

Duplicator Spoiler Games
Exposition by William Gasarch (gasarch@cs.umd.edu)

1 Introduction

Roland Fraïssé [2, 3, 4] proved theorems about logical expressibility using a back-and-forth method. Andrzej Ehrenfeucht [1] formalized Fraïssé's method by invented Duplicator-Spoiler Games¹.

In this exposition we will define Duplicator-Spoiler games for linear orderings and prove the connection to logical expressible for these games. We follow the treatment of [5] which is out of print.

2 Duplicator-Spoiler Games For Linear Orderings

Definition 2.1 A linear ordering \mathcal{L} is defined as a set L paired with an ordering $<$, denoted $\mathcal{L} = (L, <)$, such that

1. $(\forall x, y \in L) [x < y \text{ or } x \geq y, \text{ but not both}]$.
2. $(\forall x, y, z \in L)[x < y \wedge y < z \implies x < z]$.

We now define Duplicator-Spoiler Games played with linear orderings.

Definition 2.2 Let $\mathcal{L}_1 = (L_1, <)$, $\mathcal{L}_2 = (L_2, <)$. The m -round *Duplicator-Spoiler Game* on $(\mathcal{L}_1, \mathcal{L}_2)$ is defined as follows.

1. There are two players: the Spoiler and the Duplicator.
2. There are m rounds. During round i ($1 \leq i \leq m$) the Spoiler selects an element from either set and the Duplicator selects an element from the other set. The element selected from L_1 is called a_i and from L_2 , b_i .
3. If $(\forall i, j, 1 \leq i, j \leq m) [a_i < a_j \iff b_i < b_j]$, then the Duplicator wins. Otherwise, the Spoiler wins.

Definition 2.3 Let \mathcal{L}_1 and \mathcal{L}_2 be two linear orderings. Let $m \in \mathbb{N}$. If Duplicator wins the m -round Duplicator-Spoiler game on $(\mathcal{L}_1, \mathcal{L}_2)$ then \mathcal{L}_1 and \mathcal{L}_2 are m -game equivalent which we denote $\mathcal{L}_1 \equiv_m^G \mathcal{L}_2$.

Geometrically, we imagine lines being drawn from elements of one set to another, each line representing a pair of inequalities. If two lines cross, the inequalities are not consistent from set to set and the Spoiler wins. Because of this, a line, or round, reduces any game into two new ones.

Convention 2.4 Unless otherwise specified, we will assume the Spoiler and the Duplicator play optimally. In other words, if one player can implement a strategy that wins every time then the player is assumed to implement said strategy.

¹Duplicator-Spoiler games are also referred to as Ehrenfeucht-Fraïssé games.

Notation 2.5 The following notations will be used throughout.

- Let \mathcal{F}_m be the finite linear ordering of m elements.
- Let ω be the linear ordering $\{1 < 2 < 3 < \dots\}$. Note that this is equivalent to $\{2 < 4 < \dots\}$. The base set is unimportant.
- Let ω^* be the linear ordering $\{\dots < -3 < -2 < -1\}$.
- Let \mathbb{Z} be the linear ordering $\{\dots < -2 < -1 < 0 < 1 < 2 \dots\}$.
- Let \mathbb{Q} be the natural ordering of the rational numbers.
- Let \mathbb{R} be the natural ordering of the real numbers.

Definition 2.6 Let $\mathcal{L}_1 = (L_1, <_1)$, $\mathcal{L}_2 = (L_2, <_2)$. The linear ordering $\mathcal{L}_1 + \mathcal{L}_2$ is formed as follows.

We can assume that $L_1 \cap L_2 = \emptyset$ by changing the elements' labels. Let the base set for $\mathcal{L}_1 + \mathcal{L}_2$ be $L_1 \cup L_2$. Let the total order be as follows.

1. $(\forall x, y \in L_1) [x < y \iff x <_1 y]$.
2. $(\forall x, y \in L_2) [x < y \iff x <_2 y]$.
3. $(\forall x \in L_1)(\forall y \in L_2) [x < y]$.

BILL-DO EXAMPLES LEADING UP TO \mathbb{N} and $\mathbb{N} + \mathbb{Z}$.

3 The Connection to Logic

We first define formulas and two notions of the complexity of formulas: *quantifier depth* and *number of free variables*.

We will now define formulas and quantifier depth (qd) rigorously.

Definition 3.1 Our language contains the symbols $\wedge, \vee, \neg, \exists, \forall, =, <$ and variables x_1, x_2, \dots . We may use x, y, z for notational convenience.

