hence every point p is responsible for $\geq n - t$ edges. Therefore the total number of edges is $\geq n(n - t)$. Since $t \leq n^{0.9}$ we have

$$e \geq n(n - t) \geq \Omega(n^2).$$

3) Imagine that there are two vertices u, v that have m circles passing through them. For every circle there is a point $p \in P$ that is responsible for it. Let the points be $p_1, \ldots, p_m$. It is possible that two of the p_i's have the same distance to u, but no more than that. Hence there are $\geq m/2$ different distances. Hence $m/2 \leq t$, so $m = O(t)$.

4) The crossing number: Two circles intersect in at most 2 points. Hence the crossing numbers is bounded above by the square of the number of circles. The number of circles: each point causes $\leq t$ circles, hence there are $\leq nt$ circles. Therefore the crossing number is $c \leq O(n^2 t^2)$. ∎

So what to do? We need a relation between the crossing number of a multigraph and the number of vertices, edges, and multiplicity.

3 The Crossing Lemma for Graphs

We first prove a lower bound on the crossing number for graphs. We then use this to get a lower bound on the crossing number for multigraphs.

The following is well known and easy to find, so we will not prove it.

Lemma 3.1 *If $G = (V, E)$ is a planar graph with v vertices and e edges then $e \le 3v - 6$.*

Def 3.2 Let G be a graph. The *crossing number* of G is the minimal number of non-vertex crossings that the graph can be drawn with. We often denote the crossing number by c. Note that a planar graph has crossing number 0.

Lemma 3.3 *If $G = (V, E)$ is a graph with v vertices, e edges, and crossing number c then $c \ge e - 3v$.*

Proof:

First draw the graph in the plane with c non-vertex crossings. Remove the edges that cause the crossings one at a time until the graph is planar. The new graph G' has v vertices and $e - c$ edges. By the prior lemma

$$e - c \le 3v - 6$$

$$e \le 3v + c - 6.$$

$$c \ge e - 3v + 6 \ge e - 3v.$$

∎

We will get a much better lower bound on c. This result, called *The Crossing Lemma*, was proven independently by Ajtai, Chvátal, Newborn, Szemerédi [1] and Leighton [8].

Lemma 3.4 *Let $G = (V, E)$ be a graph with v vertices's, e edges, and crossing number c. If $e \ge 4v$ then $c \ge \Omega(\frac{e^3}{v^2})$.*

Proof: Let p be a probability that we will set later. For every vertex in the graph remove it with probability $1 - p$. Let the resulting graph be $G = (V', E')$. We denote the number of vertices by v', the number of edges by e', and the crossing number by c'.

$E(v') = vp$ since we retain each edge with probability p.

$E(e') = ep^2$ since we need to retain both of the endpoints to retain the edge.

$E(c') \leq cp^4$ since if you retain all four vertices then you might retain the crossing, but if you lose any one of them then you won't.

By Lemma 3.3 we have

$$c' \geq e' - 3v'.$$

By the linearity of expectation we have

$$E(c') \geq E(e') - 3E(v')$$

Combining this with what we already know about $E(v')$, $E(e')$ and $E(c')$ we obtain

$$cp^4 \geq E(c') \geq ep^2 - 3vp.$$

$$c \geq \frac{e}{p^2} - \frac{3v}{p^3}.$$

Set $p = 4v/e$ (this is where we use $e > 4v$).

Then we get

$$c \geq \frac{e^3}{64v^2} = \Omega\left(\frac{e^3}{v^2}\right).$$

∎

Note 3.5 The hypothesis $e \geq 4v$ of Lemma 3.4 can be weakened to $e \geq (3 + \epsilon)v$ for any $\epsilon > 0$.

