Polynomial With Minimal Deviance

(This work was done by Chebyshev in the 1800's.)

Def 0.1 If f is a function and $a < b$ then the **deviance of** f **on** $[a, b]$ is $\max_{x \in [a,b]} |f(x)|$.

We seek polynomials with minimum deviance. We first TRY to state this problem.
Problem: Given n, find the polynomial of degree n that has least deviance in the interval $[-1, 1]$.

There is a problem with this problem. For degree 3 take $f(x) = \frac{1}{1000}x^3$. If I made the lead coefficient even smaller we could do better, so the problem has no real answer.

Def 0.2 A polynomial is *monic* if its lead coefficient is 1.

Problem: Given n, find the monic polynomial of degree n that has least deviance in the interval $[-1, 1]$.

Example 0.3 $n = 1$. So $f(x) = x + c$ for some c. $f(x) = x$ has deviance 1. You can easily prove that if $c \neq 0$ then $f(x) = x + c$ will have larger deviance. Deviation is 1 which we write as $2^0 = 2^{n-1}$.

Example 0.4 $n = 2$. So $f(x) = x^2 + bx + c$ for some b, c. If $b = 0$ then we have $f(x) = x^2 + c$. Lets assume that $f(1) > 0$ and $f(0) < 0$.

$f(1) = 1 + c$, $|f(1)| = 1 + c$.

$f(-1) = 1 + c$, $|f(-1) = 1 + c$.

$f(0) = c$, $|f(0)| = -c$.

To get these to be minimal set them equal. $1 + c = -c$ so $c = -1/2$.

SO, a good candidate is $f(x) = x^2 - \frac{1}{2}$. Deviation is $\frac{1}{2}$ which we write as $2^(-1) = 2^{n-1}$.

Can we do better? No. Assume that $g(x)$ was a monic quadratic that did better. Since $g(1) < f(1)$, $g(0) > f(0)$, and $g(-1) < f(1)$, we have that the f and g functions cross twice. That is, there exists a, b such that $f(a) = g(a)$ and $f(b) = g(b)$. Since f, g are monic quadratic, we have $f = g$.

We now try to solve the general problem.
IDEAS: We try to find a polynomial that HUGS the deviance lines above and below. A function that LOOKS that way is Cosine. Unfortunately cosine is not a polynomial Even so, we can use this similarlity.

Lemma 0.5 *For all n, there exists f_n monic, degree n, such that $f_n(\cos\alpha) = \frac{1}{2^{n-1}}\cos n\alpha$.*

We will later prove Lemma 0.5 constructively so that the f_n's can be calculated. For now we just use Lemma 0.5.

Theorem 0.6 *Let f_n be as in Lemma 0.5. The deviance of f_n on $[-1,1]$ is $d=\frac{1}{2^{n-1}}$. The number of times that f_n hits the $y=d$ or $y=-d$ lines is exactly $n+1$.*

Proof:

The key is that the function $\cos\alpha$ is a bijection from $[0,\pi]$ to $[-1,1]$.

Let $x\in[-1,1]$. Let $\alpha\in[0,\pi]$ be such that $x=\cos\alpha$. Then

$$f_n(x)=f_n(\cos\alpha)=\frac{1}{2^{n-1}}\cos n\alpha.$$

Since cos is always between -1 and 1 we have

$$-\frac{1}{2^{n-1}}\le f_n(x)\le\frac{1}{2^{n-1}}.$$

Hence the deviance of $f_n(x)$ on $[-1,1]$ is $\frac{1}{2^{n-1}}$.

We want to know when $f_n(x)=\pm\frac{1}{2^{n-1}}$.

$$f_n(x)=\pm\frac{1}{2^{n-1}}\text{ iff}$$

$$x=\cos\alpha\text{ and }\cos n\alpha=\pm 1\text{ iff}$$

$$x=\cos\alpha\text{ and }\alpha\in\{0,\frac{\pi}{n},\frac{2\pi}{n},\ldots,\frac{n\pi}{n}\}\text{ iff}$$

$$x\in\{\cos 0,\cos\frac{\pi}{n},\cos\frac{2\pi}{n},\ldots,\cos\frac{n\pi}{n}\}.$$

There are $n+1$ of these points.