1. A variable is *free* if it is not quantified over. When we write (say) $\phi(x)$ the x is a free variable. There may be other variables; however, they are quantified over.
2. An *atomic formula* is any formula of the form $x_i < x_j$ or $x_i = x_j$. If $\phi(x_i, x_j)$ is an atomic formula then $\text{qd}(\phi(x_i, x_j)) = 0$.
3. If $\phi(\vec{x})$ is a formula then $\neg\phi(\vec{x})$ is a formula and $\text{qd}(\neg(\phi(\vec{x}))) = \text{qd}(\phi(\vec{x}))$.
4. If $\gamma(\vec{x})$ and $\theta(\vec{y})$ are formulas then
 - $\gamma(\vec{x}) \wedge \theta(\vec{y})$ is a formula and $\text{qd}(\gamma(\vec{x}) \wedge \theta(\vec{y})) = \max\{\text{qd}(\gamma(\vec{x})), \text{qd}(\theta(\vec{y}))\}$.
 - $\gamma(\vec{x}) \vee \theta(\vec{y})$ is a formula and $\text{qd}(\gamma(\vec{x}) \vee \theta(\vec{y})) = \max\{\text{qd}(\gamma(\vec{x})), \text{qd}(\theta(\vec{y}))\}$,

5. If $\phi(\vec{x}, x)$ is a formula then $(\exists x)[\phi(\vec{x}, x)]$ is a formula and $\text{qd}((\exists x)[\phi(\vec{x}, x)]) = \text{qd}(\phi(\vec{x}, x)) + 1$.
 Note that $(\exists x)[\phi(\vec{x}, x)]$ has one less free variable than $\phi(\vec{x}, x)$.

Definition 3.2 A *sentence* is a formula with no free variables.

Let ϕ be a sentence like $(\exists x)(\forall y)[x \leq y]$. Is this sentence true or false? This is a stupid question: you need to know which linear order is being talked about. The next definition gives a succinct way of saying this.

Definition 3.3 Let $\mathcal{L} = (L, <)$ be a linear ordering.

1. Let ϕ be a sentence. $\mathcal{L} \models \phi$ means that ϕ is true when interpreted in \mathcal{L} .
2. \mathcal{L}_1 and \mathcal{L}_2 are *m-truth-equivalent*, denoted $\mathcal{L}_1 \equiv_m^T \mathcal{L}_2$, if, for all ϕ with $\text{qd}(\phi) \leq m$

$$\mathcal{L}_1 \models \phi \text{ iff } \mathcal{L}_2 \models \phi.$$

We want to prove the following theorem:

Theorem 3.4 Let \mathcal{L}_1 and \mathcal{L}_2 be two linear orderings. Let $m \in \mathbb{N}$. The following are equivalent:

1. $\mathcal{L}_1 \equiv_m^G \mathcal{L}_2$.
2. $\mathcal{L}_1 \equiv_m^T \mathcal{L}_2$.

However, this is one of those cases where it is easier to prove a harder theorem. We will (1) extend Duplicator-Spoiler games to the case where some of the moves are already specified, and (2) extend the definition of \models to formulas with parameters. We will then show that those notions are equivalent.

Definition 3.5 Let $\mathcal{L}_1 = (L_1, <)$, $\mathcal{L}_2 = (L_2, <)$, $\vec{a} \in L_1^k$, and $\vec{b} \in L_2^k$. The *m-round Duplicator-Spoiler Game on $((\mathcal{L}_1, \vec{a}), (\mathcal{L}_2, \vec{b}))$* , is defined as follows.

1. There are two players: The Spoiler and The Duplicator.
2. There are m rounds. During round i ($1 \leq i \leq m$) Spoiler selects an element from either set and Duplicator selects an element from the other set. The element selected from L_1 is called a_{k+i} and from L_2 , b_{k+i} .
3. If $(\forall i, j, 1 \leq i, j \leq m+k) [a_i < a_j \iff b_i < b_j]$, then Duplicator wins. Otherwise, Spoiler wins.

Definition 3.6 If Duplicator wins the m -round game then $(\mathcal{L}_1; \vec{a}) \equiv_m^G (\mathcal{L}_2; \vec{b})$.

The game is essentially the same as the original Duplicator-Spoiler game; however, the first k rounds have already been played.

Note the following

Fact 3.7

1. $(\mathcal{L}_1; \vec{a}) \equiv_0^G (\mathcal{L}_2; \vec{b})$ iff \vec{a} and \vec{b} are of the same order type.
2. Assume $(\mathcal{L}_1; \vec{a}) \equiv_{m+1}^G (\mathcal{L}_2; \vec{b})$. If Spoiler plays a and Duplicator's winning response is b then $(\mathcal{L}_1; \vec{a}, a) \equiv_m^G (\mathcal{L}_2; \vec{b}, b)$.

