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NEAR OPTIMAL BOUNDS FOR THE ERDŐS DISTINCT DISTANCES PROBLEM IN HIGH DIMENSIONS

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We show that the number of distinct distances in a set of n points in $\mathbb{R}^d$ is $\Omega(n^{\frac{2}{d}-\frac{2}{d(d+2)}})$, $d\geq 3$. Erdős' conjecture is $\Omega(n^{2/d})$.

1. Introduction

One of the most famous and important problems in discrete geometry is the following question, posed by Erdős [7,1]:

What is the minimum number of distinct distances
determined by n points in $\mathbb{R}^d$?

Given a finite point set A, let $g(A)$ denote the number of distinct distances between the elements of A. Define $g_d(n)=\min_{A\subset\mathbb{R}^d,|A|=n} g(A)$. Erdős' question is to estimate $g_d(n)$. To this end, d is a constant and n is sufficiently large. The asymptotic notation is used under the assumption that $n\to\infty$.

To find an upper bound for $g_d(n)$, let us consider the following natural construction. Let A be the set of integral lattice points $(x_1,\dots,x_d)$ where $1\leq x_i\leq n^{1/d}$, assuming that $n^{1/d}$ is an integer. The distance between any two points in A is the square root of a positive integer less than $dn^{2/d}$. This shows that $g_d(n)=O(n^{2/d})$. Erdős and many other researchers conjecture that $g_d(n)$ is close to this upper bound.

Research on this problem has led to various new methods and concepts which are very useful for many other problems in discrete and computational

Mathematics Subject Classification (2000): 52C10 geometry. The monograph by Agarwal and Pach [1] is an excellent place to read about these developments. In the last few years, an intersting link was found between the Erdős distance problem and problems in analysis. The reader who is interested in this new direction is referred to a recent survey by Iosevich [8].

Let us now give a brief account about previous lower bounds of $g_d(n)$. Erdős proved, in 1946, that $g_2(n) = \Omega(n^{1/2})$ [7]. It is easy to show, using a variant of his argument that $g_d(n) = \Omega(n^{1/d})$, for all $d \geq 1$. There is a series of improvements for the case $d = 2$, due to Moser [11], Chung [4], Chung–Szemerédi–Trotter [5], Székely [14], Solymosi–Tóth [12] and Tardos [15]. The most current bound is $g_2(n) = \Omega(n^{0.8635})$ [15]. Little has been known for $d \geq 3$. Clarkson, Edelsbrunner, Gubias, Sharir and Welzl [6] proved that $g_3(n) = \Omega(n^{1/2})$. Very recently, Aronov, Pach, Sharir and Tardos [2] proved that $g_3(n) = \Omega(n^{77/141-\epsilon})$ for any positive constant ϵ. More general, they proved that $g_d(n) = \Omega(n^{1/(d-90/77)-\epsilon})$ for any $d \geq 3$. This result gives a non-trivial improvement for small d, compared to the previous bound $n^{1/d}$. On the other hand, as d is getting large, the exponent $1/(d-90/77)-\epsilon$ converges to $1/d$, rather than to the conjectured bound $2/d$.

Our main goal in this paper is to prove that the exponent $2/d$ is essentially best possible, as it cannot be replaced by $(2-\epsilon)/d$ for any positive constant ϵ, given that d is sufficiently large. More precisely, we show that $g_d(n) = \Omega(n^{(2-\epsilon_d)/d})$, where $\epsilon_d = O(1/d)$ tends to 0 as d tends to infinity. Our bound improves the above mentioned result by Aronov et al. for every $d \geq 3$.

Theorem 1.1. *(a)* $g_3(n) = \Omega(n^{.5643})$.

(b) For any $d \geq 4$, $g_d(n) = \Omega\left(n^{\frac{2}{d} - \frac{2}{d(d+2)}}\right)$.

This theorem is a corollary of the following stronger result, which gives a recursive estimate for $g_d(n)$.