▮

Theorem 0.7 *For all n the monic polynomial with the least deviance on $[-1,1]$ is f_n from Lemma 0.5.*

Proof: If h is a monic polynomial with better deviance then f_n then we can show that h and f_n must cross in n points, and hence are the same. (draw the picture yourself). ▮

Proof of Lemma 0.5 which we restate and elaborate on:

For all n, there exists f_n monic, degree n, such that $f_n(\cos\alpha)=\frac{1}{2^{n-1}}\cos n\alpha$.

Proof:

We first show that, for all n there exists a (not monic) polynomial g_n of degree n such that $g_n(\cos\alpha)=\cos(n\alpha)$. We will keep track of the leading coefficient of g_n. We prove this by induction on n.

Clearly $g_0(x)=1$ and $g_1(x)=x$.

Assume that $g_{n-1}(\cos\alpha) = \cos(n-1)\alpha$ and $g_n(\cos\alpha) = \cos(n)\alpha$. We get g_{n+1} in terms of g_n and g_{n-1}.

Recall that

$$\begin{aligned} \cos(x+y) = &\ \cos x \cos y - \sin x \sin y \\ \cos(x-y) = &\ \cos x \cos y + \sin x \sin y \end{aligned}$$

Hence we have

$$\begin{aligned} \cos((n+1)\alpha) = &\ \cos n\alpha \cos\alpha - \sin n\alpha \sin\alpha \\ \cos((n-1)\alpha) = &\ \cos n\alpha \cos\alpha + \sin n\alpha \sin\alpha \end{aligned}$$

We add these and isolate the $\cos(n+1)\alpha)$ term to get

$$\cos((n+1)\alpha) = 2\cos n\alpha \cos\alpha - \cos((n-1)\alpha)$$

Inductively $\cos(n\alpha) = g_n(\cos\alpha)$ and $\cos((n-1)\alpha) = g_{n-1}(\cos\alpha)$.

So we have

$$\cos((n+1)\alpha) = \ 2 \cdot g_n(\cos\alpha)\cos\alpha - g_{n-1}(\cos\alpha)$$

So

$g_{n+1}(x) = 2xg_n(x) - g_{n-1}(x)$.

Note that

$$\begin{aligned} g_0(x) = &\ 1 \\ g_1(x) = &\ x \\ g_2(x) = &\ 2x^2 - 1 \\ g_3(x) = &\ 4x^3 - 2x - x = 4x^3 - 3x \\ g_4(x) = &\ 8x^4 - 6x^2 - 2x^2 + 1 = 8x^4 - 8x^2 + 1 \end{aligned}$$

One can easily prove by induction that, for all $n \geq 1$, g_n has leading coeffient 2^{n-1}. Note that g_0 does not fit this pattern.

For $n \geq 1$ let $f_n(x) = \frac{1}{2^{n-1}} g_n(x)$. Note that f_n is monic, degree n, and $f_n(x) = \frac{1}{2^{n-1}}\cos(n\alpha)$. ∎

Note 0.8 From the recurrence we can deduce some things about g_n. One can easily prove by induction that (1) if n is even then g_n only uses even powers and if n is odd then g_n only uses odd powers, and (2) if $n \equiv 0 \pmod 4$ then the constant term is 1, if $n \equiv 2 \pmod 4$ then the constant term is -1, and if $n \equiv 1, 3 \pmod 4$ then the constant term is 0.

Example 0.9

$$\begin{aligned} f_1(x) = &\ x \\ f_2(x) = &\ x - \tfrac{1}{2} \\ f_3(x) = &\ x^3 - \tfrac{3}{4}x \\ f_4(x) = &\ x^4 - x^2 + \tfrac{1}{8} \end{aligned}$$

Is there a way to compute these without using the recurrence? There is— we derive it using generating functions. We first derive a closed form for g_n then modify it for f_n. We write g_n instead of g_n.