Note 3.6 The above proof gives $c \geq \frac{e^3}{64v^2} \sim 0.0156\frac{e^3}{v^2}$. The best result known to date is by Pach, Radoicic, Tardos, and Toth [11, 12] who have that if $e \geq 7n$ then $c \geq 0.032\frac{e^3}{v^2}$. It is know that there are an infinite number of n such that there is a graph on n vertices with graphs with $e \geq 7n$ and $c \leq 0.09\frac{e^3}{v^2}$.

4 The Crossing Lemma for Multigraphs

The following lemma is due to Székely [13].

Lemma 4.1 *Let $G = (V, E)$ be a multigraph with v vertices, e edges, multiplicity m and crossing number c. If $e \geq 9mn$ then $c \geq \Omega(e^3/mv^2)$.*

Proof:

Some of the edges have high multiplicity which leads to lots of crossings (good), but also lots of edges (bad). Some of the edges have low multiplicity (bad), but also not that many edges (good). How to balance the two?

We assume that m is a power of 2. For $0 \le i \le \lg m$ let $G_i = (V, E_i)$ be the multigraph that only uses those edges that have multiplicity in $[2^i, 2^{i+1})$. For example, if between u and v there are 10 edges then in G_3 there are 10 edges between u and v but in any other G_i there are no edges between u and v. Let $e_i = |E_i|$ and let $c_i = c(G_i)$. Note that G_i has v vertices.

It is easy to see that

$$c(G) \ge \sum_{i=0}^{\lg m} c(G_i).$$

We will now estimate $c(G_i)$. This will be easier than estimating $c(G)$ since we have much more information about the multiplicities.

Let $G_i^* = (V, E_i^*)$ be formed by collapsing all of the multiedges of G_i into edges. Let $e_i^* = |E_i^*|$ and $c_i^* = c(G_i^*)$. The following are easy to see:

Fact 4.2

1. *$e_i^* \ge e_i/2^{i+1}$ (since the multiplicity of G_i is $\le 2^{i+1}$).*
2. *$c_i \ge 2^{2i} c_i^*$ (since the multiplicity of G_i is $\ge 2^i$).*

We would like to apply Lemma 3.4 to the graph G_i^*. However, to do this we would need $e_i^* \ge 4v$. This could easily not be the case.

Note that by Fact 4.2

$$e_i^* < 4v \implies e_i/2^{i+1} < 4v \implies e_i < 2^{i+3}v.$$

Hence

$$e_i \ge 2^{i+3}v \implies e_i^* \ge 4v.$$

Let

$$\begin{aligned} A = & \quad \{i : e_i \ge 2^{i+3}v\} \\ B = & \quad \{i : e_i < 2^{i+3}v\} \end{aligned}$$

We will only deal with $i \in A$ for which we clearly have $e_i^* \ge 4v$ and hence can apply Lemma 3.4.

We will need $\sum_{i \in A} e_i^* \ge e/9$ (any constant would work). Note that

$$e = \sum_{i \in A} e_i + \sum_{i \in B} e_i.$$

By the definition of B we have

$$\sum_{i\in B} e_i \le \sum_{i\in B} 2^{i+3}v \le 8v\sum_{i\in B} 2^i \le 8v\sum_{i=0}^{\lg m} 2^i \le 8nm.$$

Since $e \ge 9nm$ we have

$$\sum_{i\in B} e_i \le 8nm \le 8e/9.$$

Hence

$$\sum_{i\in A} e_i = e - \sum_{i\in B} e_i \ge e - 8e/9 = e/9 = \Omega(e) = \Omega(n^2).$$

Let $i \in A$. Then, by Lemma 3.4.

$$c_i^* \ge \Omega\left(\frac{(e_i^*)^3}{v^2}\right).$$

By Fact 4.2 we get

$$\frac{c_i}{2^{2i}} \ge c_i^* \ge \Omega\left(\frac{(e_i^*)^3}{v^2}\right) \ge \Omega\left(\frac{e_i^3}{v^2 2^{3i}}\right).$$

$$c_i \ge \Omega\left(\frac{e_i^3}{v^2 2^i}\right).$$

Hence we have

$$c(G) \ge \sum_{i\in A}\Omega\left(\frac{e_i^3}{v^2 2^i}\right) \ge \Omega\left(\sum_{i\in A}\frac{e_i^3}{v^2 2^i}\right) \ge \frac{1}{v^2}\Omega\left(\sum_{i\in A}\frac{e_i^3}{2^i}\right) \ge \frac{1}{v^2}\Omega\left(\sum_{i\in A}\left(\frac{e_i}{2^{i/3}}\right)^3\right).$$