Theorem 1.2. *(a) If* $g_{d_0}(n) = \Omega(n^{\alpha_{d_0}})$, *then for all* $d \geq d_0$

$$g_d(n) = \Omega\left(n^{\frac{2d}{(d+d_0+1)(d-d_0)+2d_0/\alpha_{d_0}}}\right). \tag{1}$$

(b) If $g_{d_0}(n) = \Omega(n^{\alpha_{d_0}})$, *then for all* $d \geq d_0$, $d - d_0$ *even*

$$g_d(n) = \Omega\left(n^{\frac{2(d+1)}{(d+d_0+2)(d-d_0)+2(d_0+1)/\alpha_{d_0}}}\right). \tag{2}$$

Tardos result [15] asserts that one can set $\alpha_2 = .8635$. Applying (1) with $d_0 = 2$, $d = 3$ and $\alpha_2 = .8635$ gives $g_3(n) = \Omega(n^{.5643})$, proving part (a) of Theorem 1.1. This estimate improves the bound $\Omega(n^{77/141-\epsilon})$ by Aronov et al. as $77/141 < .5461$. This bound on $g_3(n)$ can be further improved to $n^{.566}$ using additional arguments. The details will appear later.

Part (b) of Theorem 1.2 implies:

Corollary 1.3. *For any even d,*

$$g_d(n) = \Omega\left(n^{\frac{2(d+1)}{d^2+2d-8+6/\alpha_2}}\right). \tag{3}$$

For any odd $d \geq 3$

$$g_d(n) = \Omega\left(n^{\frac{2(d+1)}{d^2+2d-15+8/\alpha_3}}\right). \tag{4}$$

As mentioned above, we can set $\alpha_2 = .8635$ and $\alpha_3 = .5643$. With these values, the exponents in Corollary 1.3 are larger than $\frac{2}{d} - \frac{2}{d(d+2)}$ in both cases. This proves part (b) of Theorem 1.1.

We would like to point out that for those d where $d - d_0$ is an even positive integer, the bound in part (b) of Theorem 1.2 is superior to the bound in part (a). We leave the details as an exercise.

Finally, let us mention that recently several variants of Erdős distance problem have been raised by analysts. The method developed in this paper helps us to obtain new results concerning these problems. The details will appear in a future paper.

The rest of the paper is organized as follows. In the next section, we present two recursive theorems and use them to obtain Theorem 1.2. The next section, Section 3, discusses a lemma that we need in the proof of these recursive theorems. The full proofs of these theorems follow in Sections 4 and 5, respectively. The final section, Section 6, is devoted to concluding remarks.

2. Recursions

For a finite set A we denote by $t(A)$ the maximum number of distinct distances measured from a point in A. Furthermore, define

$$t_d(n) = \min_{A \subset \mathbb{R}^d, |A| = d} t(A).$$

It is clear that $t_d(n) \leq g_d(n)$. Instead of lower bounding $g_d(n)$, we are going to bound $t_d(n)$ from below. All theorems and corollaries in this section hold, with the same proofs, if we replace $t_d(n)$ by $g_d(n)$.

Theorem 2.1. *Let A be a set of n points in $\mathbb{R}^d$ $(d\geq 3)$ and m be the maximum cardinality of the intersection of A with a hyperplane of co-dimension 1. Then*

$$t(A) = \Omega\Big(\max\Big\{\frac{n}{m^{(d-1)/d}}, t_{d-1}(m)\Big\}\Big). \tag{5}$$

Theorem 2.2. *Let A be a set of n points in $\mathbb{R}^d$ $(d\geq 3)$ and m be the maximum cardinality of the intersection of A with a hyperplane of co-dimension 2. Then*

$$t(A) = \Omega\left(\max\left\{\frac{n^{(d+1)/2d}}{m^{(d-1)/2d}}, t_{d-2}(m)\right\}\right). \tag{6}$$

2.1. A recursion using Theorem 2.1

In this subsection, we use Theorem 2.1 to obtain part (a) of Theorem 1.2. First, we can prove the following general result.

Corollary 2.3. *Let α be a positive constant such that $t_{d-1}(n) = \Omega(n^\alpha)$, then*

$$t_d(n) = \Omega\Big(n^{\frac{d\alpha}{d\alpha+(d-1)}}\Big). \tag{7}$$

Proof. Theorem 2.1 implies that

$$t_d(n) = \Omega\Big(\frac{n}{m^{(d-1)/d}} + t_{d-1}(m)\Big) = \Omega\Big(\frac{n}{m^{(d-1)/d}} + m^\alpha\Big). \tag{8}$$

Set $\theta = \frac{d\alpha}{d\alpha+(d-1)}$. By convexity,

$$\frac{n}{m^{(d-1)/d}} + m^\alpha \geq \Big(\frac{n}{m^{(d-1)/d}}\Big)^\theta (m^\alpha)^{1-\theta} = \Omega(n^\theta) = \Omega\Big(n^{\frac{d\alpha}{d\alpha+(d-1)}}\Big), \tag{9}$$

completing the proof. ∎

Corollary 2.3 gives rise to the following recursive estimate. Assume that for some $d_0\geq 1$ there is a constant α_{d_0} such that $t_{d_0}(n) = \Omega(n^{\alpha_{d_0}})$. Define

$$\alpha_d = \frac{d\alpha}{d\alpha_{d-1} + (d-1)} \tag{10}$$

for $d\geq d_0+1$.

