Distinct Distances In Homogeneous Sets

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ABSTRACT

We show that the number of distinct distances in a well-distributed set of n points in $\mathbb{R}^d$ is $\Omega(n^{2/d-1/d^2})$ which is not far from the best known upper bound $O(n^{2/d})$.

Categories and Subject Descriptors

G.2.1 [**Mathematics of Computing**]: Discrete Mathematics—*Combinatorics*

General Terms

Theory

Keywords

distance sets

1. INTRODUCTION

One of the most famous questions of Erdős in discrete geometry is the following [3]: what is the minimum number of distinct distances determined by n points in $\mathbb{R}^d$? The question was posed in 1946. Today, 57 years later, even the planar case ($d = 2$) is still open. (see in [6] and [5]) Improving an result of Clarkson, Edelsbrunner, Gubias, Sharir and Welzl [2], very recently Aronov, Pach, Sharir, and Tardos [1] showed that the number of distinct distances determined by a set of n points in three-dimensional space is $\Omega(n^{77/141-\epsilon})$, for any $\epsilon > 0$. They conjectured that the lower bound should be close to $\Omega(n^{2/3})$ which is the best known upper bound given by the vertices of an $n^{1/3} \times n^{1/3} \times n^{1/3}$ integer lattice. As the first challenge they posed the problem of proving a better lower bound $\Omega(n^{5/9})$. In this paper we prove a general bound $\Omega(n^{2/d-1/d^2})$ for homogeneous sets in $\mathbb{R}^d$, for any $d \geq 2$. For the special case $d = 3$, our bound is $\Omega(n^{5/9})$. Homogeneous sets are interesting for at least two reasons. First, the only known point sets providing small distance sets are homogeneous. Second, homogeneous sets play an important role in analysis (see [4], for instance). Prior to our result, the best bound for homogeneous sets was due to Iosevich, who [4] showed that the number of distinct distances determined by a homogeneous set of n points in $\mathbb{R}^d$ is $\Omega(n^{3/2d})$.

SoCG'03, June 8–10, 2003, San Diego, California, USA.

2. THE RESULT

Let n be a large positive integer and C be the cube of volume n centered at the origin in $\mathbb{R}^d$, whose edges are parallel to the axis. Let A be a set of n points contained in the interior of C. We say that A is *homogeneous* if any unit cube in $\mathbb{R}^d$ contains only $O(1)$ points from A. We denote by Dist (A) the set of different distances between two points in A.

THEOREM 1. *For a set A as above, Dist (A) has cardinality $\Omega(n^{2/d-1/d^2})$.*

The exponent $2/d - 1/d^2$ is near best possible, as one can construct a homogenous set where Dist $(A) = O(n^{2/d})$ (the lattice, for example). The asymptotic notation is used under the assumption that $n \to \infty$; d, the dimension, is a constant.

Before presenting the proof, let us mention that our method also works without the homogeneity assumption. Using this method and some additional arguments, we obtained new bounds for the number of distinct distances of an arbitrary set. Details will appear in a subsequent paper.

3. PROOF OF THEOREM 1

In the following, r is a parameter to be determined. We call a triangle r-small if its longest edge has length at most r. Let N_r be the number of r-small triangles with vertices in A. This quantity will play a crucial role in our proof. We shall estimate N_r from both above and below and the bound in Theorem 1 will follow as a consequence of these estimates, given an appropriate choice of r.

LEMMA 2. *We have $N_r = O(nr^{2d})$, for any $r \geq 1$.*

Proof of Lemma 2. Let $\mathcal{P}(x)$ be the canonical partition of $\mathbb{R}^d$ into the union of cubes of length x. We use the following simple observation. There is a constant $l = l(d)$ such that there are l translations of $\mathcal{P}(x)$, $\mathcal{P}_1(x), \dots, \mathcal{P}_l(x)$ such that for any ball B of radius $x/8$ there is a cube in $\mathcal{P}_i$ containing B, for some $1 \leq i \leq l$.

For our purpose, choose $x = 8r$. Since any r-small triangle is contained in a ball of radius r, the above observation then implies that

$$N_r \leq N_{1,r} + \cdots + N_{l,r} \tag{1}$$

where $N_{i,r}$ denotes the number of r-small triangles whose vertices are in some cube of $\mathcal{P}_i(8r)$. Fix an i between 1 and l, the volume of the cubes in $\mathcal{P}_i$ is r^d; moreover, these cubes are disjoint, so there are only $O(n/r^d)$ cubes of $\mathcal{P}_i$ with a non-empty intersection with A. Furthermore, as A is homogeneous, each cube contains $O(r^d)$ points from A. So, the number of triangles in each cube is $O(r^{3d})$. Thus for each i, $N_{i,r} = O(\frac{n}{r^d} r^{3d}) = O(nr^{2d})$. As the number of i's is a constant, our proof is complete. □

LEMMA 3. *For any fixed d there are positive constants c and b such that the following holds. Let X be a set of m points on a sphere with surface area S in $\mathbb{R}^d$. Assume that r satisfies $cr^{d-1}m \geq S$. Then X contains at least bm r-small triangles.*

Proof of Lemma 3. Define a graph G on X as follows: Two vertices of X are adjacent if their distance is at most $r/2$. By the triangle inequality, the vertices of any path of length 2 of G form an r-small triangle. Moreover, any connected graph H contains at least $|V(H)| - 2$ different paths of length 2, so any connected component H of G gives rise to at least $|V(H)| - 2$ r-small triangles. It now suffices to show that the connected components of size at least 3 in G contains at least half of X. Here and later $V(H)$ denotes the vertex set of H.