We need to lower bound

$$\sum_{i\in A}\left(\frac{e_i}{2^{i/3}}\right)^3.$$

Hence we want to lower bound the min this sum can achieve. We do not know what the e_i's, though we do know that $\sum_{i\in A} e_i \ge e/9$. We will use Hölder's inequality.

BILL- ask RJB how to do this easier.

Lemma 4.3 *(Hölder's Inequality) Let $x_1,\ldots,x_L,y_1,\ldots,y_L$ be nonnegative reals. Let p,q be such that $1/p+1/q=1$. Then*

$$\sum_{i=1}^{L} x_i y_i \le (\sum_{i=1}^{L} x_i^p)^{1/p}(\sum_{i=1}^{L} y_i^q)^{1/q}$$

Proof: See Appendix. ▮

We will use Hölder's inequality in the following form:

$$\sum_{i=1}^{L} x_i^p \geq \left(\frac{\sum_{i=1}^{L} x_i y_i}{(\sum_{i=1}^{L} y_i^q)^{1/q}} \right)^p$$

We will use it with $p = 3$, $q = 3/2$, and $i \in A$ instead of $i = 0$ to L.

$$\sum_{i \in A} x_i^3 \geq \left(\frac{\sum_{i \in A} x_i y_i}{(\sum_{i \in A} y_i^{3/2})^{2/3}} \right)^3$$

Recall that we want to minimize

$$\sum_{i \in A} \left(\frac{e_i}{2^{i/3}} \right)^3$$

with the constraint that $\sum_{i \in A} e_i \geq e/9$. Let $x_i = e_i/2^{i/3}$ and $y_i = 2^{i/3}$.

Note that

$$\sum_{i \in A} x_i y_i = \sum_{i \in A} e_i \geq e/9 = \Omega(e)$$

$$(y_i)^{3/2} = (2^{i/3})^{3/2} = 2^{i/2}$$

$$\sum_{i \in A} y_i^{3/2} = \sum_{i \in A} 2^{i/2} = O(2^{(\lg m)/2}) = O(m^{1/2})$$

$$(\sum_{i \in A} y_i^{3/2})^{2/3} = O(m^{1/3})$$

By Hölder's inequality we get

$$\sum_{i \in A} \left(\frac{e_i}{2^{i/3}} \right)^3 \geq \left(\frac{e}{m^{1/3}} \right)^3 \geq \Omega\left(\frac{e^3}{m} \right)$$

Recall that we had:

$$c(G) \geq \sum_{i \in A} \Omega\left(\frac{e_i^3}{v^2 2^i} \right) \geq \Omega\left(\sum_{i \in A} \frac{e_i^3}{v^2 2^i} \right) \geq \frac{1}{v^2} \Omega\left(\sum_{i \in A} \frac{e_i^3}{2^i} \right) \geq \frac{1}{v^2} \Omega\left(\sum_{i \in A} \left(\frac{e_i}{2^{i/3}} \right)^3 \right).$$

We now have

$$\sum_{i \in A} \left(\frac{e_i}{2^{i/3}} \right)^3 \geq \left(\frac{e}{m^{1/3}} \right)^3 \geq \Omega\left(\frac{e^3}{m} \right).$$

Hence

$$c(G) \geq \Omega\left(\frac{e^3}{mv^2}\right).$$

▮

Note 4.4 Lemma 4.1 and its proof are from [13]. In that paper the Lemma uses the premise $e \geq 5mn$. This appears to be a typo in that paper.