Corollary 2.4. *With the above assumption and notation, we have*

$$t_d(n) = \Omega(n^{\alpha_d}). \tag{11}$$

We have an exact formula for α_d, given α_{d_0}.

Fact 2.5. *For any* $d \geq d_0$

$$\alpha_d = \frac{2d}{(d+d_0+1)(d-d_0)+2d_0/\alpha_{d_0}}. \tag{12}$$

Corollary 2.4 and Fact 2.5 imply statement (a) of Theorem 1.2.

Proof. Define $\gamma_d = 1/\alpha_d$; (10) implies

$$\gamma_d = 1 + \frac{d-1}{d}\gamma_{d-1}. \tag{13}$$

Using induction, it is easy to show that for any $d \geq d_0$

$$\gamma_d = \frac{(d+d_0+1)(d-d_0)}{2d} + \frac{d_0}{d}\gamma_{d_0}, \tag{14}$$

which is equivalent to (12). ∎

2.2. A recursion using Theorem 2.2

The arguments here are very similar to the arguments in the previous subsection. As an analogue of Corollary 2.3, we have:

Corollary 2.6. *Let* α *be a positive constant such that* $t_{d-2}(n) = \Omega(n^\alpha)$, *then*

$$t_d(n) = \Omega\left(n^{\frac{(d+1)\alpha}{2d\alpha+(d-1)}}\right). \tag{15}$$

Proof. Theorem 2.2 implies that

$$t_d(n) = \Omega\left(\frac{n^{(d+1)/2d}}{m^{(d-1)/2d}} + t_{d-2}(m)\right) = \Omega\left(\frac{n^{(d+1)/2d}}{m^{(d-1)/2d}} + m^\alpha\right). \tag{16}$$

Set $\theta = \frac{2d\alpha}{2d\alpha+(d-1)}$. By convexity,

$$\begin{aligned} \frac{n^{(d+1)/2d}}{m^{(d-1)/2d}} + m^\alpha \geq \left(\frac{n^{(d+1)/2d}}{m^{(d-1)/2d}}\right)^\theta (m^\alpha)^{1-\theta} &= \Omega(n^{\theta(d+1)/2d}) \\ &= \Omega\left(n^{\frac{(d+1)\alpha}{2d\alpha+(d-1)}}\right), \end{aligned} \tag{17}$$

completing the proof. ∎

Assume that for some $d_0 \geq 1$ there is a constant α_{d_0} such that $t_{d_0}(n) = \Omega(n^{\alpha_{d_0}})$. Define

$$\alpha_d = \frac{(d+1)\alpha_{d-2}}{2d\alpha_{d-2}+(d-1)}, \tag{18}$$

for $d = d_0+2, d_0+4$, etc.

Corollary 2.7. *With the above assumption and notation, we have*

$$t_d(n) = \Omega(n^{\alpha_d}). \tag{19}$$

For a fixed pair of d_0 and α_{d_0}, we can give an explicit formula for α_d.

Fact 2.8. *For any $d \geq d_0$ and $d - d_0$ even*

$$\alpha_d = \frac{2(d+1)}{(d+d_0+2)(d-d_0)+2(d_0+1)/\alpha_{d_0}}. \tag{20}$$

Proof. Define $\gamma_d = 1/\alpha_d$; (18) implies

$$\gamma_d = \frac{2d}{d+1} + \frac{d-1}{d+1}\gamma_{d-2}. \tag{21}$$

Using induction, it is easy to show that for any $d \geq d_0$ and $d - d_0$ even

$$\gamma_d = \frac{(d+d_0+2)(d-d_0)}{2(d+1)} + \frac{d_0+1}{d+1}\gamma_{d_0}, \tag{22}$$

which is equivalent to (20). ∎

Corollary 2.7 and Fact 2.8 together imply part (b) of Theorem 1.2.