Let $\phi(\vec{x})$ be a formula with k free variables. For example, if $k = 1$ then $(\exists x)[x \leq y]$ would be such a formula. Is this sentence true or false? This is an even stupider question than the one about sentences: you need to know which linear order $\mathcal{L} = (L, <)$ is being talked about AND you need to know the k elements of L that you intend to plug into ϕ . The next definition gives a succinct way of saying this.

Did I say succinct? If by *succinct* I mean *long and boring inductive definition* then yes, it is succinct. All we really *want* to say is

$(\mathcal{L}; \vec{a}) \models \phi(\vec{x})$ iff the statement $\phi(\vec{a})$ is true in \mathcal{L} .

Why don't we just say that? Because we need the inductive definition in order to prove things about \models . Alas, in order to prove things rigorously we must be a bit pedantic.

Definition 3.8 Let $\mathcal{L} = (L, <)$ and $\vec{a} \in L^k$. Let $\phi(\vec{x})$ have k free variables. We define $(\mathcal{L}; \vec{a}) \models \phi(\vec{x})$ inductively.

1. $(\mathcal{L}; a_1, a_2) \models (x < y)$ holds iff $a_1 < a_2$.
2. $(\mathcal{L}; a_1, a_2) \models (x = y)$ holds iff $a_1 = a_2$.
3. $(\mathcal{L}; \vec{a}) \models \neg\phi(\vec{x})$ holds iff it is NOT the case that $(\mathcal{L}, \vec{a}) \models \phi(\vec{x})$.
4. $(\mathcal{L}; \vec{a}) \models \phi_1(\vec{x}) \wedge \phi_2(\vec{x})$ holds iff $(\mathcal{L}, \vec{a}) \models \phi_1(\vec{x})$ and $(\mathcal{L}, \vec{a}) \models \phi_2(\vec{x})$.
5. $(\mathcal{L}; \vec{a}) \models \phi_1(\vec{x}) \vee \phi_2(\vec{x})$ holds iff $(\mathcal{L}, \vec{a}) \models \phi_1(\vec{x})$ or $(\mathcal{L}, \vec{a}) \models \phi_2(\vec{x})$.
6. $(\mathcal{L}, \vec{a}) \models (\exists x)[\phi(\vec{x}, x)]$ holds iff there is an $a \in L$ such that $(\mathcal{L}; \vec{a}, a) \models \phi(\vec{a}, a)$.
7. To summarize: If $\phi(\vec{x})$ has k free variables and $\vec{a} \in L^k$ then $(\mathcal{L}; \vec{a}) \models \phi(\vec{x})$ means that $\phi(\vec{a})$ is true in \mathcal{L} .

Definition 3.9 Let $\vec{a} \in L_1^k$ and $\vec{b} \in L_2^k$. $(\mathcal{L}_1; \vec{a}) \equiv_m^T (\mathcal{L}_2; \vec{b})$ if, for all $\phi(\vec{x})$ with $\text{qd}(\phi) \leq m$ and k free variables,

$$(\mathcal{L}_1; \vec{a}) \models \phi(\vec{x}) \text{ iff } (\mathcal{L}_2; \vec{b}) \models \phi(\vec{x}).$$

Definition 3.10 Let $m \geq 1$. A formula is *m-simple* if it is of the form $(\exists x)[\psi(\vec{x}, x)]$ where $\text{qd}(\psi(\vec{x}, x)) \leq m - 1$.

We leave the proof of the following easy lemma to the reader.

Lemma 3.11 Let $m \geq 1$. If $\text{qd}(\phi(\vec{x})) = m$ then $\phi(\vec{x})$ can be written as a boolean combination of *m-simple* formulas.

The following is our main theorem.

Theorem 3.12 *Let $\mathcal{L}_1 = (L_1, <)$ and $\mathcal{L}_2 = (L_2, <)$ be two linear orderings. For all $m \in \mathbb{N}$, for all $k \in \mathbb{N}$, for all $\vec{a} \in L_1^k$ and $\vec{b} \in L_2^k$. The following are equivalent:*

1. $(\mathcal{L}_1; \vec{a}) \equiv_m^G (\mathcal{L}_2; \vec{b})$.
2. $(\mathcal{L}_1; \vec{a}) \equiv_m^T (\mathcal{L}_2; \vec{b})$.