5 $g'(n) \geq \Omega(n^{2/3})$

Theorem 5.1 *$g'(n) \geq \Omega(n^{2/3})$. Hence $g(n) \geq \Omega(n^{2/3})$.*

Proof: Let $G = G_P$. By Lemma 2.3

1. $v = n$.
2. $e = \Omega(n^2)$.
3. $m = O(t)$.
4. $c = O(n^2t^2)$.

By Lemma 4.1 and $c = O(n^2t^2)$ we have

$$n^2t^2 \geq c \geq \Omega\left(\frac{e^3}{mv^2}\right)$$

Hence

$$n^2t^2 \geq \Omega\left(\frac{e^3}{mv^2}\right) \geq \Omega\left(\frac{n^6}{tn^2}\right) = \Omega\left(\frac{n^4}{t}\right)$$

$$t^3 = \Omega(n^2)$$

$$t = \Omega(n^{2/3}).$$

▮

6 Motivation For How to Proceed

Look at Theorem 5.1. The GOOD NEWS is that we got $\Omega(n^{2/3})$. The BAD NEWS is that we didn't do any better than that.

How can we do better? We need to get a *better* handle on the multiplicities. How many edges really have that high a multiplicity? We will (much later) find a cutoff point k and remove from the graph all edges with multiplicity $\geq k$. That is, if there are $\geq k$ edges between u and v then they will *all* be removed. The resulting graph will still have $\Omega(n^2)$ edges but only have multiplicity $\leq k$.

Lets look more carefully at an edge of high multiplicity. We said (correctly) that if there is an edge of multiplicity m then there are $\Theta(m)$ points in P that cause those edges. Note that *those points are on the same line*!! See Figure 2.

INSERT A FIGURE.

They are all on the perpendicular bisector of the line from u to v. And there are at most $t+1$ of these points. This inspires us to look at a new problem:

Given a set P of n points in the plane, and a parameter k, how many incidences are there between those points and lines in

$$L = \{\ell : \ell \textit{ is incident to} \geq k \textit{ points of } P\ \}.$$

To get a good bound on this number we take a rather long detour. In particular, we prove the Szemerédi -Trotter Theorem. We already have one of the ingredients for that proof, namely the Crossing Lemma (Lemma 3.3).

7 The Szemerédi-Trotter Theorem

If you have a set of points P, and a set of lines L, how many times do a point and a line meet? They could of course meet 0 times. What is the maximum amount of times they could meet?

Def 7.1 Let P be a set of points and L be a set of lines. An *incidence of P and L* is a pair $(p, \ell) \in P \times L$ such that point p is on line ℓ. Let

$$I_{P,L} = \{(p, \ell) : p \in P, \ell \in L \text{ and } p \text{ is on } \ell\}.$$

We will leave out the subscripts if they are understood.

We will prove the following theorem:

$$|I| = O(|P| + |L| + (|L||P|)^{2/3}).$$

This was first proven by Szemerédi and Trotter [14]. Different proofs can be found in [4] and [10]. We present the simplest known proof, due to Székely [13].

Theorem 7.2 *For any set of P points and L lines in the plane,*

$$|I| \leq O(|P| + |L| + (|L||P|)^{2/3}).$$

Proof:

Define a graph $G = (V, E)$ as follows:
$V = P$, the set of points.
$E = \{(x, y) : x$ and y are both on some line $\ell \in L$ and are adjacent $\}$.

Let $v = |V|$ and $e = |E|$. It is easy to see that $v = P$ The number of edges is harder to determine. Let the lines be $\ell_1, \ell_2, \ldots, \ell_L$. Assume that ℓ_i has p_i points of P on it. Then ℓ_i is responsible for $p_i - 1$ edges. Hence the total number of edges is

$$\sum_{i=1}^{|L|}(p_i - 1) = (\sum_{i=1}^{|L|} p_i) - |L| = |I| - |L|.$$

Hence

$$e = |I| - |L|.$$

Look at the natural way to draw the graph— placing the points where they are naturally. Where there is a crossing you must have two of the lines intersecting. Hence there are at most $|L|^2$ crossings. Hence

$$c \leq |L|^2.$$

We want to apply Lemma 3.4. However, for this we need $e \geq 4v$. But this might not be true. Hence we have two cases.