Let $\Phi = \sum_{n=0}^{\infty} g_n z^n$. We will obtain another expression for Φ so that we can read off the coeff of z^n which will be a poly in x of degree n. of degree n.

$$
\begin{array}{rl}
\Phi = & \sum_{n=0}^{\infty} g_n z^n = g_0 + g_1 z + \sum_{n\ge 2}^{\infty} g_n z^n \\
& g_0 + g_1 z + \sum_{n\ge 2}^{\infty}(2x g_{n-1} - g_{n-2}) z^n \\
& g_0 + g_1 z + 2x \sum_{n\ge 2}^{\infty} g_{n-1} z^n - \sum_{n\ge 2} g_{n-2} z^n \\
& g_0 + g_1 z + 2xz \sum_{n\ge 2}^{\infty} g_{n-1} z^{n-1} - z^2 \sum_{n\ge 2} g_{n-2} z^{n-2} \\
& g_0 + g_1 z + 2xz \sum_{n\ge 1}^{\infty} g_n z^n - z^2 \sum_{n\ge 0} g_n z^n \\
& g_0 + g_1 z + 2xz(-g_0 + \sum_{n\ge 0}^{\infty} g_n z^n) - z^2 \sum_{n\ge 0} g_n z^n \\
& g_0 + g_1 z + 2xz(-g_0 + \Phi) - z^2\Phi \\
& 1 + xz + 2xz(-1 + \Phi) - z^2\Phi \\
& 1 + xz - 2xz + 2xz\Phi - z^2\Phi \\
& 1 - xz + 2xz\Phi - z^2\Phi \\
\Phi - 2xz\Phi + z^2\Phi & = 1 - xz \\
\Phi(1 - 2xz + z^2) = & 1 - xz \\
\Phi = & \frac{1-xz}{1-2xz+z^2} \\
\Phi = & \frac{1-xz}{1-(2xz-z^2)}
\end{array}
$$

Now we want to look at

$$
\begin{array}{rl}
\frac{1}{1-(2xz-z^2)} & = \sum_{n=0}^{\infty}(2xz - z^2)^n \\
& = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} (z^2)^i (2xz)^{n-i} (-1)^i \\
& = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} z^{2i} (2x)^{n-i} z^{n-i} (-1)^i \\
& = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} z^{n+i} (2x)^{n-i} (-1)^i \\
& = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} (-1)^i (2x)^{n-i} z^{n+i}
\end{array}
$$

We rewrite this so that $n+i$ is the outer sum. Let $m = n+i$. As m goes from 0 to infinity, i goes from 0 to $\lfloor m/2 \rfloor$.

$$
\begin{array}{rl}
\frac{1}{1-(2xz-z^2)} = & \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} (-1)^i (2x)^{n-i} z^{n+i} \\
& \sum_{m=0}^{\infty} \sum_{i=0}^{\lfloor m/2 \rfloor} \binom{m-i}{i} (-1)^i (2x)^{m-2i} z^m \\
& \sum_{m=0}^{\infty} z^m \sum_{i=0}^{\lfloor m/2 \rfloor} \binom{m-i}{i} (-1)^i (2x)^{m-2i} \\
& \sum_{n=0}^{\infty} z^n \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} (-1)^i (2x)^{n-2i}
\end{array}
$$

Let $h_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} (-1)^i (2x)^{n-2i}$. Then $\frac{1}{1-(2xz-z^2)} = \sum_{n\ge 0} h_n(x) z^n$. But we are interested in $\frac{1-xz}{1-(2xz-z^2)} = (1-xz)\sum_{n\ge 0} h_n(x) z^n$.