3. Partition of spaces

In this section, we present a lemma which we shall need in the proofs of Theorems 2.1 and 2.2. The development of this lemma was motivated by practical problems in geometric searching. One of the main techniques for doing a search is divide-and-conquer. In many problems, the situation looks as follows: Given a set B of hyperplanes (of co-dimension 1) in $\mathbb{R}^d$, one would like to partition $\mathbb{R}^d$ in not too many parts so that each part intersects only few hyperplanes. The following lemma, due to Chazelle and Friedman [3] was discovered along these lines. The reader who is interested in this lemma and its applications is referred to Section 6 of Matousek's monograph [10], which contains a detailed discussion about this lemma and its origin.

Definition 3.1. A hyperplane H strongly intersects a set P if $H \cap P$ is not empty and P has a point on both side of H.

Lemma 3.2. *Let B be a set of k hyperplanes in $\mathbb{R}^d$. For any $1 \leq r \leq k$, one can partition $\mathbb{R}^d$ into r sets $P_1, \dots, P_r$ such that for each $1 \leq i \leq r$, there are only $O(k/r^{1/d})$ planes which strongly intersect P_i.*

The bound $O(k/r^{1/d})$ is best possible; the hidden constants in O depend on d but not on r. One can also guarantee that the sets P_i are generalized simplices. Strong intersection actually means intersection with the interior (see [10]), but this information is not important to our proofs. Let us now consider a little bit more complex situation when beside B we also have a set A of n points. We can require, in addition, that each part contains not too many points.

Lemma 3.3. *Let A be a set of n points and B be a set of k hyperplanes in $\mathbb{R}^d$. For any $1 \le r \le k$, one can partition $\mathbb{R}^d$ into r sets $P_1, \dots, P_r$ such that for each $1 \le i \le r$, $|P_i \cap A| \le 2n/r$ and P_i strongly intersects $O(k/r^{1/d})$ planes.*

Proof. Let us assume, without loss of generality, that r is even and $2n/r$ is an integer. Apply Lemma 3.2 with $r' = r/2$. If $|P_i \cap A| \le 2n/r$ for all $i = 1, \dots, r'$ then we are done. Otherwise, for each i where $|P_i \cap A| > 2n/r$, partition P_i into smaller parts so that all but at most one of them have exactly $2n/r$ points. The resulting finer partition has at most $r' + r/2 = r$ parts and each part satisfies the requirement of the lemma. ∎

Lemma 3.2 is not restricted to hyperplanes. It is known that this lemma still holds if we replace a family of hyperplanes by a family of surfaces satisfying certain topological conditions. In particular, the lemma holds if we replace hyperplanes by (full dimensional) spheres (see Section 6.5 of [10]). As an analogue of Lemma 3.3, we obtain the following lemma, which we shall use in the next proof.

Definition 3.4. A sphere S strongly intersects a set P if $S \cap P$ is not empty and P has a point on both side of S.

Lemma 3.5. *Let A be a set of n points and B be a set of k spheres in $\mathbb{R}^d$. For any $1 \le r \le k$, one can partition $\mathbb{R}^d$ into r sets $P_1, \dots, P_r$ such that for each $1 \le i \le r$, $|P_i \cap A| = O(n/r)$ and there are only $O(k/r^{1/d})$ spheres which strongly intersect P.*

4. Proof of Theorem 2.1

Since m is the maximum number of points of A on a hyperplane, there is a hyperplane of dimension $d-1$ containing m points of A and thus $t(A) \ge t_{d-1}(m)$. The non-trivial half of the bound is to show $t(A) = \Omega(\frac{n}{m^{(d-1)/d}})$.

Set $t = t(A)$. Since there are at most t distinct distances from v, all points in A (except v) are contained on t spheres $S_1(v), \dots, S_t(v)$ centered at v (we can add a few dummy spheres which contain no points from A). Together we have $k=nt$ spheres. We apply Lemma 3.5 to A and the collection of these k spheres. The sets $P_1,\dots,P_r$ in the partition will be referred to as *cells*.

We call a pair (u,v), $u\in A, v\in A$, *consistent* if u and v belong to the same cell. Let M_r denote the number of consistent pairs. We are going to estimate M_r from both above and below. The statement of the theorem will follow from these estimates, under a proper choice of r.

Since $|P_i\cap A|=O(n/r)$ for all $1\le i\le r$,

$$M_r = O\left(r\binom{n/r}{2}\right) = O\left(\frac{n^2}{r}\right). \tag{23}$$

To lower bound M_r, let us consider a point $v\in A$. If a cell P has a common point with $S_i(v)$ but does not intersect $S_i(v)$ strongly, then we say that it intersects $S_i(v)$ *weakly*. Let $s_i(v)$ be the number of cells intersecting $S_i(v)$ (either strongly or weakly).