Case 1: $e < 4v$. Hence $|I| - |L| \leq 4|P|$, so $|I| \leq 4|P| + |L| = O(|P| + |L| + (|L||P|)^{2/3})$.

Case 2: $e \geq 4v$. We apply Lemma 3.4 to obtain

$$|L|^2 \geq c \geq \Omega\left(\frac{e^3}{v^2}\right) = \Omega\left(\frac{(|I| - |L|)^3}{|P|^2}\right)$$

$$(|L||P|)^2 \geq \Omega((|I| - |L|)^3)$$

$$(|L||P|)^{2/3} \geq \Omega(|I| - |L|)$$

$$|I| \leq O((|L||P|)^{2/3} + |L|) \leq O(|P| + |L| + (|L||P|)^{2/3}).$$

∎

Note 7.3 The best known upper and lower bounds on I are due to Pach, Radoicic, Tardos, and Toth [11], [12]. They are

$$0.42(|L||P|)^{2/3} + |L| + |P| \leq |I| \leq 2.5(|L||P|)^{2/3} + |L| + |P|.$$

8 Corollaries of the Szemerédi-Trotter Theorem

We state two corollaries of The Szemerédi-Trotter Theorem. We call them Lemmas since we will use them to prove lower bounds on $g(n)$.

BILL- CHECK IF YOU USE THIS LEMMA FOR THE $n^{0.85}$ RESULT MUCH LATER.

Lemma 8.1 *Let P be a set of points and let $k \in \mathsf{N}$. (We assume k is bigger than any constant we may encounter.) Let*

$$L = \{\ell : \ell \text{ has at least } k \text{ points from } P \text{ on it}\}.$$

Then

$$|L| = O\left(\max\left\{\frac{|P|}{k}, \frac{|P|^2}{k^3}\right\}\right).$$

Proof: Note that the number of incidences of P and L is at least $k|L|$. Hence

$$|I| \geq k|L|.$$

Using this and Theorem 7.2 to P and L to obtain

$$k|L| \leq |I| \leq O(|P| + |L| + (|L||P|)^{2/3}).$$

There are two cases.

Case 1: $(|L||P|)^{2/3} \leq |P| + |L|$.

$$k|L| \leq O(|P| + |L|) \leq O(|P|) + O(|L|)$$

$$|L| \leq O\left(\frac{|P|}{k}\right).$$

Case 2: $|P| + |L| \leq (|L||P|)^{2/3}$.

$$k|L| \leq O((|L||P|)^{2/3})$$

$$k|L|^{1/3} \leq O(|P|^{2/3})$$

$$k^3|L| \leq O(|P|^2)$$

$$|L| \leq O\left(\frac{|P|^2}{k^3}\right)$$

Combining the two cases yields

$$|L| = O\left(\max\left\{\frac{|P|}{k}, \frac{|P|^2}{k^3}\right\}\right).$$

▮

We now want to bound $|I|$, the number of incidences between P and L. Theorem 7.2 does not give us a good upper bound on $|I|$. It implies that $|I|$ is bounded above by the max of the following two quantities.

- $O(|P| + |P|/k + (|P|^2/k)^{2/3})$
- $(|P| + |P|^2/k^3 + (|P|^3/k^3)^{2/3}) = O(|P| + |P|^2/k^3 + |P|^2/k^2) = O(|P| + |P|^2/k^2)$

The second bound is good enough for our later purpose; however, the first one is not. Hence we need and obtain a better bound.