$$\begin{aligned}
(1-xz)\textstyle\sum_{n\geq 0} h_n(x)z^n = & \textstyle\sum_{n\geq 0} h_n(x)z^n - xz\sum_{n\geq 0} h_n(x)z^n \\
= & \textstyle\sum_{n\geq 0} h_n(x)z^n - \sum_{n\geq 0} xh_n(x)z^{n+1} \\
= & \textstyle\sum_{n\geq 0} h_n(x)z^n - \sum_{m\geq 1} xh_{m-1}(x)z^m \\
= & \textstyle\sum_{n\geq 0} h_n(x)z^n - \sum_{n\geq 1} xh_{n-1}(x)z^n \\
= & \textstyle h_0 + \sum_{n\geq 1} h_n(x)z^n - \sum_{n\geq 1} xh_{n-1}(x)z^n \\
= & \textstyle h_0 + \sum_{n\geq 1}(h_n(x) - xh_{n-1}(x))z^n
\end{aligned}$$

Recall that we want the coef of z^n. Hence we want $h_n(x) - xh_{n-1}(x))$. We first get a neater form for h_n. There are two cases. We do the first case, m even, and leave the second case, m odd, to the reader.

Case 1: n is even. So $n = 2m$ and $\lfloor n/2 \rfloor = m$, $\lfloor n-1/2 \rfloor = m-1$.

$$\begin{aligned}
h_n(x) = & \textstyle\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}(-1)^i(2x)^{n-2i} \\
= & \textstyle\sum_{i=0}^{m} \binom{2m-i}{i}(-1)^i(2x)^{2m-2i} \\
= & \textstyle\sum_{i=0}^{m} \binom{2m-i}{i}(-1)^i(2x)^{2(m-i)} \text{ now set } j = m-i \\
= & \textstyle\sum_{j=0}^{m} \binom{m+j}{m-j}(-1)^{m-j}(2x)^{2j} \\
= & \textstyle\sum_{i=0}^{m} \binom{m+i}{m-i}(-1)^{m-i}2^{2i}x^{2i}
\end{aligned}$$

$$\begin{aligned}
h_{n-1}(x) = & \textstyle\sum_{i=0}^{m-1} \binom{2m-i-1}{i}(-1)^i(2x)^{2m-2i-1} \\
= & \textstyle\sum_{i=0}^{m-1} \binom{2m-i-1}{i}(-1)^i(2x)^{2m-2i-1} \\
= & \textstyle\sum_{i=0}^{m-1} \binom{2m-i-1}{i}(-1)^i(2x)^{2(m-i)-1} \text{ now set } j = m-i \\
= & \textstyle\sum_{j=1}^{m} \binom{m+j-1}{m-j}(-1)^{m-j}(2x)^{2j-1} \\
= & \textstyle\sum_{i=1}^{m} \binom{m+i-1}{m-i}(-1)^{m-i}2^{2i-1}x^{2i-1}
\end{aligned}$$

$xh_{n-1}(x) = \sum_{i=0}^{m-1} \binom{m+i-1}{m-i}(-1)^{m-i}2^{2i-1}x^{2i}$

So

$$\begin{aligned}
h_n(x) - xh_{n-1}(x) = & \textstyle\sum_{i=0}^{m} \binom{m+i}{m-i}(-1)^{m-i}2^{2i}x^{2i} - \sum_{i=1}^{m} \binom{m+i-1}{m-i}(-1)^i 2^{2i-1}x^{2i} \\
= & \textstyle(-1)^m + \sum_{i=1}^{m}(\binom{m+i}{m-i}(-1)^{m-i}2^{2i} - \binom{m+i-1}{m-i}(-1)^i 2^{2i-1})x^{2i}
\end{aligned}$$

Hence if $m = 2n$ then

$g_n(x) = (-1)^m + \sum_{i=1}^{m}(\binom{m+i}{m-i}(-1)^{m-i}2^{2i} - \binom{m+i-1}{m-i}(-1)^i 2^{2i-1})x^{2i}$

Thus

$f_n(x) = \frac{1}{2^{n-1}} g_n(x) =$

$\frac{(-1)^m}{2^{n-1}} + \sum_{i=1}^{m}(\binom{m+i}{m-i}(-1)^{m-i}2^{2i-n+1} - \binom{m+i-1}{m-i}(-1)^i 2^{2i-n})x^{2i}$