Consider a sphere $S_i(v)$. Without loss of generality, we can assume that the cells intersecting $S_i(v)$ are $P_1,\dots,P_l$. Let $x_j=|P_j\cap S_i(v)|$, $1\le j\le l$. The number of consistent pairs on S_i is

$$\sum_{j=1}^{l}\binom{x_j}{2} \ge \sum_{x_j\ge 1}(x_j-1) = |A\cap S_i(v)| - s_i(v). \tag{24}$$

Summing the above estimate over all spheres $S_i(v)$ centered at v and then summing over all $v\in A$ give us

$$\sum_{v\in A}\sum_{S_i(v)}(|A\cap S_i(v)| - s_i(v))$$

consistent pairs. However, this is not yet an estimate for M_r, as a pair can be counted many times. Indeed, if the vertices of a pair are of the same distance from p points in A, then the pair is counted p times. The points which are at the same distance from the vertices of a pair lie on a hyperplane. We assume that a hyperplane contains at most m points from A, so the multiplicity of any pair is at most m. It follows that

$$M_r \ge \frac{1}{m}\sum_{v\in A}\sum_{S_i(v)}(|A\cap S_i(v)| - s_i(v)). \tag{25}$$

Now we are going to bound the right hand side of (25) from below. First of all, it is trivial that for any $v\in A$

$$\sum_{S_i(v)}|A\cap S_i(v)| = |A\setminus\{v\}| = n-1,$$

so

$$\sum_{v\in A}\sum_{S_i(v)}|A\cap S_i(v)| = n(n-1). \tag{26}$$

To estimate $\sum_{v\in A}\sum_{S_i(v)} s_i(v)$, we split each $s_i(v)$ as the sum of two terms $s'_i(v)$ and $s''_i(v)$ which count the number of strong and weak intersections, respectively. It follows that

$$\sum_{v\in A}\sum_{i=1}^{t} s_i(v) = \sum_{v\in A}\sum_{i=1}^{t} s'_i(v) + \sum_{v\in A}\sum_{i=1}^{t} s''_i(v). \tag{27}$$

The sum $\sum_{v\in A}\sum_{i=1}^{t} s'_i(v)$ counts the total number of strong intersections between the spheres and the cells. Since there are r cells and for each cell there are only $O(k/r^{1/d})$ spheres strongly intersect it, it follows that

$$\sum_{v\in A}\sum_{i=1}^{t} s'_i(v) = O\Big(r\frac{k}{r^{1/d}}\Big) \le cntr^{(d-1)/d}, \tag{28}$$

for some constant c.

The sum $\sum_{v\in A}\sum_{i=1}^{t} s''_i(v)$ counts the total number of weak intersections between the spheres and the cells. To bound this number, notice that for a fixed point $v\in A$ and a fixed cell P, there are at most two spheres centered at v which intersect P weakly (if P weakly intersects S then either P is inside S or P is outside S). Thus we have

$$\sum_{v\in A}\sum_{i=1}^{t} s''_i(v) \le 2nr. \tag{29}$$

The estimates in (25–29) together imply that

$$\begin{aligned} M_r &\ge \frac{1}{m}\Big(\sum_{v\in A}\sum_{i=1}^{t}|A\cap S_i(v)| - cntr^{(d-1)/d} - 2nr\Big) \\ &= \frac{1}{m}\Big(n(n-1) - cntr^{(d-1)/d} - 2nr\Big). \end{aligned} \tag{30}$$

This, together with the upper bound (23), yields

$$\frac{n^2}{r} = \Omega\Big(\frac{1}{m}\Big(n(n-1) - cntr^{(d-1)/d} - 2nr\Big)\Big). \tag{31}$$

Let us choose $r=\epsilon(\frac{n}{t})^{d/(d-1)}$, where ϵ is a positive constant. Setting ϵ sufficiently small compared to $1/c$, we have that $cntr^{(d-1)/d} \le n^2/3$ and also that $2nr \le n^2/6$ (the second inequality is due to the fact that $t = \Omega(n^{1/d})$, mentioned in the introduction). So with this setting of r, we have

$$n(n-1) - cntr^{(d-1)/d} - 2nr \ge n(n-1) - n^2/2 \ge n^2/3.$$

So with this choice of r, (31) implies

$$\frac{n^2}{\epsilon(\frac{n}{t})^{d/(d-1)}} = \Omega\left(\frac{n^2}{m}\right). \tag{32}$$

It follows that

$$t^{d/(d-1)} = \Omega\left(\frac{n^{d/(d-1)}}{m}\right), \tag{33}$$

namely,

$$t = \Omega\left(\frac{n}{m^{(d-1)/d}}\right), \tag{34}$$

concluding the proof. ∎

5. Proof of Theorem 2.2

This proof, in spirit, is very similar to the previous one. The main (and only) difference here is that we now consider triplets instead of pairs. We only need to show that

$$t(A) = \Omega\left(\frac{n^{(d+1)/2d}}{m^{(d-1)/2d}}\right).$$

We call a triplet in A *consistent* if its three elements belong to the same cell. Let N_r denote the number of consistent triplets. Similar to the previous proof, we are going to estimate N_r from both above and below.