Corollary 8.2 *Let P be a set of points and let $k \in \mathsf{N}$. (We assume k is bigger than any constant we encounter.) Let*

$$L = \{\ell : \ell \textit{ has at least } k \textit{ points from } P \textit{ on it}\}.$$

Let I be the set of incidences between P and L. Then

$$|I| \le O\left(|P| \log |P| + \frac{|P|^2}{k^2}\right).$$

Proof: We encounter the same problem, and use the same solution, as in the proof of Lemma 4.1. The problem is that some of the lines have a 'small' number of points of P on them, say roughly k, while others may have many, say roughly P. We will partition the lines into types we can more easily reason about.

We assume that $|P|$ is a power of 2.

If ℓ is a line then let $I(\ell)$ be the set of (number of) incidences between ℓ and P.

For $0 \le i \le L$ let

$$L_i = \{\ell : i \le I(\ell) < 2i\}.$$

$$I_i = \bigcup_{\ell \in L_i} I(\ell).$$

We are only concerned with $i = 2^0 k, 2^1 k, 2^2 k, \ldots, 2^{\lg |P|} k$. (Actually we are really only concerned with $2^0 k, \ldots, 2^{\lg |P| - \lg k} k$; however, this refinement will not help us.) In particular we will use

$$|I| \le \sum_{i=0}^{\lg |P|} |I_{2^i k}|.$$

However, we state and proof some facts about $|I_i|$, $|L_i|$ in general. We will later use these facts with $i = 2^0 k, \ldots, 2^{\lg |P|} k$.

- $i|L_i| \le |I_i| \le 2i|L_i|$ by the definition of $|L_i|$.
- $|I_i| \le O(|P| + |L_i| + (|P||L_i|)^{2/3})$ by Theorem 7.2.
- $i|L_i| \le O(|P| + |L_i| + (|L||P|_i)^{2/3})$ by the last two items.

We bound $|L_i|$ and then later plug in $2^i k$ for i. By the last item we have

$$|L_i| \le O\left(\frac{|P|}{i} + \frac{(|L||P_i|)^{2/3}}{i}\right)$$

There are two cases;

Case 1: $\frac{|P|}{i} \ge \frac{(|L||P_i|)^{2/3}}{i}$

$$|L_i| \le O\left(\frac{|P|}{i}\right).$$

Case 2: $\frac{(|L||P_i|)^{2/3}}{i} \ge \frac{|P|}{i}$

$$|L_i| \le O\left(\frac{(|L||P_i|)^{2/3}}{i}\right).$$

$$|L_i|^{1/3} \le O\left(\frac{|P|^{2/3}}{i}\right).$$

$$|L_i| \le O\left(\frac{|P|^2}{i^3}\right).$$

We combine the two cases to get the more easily managed equation

$$|L_i| \le O\left(\frac{|P|}{i} + \frac{|P|^2}{i^3}\right).$$

Recall that $|I|_i \le 2i|L_i|$. Hence

$$|I_i| \le 2i|L_i| \le O\left(|P| + \frac{|P|^2}{i^2}\right).$$

Our concern is

$$|I_{2^i k}| \le O\left(|P| + \frac{|P|^2}{2^{2i}k^2}\right).$$

$$|I| \le \sum_{i=0}^{\lg |P|} |I_{2^i k}| \le \sum_{i=0}^{\lg |P|} O\left(|P| + \frac{|P|^2}{2^{2i}k^2}\right) \le O(|P| \log |P|) + O\left(\sum_{i=0}^{\lg |P|} \frac{|P|^2}{2^{2i}k^2}\right)$$

$$\leq O(|P|\log|P|) + O\left(\frac{|P|^2}{k^2}\sum_{i=0}^{\lg|P|}\frac{1}{2^{2i}}\right) \leq O\left(|P|\log|P| + \frac{|P|^2}{k^2}\right).$$

∎

9 $g'(n) = \Omega(n^{0.8})$

Theorem 9.1 *$g'(n) \geq \Omega(n^{4/5})$. Hence $g(n) \geq \Omega(n^{4/5})$.*

Proof: Let $G = G_P$. By Lemma 2.3

1. $v = n$.
2. $e = \Omega(n^2)$.
3. $m = O(t)$.
4. $c = O(n^2t^2)$.