Observe that if H and H' are two different connected components, then the $r/(2d)$-neighborhoods of $V(H)$ and $V(H')$ are disjoint. It is clear that each neighborhood intersects the sphere in a region with area at least $a(d)r^{d-1}$, where $a(d)$ is a constant depending on d. We can conclude that there are at most $S/a(d)r^{d-1}$ mutually disjoint neighborhoods. Thus the number of vertices contained in connected components of size at most 2 is upper bounded by

$$2\frac{S}{a(d)r^{d-1}} \leq \frac{|X|}{2}, \tag{2}$$

provided that we set $c \leq a(d)/4$. □

Now we are in position to finish the proof. Assume that Dist $(A) = t$. Fix a point $v \in A$, there are t spheres centered at v such that every other vertex of A is on one of these spheres. We are going to count the number of r-small triangles with vertices on the same sphere. A sphere is *good* if it has at least $n/2t$ points from A. At least half of the points in A are contained on the good spheres and from now on we consider these spheres only.

Let S be the surface area of the sphere with radius equals the diameter of C. Choose r (as in the pervious lemma) such that $cr^{d-1}\frac{n}{2t} = S$. As the surface of any good sphere is at most S, Lemma 3 implies that on any good sphere B, there are $\Omega(|A \cap B|)$ r-small triangles. Therefore, the good spheres centered at v contain at least $\Omega(|A|)$ r-small triangles. Repeating this estimate with respect to the spheres centered at the other vertices of A, we obtain altogether $\Omega(|A|^2) = \Omega(n^2)$ r-small triangles. This is not yet a lower bound on N_r as some of these triangles are the same. On the other hand, the multiplicities are not too large, thanks to the homogeneity of the set A. Indeed, the multiplicity of a triangle T is the number of points of A which have the same distance to the three vertices of T. If the three vertices of T are co-linear, there is no point with equal distance to these vertices. Otherwise, the points with equal distance to the vertices of T lie in the intersection of a hyperplane of codimension 2 and C. As C has volume n, this intersection can be covered by $O(n^{(d-2)/d})$ unit hypercubes. By the homogeneous assumption, it follows that the multiplicity of T is $O(n^{(d-2)/d})$. So we can conclude that the number of small triangles is at least $\Omega(\frac{n^2}{n^{(d-2)/d}}) = \Omega(n^{(d+2)/d})$.

Together with Lemma 2, we have the following double inequality

$$O(nr^{2d}) = N_r = \Omega(n^{(d+2)/d}), \tag{3}$$

which implies

$$r = \Omega(n^{1/d^2}). \tag{4}$$

On the other hand, we choose r such that $r^{d-1}\frac{n}{2t} = \Theta(S)$, where $S = \Omega(n^{(d-1)/d})$ is the surface area of the sphere with radius equals the diameter of C (which is $d^{1/2}n^{1/d}$). This implies $t = \Theta(r^{d-1}n^{1/d})$. Together with (4), we have

$$t = \Theta(r^{d-1}n^{1/d}) = \Omega(n^{(d-1)/d^2}n^{1/d}) = n^{2/d-1/d^2}, \tag{5}$$

concluding the proof. □

4. ACKNOWLEDGMENTS

We would like to thank Gábor Tardos for pointing out an error in an earlier version. The work of the first author was partially supported by grants OTKA T043631 and T032452. The second author is supported by an A. Sloan fellowship and by NSF grant DMS-0200357.

5. REFERENCES

[1] Aronov, Pach, Sharir, and Tardos. Distinct distances in three and higher dimensions. Manuscript, 2003.

[2] Clarkson, Edelsbrunner, Guibas, Sharir, and Welzl. Combinatorial complexity bounds for arrangements of curves and spheres. *Discrete and Computational Geometry*, 5, 1990.

[3] Erdős. On sets of distances of n points. *Amer. Math. Monthly*, 53:248–250, 1946.

[4] Iosevich. Curvature, combinatorics, and the fourier transform. *Notices of the American Mathematical Society*, 48, 2001.

[5] Solymosi, Tardos, and Toth. The k most frequent distances in the plane. *Discrete and Computational Geometry*, 28, 2002.

[6] Solymosi and Toth. Distinct distances in the plane. *Discrete and Computational Geometry*, 25, 2001.