Since $|P_i \cap A| = O(n/r)$ for all $1 \le i \le r$,

$$N_r = O\left(r\binom{n/r}{3}\right) = O\left(\frac{n^3}{r^2}\right). \tag{35}$$

To lower bound N_r, again let us consider a point $v \in A$. Consider a sphere $S_i(v)$. Without loss of generality, we can assume that the cells intersecting $S_i(v)$ are $P_1, \ldots, P_l$. Let $x_j = |P_j \cap S_i(v)|$, $1 \le j \le l$. The number of consistent triplets on S_i is

$$\sum_{j=1}^{l} \binom{x_j}{3} \ge \sum_{j=1}^{l} (x_j - 2) = |A \cap S_i(v)| - 2s_i(v). \tag{36}$$

Summing the above estimate over all spheres $S_i(v)$'s and then summing over all $v \in A$ give us

$$\sum_{v\in A} \sum_{S_i(v)} (|A\cap S_i(v)| - 2s_i(v))$$

consistent triplets. Similar to the previous proof, this is not yet an estimate for N_r, as a triplet can be counted many times. Notice that if the three vertices of a consistent triplet T are colinear, then there is no point which is at the same distance from the vertices of T. Otherwise, the points which are at the same distance from the vertices of T lie on a hyperplane of co-dimension 2. By the assumption of the theorem, a hyperplane of co-dimension 2 contains at most m points from A, so the multiplicity of T is at most m. It follows that

$$N_r \geq \frac{1}{m} \sum_{v\in A} \sum_{S_i(v)} (|A\cap S_i(v)| - 2s_i(v)). \tag{37}$$

The estimates in (25–29) from the previous proof imply that

$$\begin{aligned} N_r &\geq \frac{1}{m}\left(\sum_{v\in A} \sum_{i=1}^{t} |A\cap S_i(v)| - cntr^{(d-1)/d} - cnr\right) \\ &= \frac{1}{m}\left(n(n-1) - cntr^{(d-1)/d} - cnr\right), \end{aligned} \tag{38}$$

for some constant c. This, together with the upper bound (35), yields

$$\frac{n^3}{r^2} = \Omega\left(\frac{1}{m}\left(n(n-1) - cntr^{(d-1)/d} - cnr\right)\right). \tag{39}$$

We set r as before: $r = \epsilon(\frac{n}{t})^{d/(d-1)}$, where ϵ is a small positive constant. With this choice of r, (39) implies

$$\frac{n^3}{\epsilon^2(\frac{n}{t})^{2d/(d-1)}} = \Omega\left(\frac{n^2}{m}\right). \tag{40}$$

It follows that

$$t^{2d/(d-1)} = \Omega\left(\frac{n^{(d+1)/(d-1)}}{m}\right), \tag{41}$$

namely,

$$t = \Omega\left(\frac{n^{(d+1)/2d}}{m^{(d-1)/2d}}\right), \tag{42}$$

concluding the proof. ∎

6. Concluding remarks

Distinct distances in homogeneous sets. A set A of cardinality n is homogeneous if it is a subset of a full dimensional hypercube of volume n and any unit hypercube contains only $O(1)$ elements of A. If A is homogeneous, then a hyperplane of co-dimension 1 contains only $O(n^{(d-1)/d})$ elements of A. Thus, in Theorem 2.1, we can set $m=n^{(d-1)/d}$ to get

$$t(A) = \Omega\left(n^{2/d-1/d^2}\right), \qquad (43)$$

for any $d \geq 3$. This estimate improves a results of Iosevich [8,9], who used a stronger definition of homogeneity. Applying Theorem 2.2 instead of Theorem 2.1 results in the same bound. For the special case $d = 3$, we can obtain the bound $\Omega(n^{.5794})$ (see [13]) for details. The homogeneity assumption is very popular among analysts, since their finite sets are usually the discretized versions of continuous domains.

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