As we saw in the proof of Theorem 5.1 the above facts can be used to show $g(n) \geq n^{2/3}$. We need a more careful argument to obtain $g(n) \geq n^{4/5}$.

Let k be a parameter to be chosen later. We will are going to *remove* all edges of multiplicity $\geq k$ to create a new graph G'. G' has vertex set $V = P$ and all the edges of multiplicity $\leq k$. Let e' be the number of edges in G' and c' be the crossing number of G'. Note that

$$c' = \Omega\left(\frac{(e')^3}{kn^2}\right).$$

We want c' to be large. We have two conflicting requirements on k

- We want k small so that c' has a small denominator and hence is large.
- We want k large so that e' is large so that c' has a large numerator and hence is large.

What we really need to know is, how many edges have high multiplicity?

Claim 1: The number of edges of multiplicity $\geq k$ is at most

$$O\left(tn\log n + \frac{tn^2}{k^2}\right).$$

Proof of Claim 1: Let

$$L = \{\ell : \ell \text{ has at least } k \text{ points from } P \text{ on it}\}.$$

Let I be the set of incidences between P and L. By Corollary 8.2

$$|I| \le O\left(n\log n + \frac{n^2}{k^2}\right).$$

Let (u,v) be an edge of multiplicity $\ge k$. It exists because of the incidence of a point of P on the the perpendicular bisector of (u,v). Map (u,v) to the incidence of that point on that line. See Figure 3.

INSERT A FIGURE

Note that that line has $\ge k$ points of P on it, so its in I. Hence we are mapping edges of multiplicity $\ge k$ to elements of I. (Note that I is a set of *incidences*, not a set of points.) An element of I can be mapped to at most t times. Hence the number of edges of multiplicity $\ge k$ is at most

$$t|I| \le O\left(tn\log n + \frac{tn^2}{k^2}\right).$$

End of Proof of Claim 1

Let G' be the graph G with all of the edges of multiplicity $\ge k$ removed. Let e' be the number of edges in G' and c' be the crossing number of G'. Note that G' has n vertices. We want to pick a value of k such that $e' \ge \Omega(n^2)$.

By Claim 1 there is a constant b so that we have removed no more than

$$\le btn\log n + \frac{btn^2}{k^2} \text{ edges .}$$

Recall that $e = \Theta(n^2)$. Let a be a constant such that $e \ge an^2$. Hence we have

$$e' \ge an^2 - btn\log n - b\frac{tn^2}{k^2}$$

Recall that $t \le n^{0.9}$. Hence we can ignore the $btn\log n$ term by lowering the a just a little (we do not bother to rename a).

$$e' \ge an^2 - \frac{btn^2}{k^2}.$$

To get $e' = \Omega(n^2)$ it will suffice to have

$$k \ge \sqrt{\frac{2bt}{a}}.$$

Recall that we had two goals: keep e' large and k small. Hence it makes sense to take $k = \left\lceil \sqrt{\frac{2bt}{a}} \right\rceil$. With this value of k we have $e' = \Omega(n^2)$ and $k = \Omega(t^{1/2})$. We would like to apply 4.1. Recall that one of the premises was $e \ge 9mv$. Since we have $e = \Omega(n^2)$, $v = n$, and $m \le O(t^{1/2})$. Recall that we assume $t = n^{0.9}$, hence $m \le O(n^{0.45})$. Hence $9mv = O(n^{1.45})$. Clearly $e \ge 9mv$.

By Lemma 4.1 we have the following.

$$c' \geq \Omega\left(\frac{e'^3}{kn^2}\right) = \Omega\left(\frac{(n^2)^3}{t^{1/2}n^2}\right) = \Omega\left(\frac{n^6}{t^{1/2}n^2}\right) = \Omega\left(\frac{n^4}{t^{1/2}}\right).$$

Recall that we also have $c' \leq n^2t^2$, so we have

$$n^2t^2 \geq c' \geq \Omega\left(\frac{n^4}{t^{1/2}}\right).$$

$$t^{2.5} \geq \Omega(n^2)$$

$$t \geq \Omega(n^{2/2.5}) = \Omega(n^{0.8}).$$

∎

References

[1] M. Ajtai, V. Chvátal, M. Newborn, and E. Szemerédi. Crossing free subgraphs. *Annals of Discrete Mathematics*, 12:9–12, 1982.

[2] F. Chung. The number of different distances determined by n points in the plane. *Journal of Combinatorial Theory, Series A*, 36:342–354, 1984. `http://math.ucsd.edu/~fan/` or `http://www.cs.umd.edu/~/erdos\_dist/erdos\_dist.html`.

[3] F. Chung, E. Szemerédi, and W. Trotter. The number of different distances determined by a set of points in the Euclidean plane. *Discrete & Computational Geometry*, 7:1–11, 1992. `http://math.ucsd.edu/~fan/` or `http://www.cs.umd.edu/~/erdos\_dist/erdos\_dist.html`.

[4] K. Clarkson, H. Edelsbrunner, L. Guibas, M. Sharir, and E. Welzl. Combinatorial complexity bounds for arrangements of curves and spheres. *Discrete & Computational Geometry*, 5:99–160, 1990.

[5] P. Erdös. On sets of distances of n points. *The American Mathematical Monthly*, 53:248–250, 1946. `http://www.cs.umd.edu/~/erdos\_dist/erdos\_dist.html` or `http://www.jstor.org/`.

[6] J.Solymosi and C.D.Toth. Distinct distances in the plane. *Discrete & Computational Geometry*, 25:629–634, 2001. `http://www.cs.umd.edu/~/erdos\_dist/erdos\_dist.html` or `http://citeseer.ist.psu.edu/`.

[7] N. Katz and G. Tardos. A new entropy inequality for the Erdös distance problem. In *Towards a theory of Geometric Graphs*, volume 342 of *Contemporary Mathematics*. American Mathematical Society, 2004. `http://www.renyi.hu/~tardos/` or `http://www.cs.umd.edu/~/erdos\_dist/erdos\_dist.html`.

[8] F. T. Leighton. *Complexity issues in VLSI*. MIT Press, 1983.

[9] L. Moser. On the different distances determined by n points. *The American Mathematical Monthly*, 59:85–91, 1952. `http://www.cs.umd.edu/~/erdos\_dist/erdos\_dist.html` or `http://www.jstor.org/`.

[10] J. Pach and P. Agarwal. *Combinatorial Geometry*. Wiley, 1995.

[11] J. Pach, R. Radoicic, G. Tardos, and T. Toth. Improving the Crossing Lemma by finding more crossings in sparse graphs. *Discrete & Computational Geometry*, pages 527–552, 2006. Earlier version in 20th Annual Symposium on Comp. Geom. Current version at `http://www.renyi.hu/~tardos/`.

[12] J. Pach and G. Toth. Graphs drawn with few crossings. *Combinatorica*, 17, 1997.

[13] L. Székely. Crossing numbers and hard Erdös problems in discrete geometry. *Combinatorics, Probability and Computing*, 11:1–10, 1993. `http://www.cs.umd.edu/~/erdos\_dist/erdos\_dist.html`.

[14] E. Szemerédi and W. Trotter. Extremal problems in discrete geometry. *Combinatorica*, 3:381–392, 1983.

[15] G. Tardos. On distinct distances and distinct distances. *Advances in Mathematics*, 180:275–289, 2003. `http://www.renyi.hu/~tardos/` or `http://www.cs.umd.edu/~/erdos\_dist/erdos\_dist.html`.