Proof: We prove this by induction on m .

Base Case: $m = 0$. Assume \vec{a} and \vec{b} are k -tuples. We prove two implications.

First Implication:

$$(\mathcal{L}_1; \vec{a}) \equiv_0^G (\mathcal{L}_2; \vec{b}) \implies (\mathcal{L}_1; \vec{a}) \equiv_0^T (\mathcal{L}_2; \vec{b}).$$

Assume $(\mathcal{L}_1; \vec{a}) \equiv_0^G (\mathcal{L}_2; \vec{b})$. Then, for all $1 \leq i, j \leq k$

$$a_i < a_j \text{ iff } b_i < b_j.$$

Hence $(\mathcal{L}_1; \vec{a})$ and $(\mathcal{L}_2; \vec{b})$ have the same order type.

We need to show, for all $\phi(\vec{x})$ of quantifier depth 0 that have $|\vec{a}|$ free variables, $(\mathcal{L}_1; \vec{a}) \models \phi(\vec{x})$ iff $(\mathcal{L}_2; \vec{b}) \models \phi(\vec{x})$. This means that they agree on all formulas of quantifier depth 0. Formulas of quantifier depth 0 are Boolean combinations of atomic formulas. One can easily show but induction on formation that since $(\mathcal{L}_1; \vec{a})$ and $(\mathcal{L}_2; \vec{b})$ have the same order type they will agree on any boolean combination of atomic formulas on $|\vec{a}|$ free variables.

Second Implication:

$$(\mathcal{L}_1; \vec{a}) \equiv_0^T (\mathcal{L}_2; \vec{b}) \implies (\mathcal{L}_1; \vec{a}) \equiv_0^G (\mathcal{L}_2; \vec{b}).$$

Assume $(\mathcal{L}_1; \vec{a}) \equiv_0^T (\mathcal{L}_2; \vec{b})$. Then for all $1 \leq i, j \leq k$

$$(\mathcal{L}_1; \vec{a}) \models x_i < x_j \text{ iff } (\mathcal{L}_2; \vec{b}) \models x_i < x_j.$$

Hence $a_i < a_j$ iff $b_i < b_j$. Therefore $(\mathcal{L}_1; \vec{a}) \equiv_0^G (\mathcal{L}_2; \vec{b})$.

Induction Step:

Induction Hypothesis: The statement of the theorem. However, we state it with the parameters that we need: For all $\vec{a} \in L_1^k$, $a \in L_1$, $\vec{b} \in L_2^k$, $b \in L_2$ the following are equivalent:

- $(L_1; \vec{a}, a) \equiv_m^T (L_2; \vec{b}, b)$.
- $(L_1; \vec{a}, a) \equiv_m^G (L_2; \vec{b}, b)$.

Let $k \in \mathbb{N}$, $\vec{a} \in L_1^k$, $\vec{b} \in L_2^k$. We need to prove two implications.

First Implication:

$$(L_1; \vec{a}) \equiv_{m+1}^G (L_2; \vec{b}) \implies (L_1; \vec{a}) \equiv_{m+1}^T (L_2; \vec{b}).$$

Assume $(L_1; \vec{a}) \equiv_{m+1}^G (L_2; \vec{b})$. (We won't use this until later.) We need to show that $(L_1; \vec{a}) \equiv_{m+1}^T (L_2; \vec{b})$.

Now we need to show, for all formulas $\phi(\vec{x})$ with $|\vec{a}|$ free variables, of quantifier depth $\leq m+1$, $(L_1; \vec{a}) \models \phi(\vec{z})$ iff $(L_2; \vec{b}) \models \phi(\vec{z})$. We prove this by induction on the formation of $\phi(\vec{x})$.

CASE 1: If $\phi(\vec{x})$ has quantifier depth $\leq m$ then, by $(L_1; \vec{a}) \equiv_m^G (L_2; \vec{b}) \implies (L_1; \vec{a}) \equiv_m^T (L_2; \vec{b})$, (a weaker statement than we have) and the induction hypothesis we have $(L_1; \vec{a}) \models \phi(\vec{z})$ iff $(L_2; \vec{b}) \models \phi(\vec{z})$.

CASE 2: (This is the main case of interest.) $\phi(\vec{x})$ is m -simple. $\phi(\vec{x}) = (\exists x)[\psi(\vec{x}, x)]$.

$$(\mathcal{L}_1; \vec{a}) \models \phi(\vec{x})$$

iff

$$(\mathcal{L}_1; \vec{a}) \models (\exists x)[\psi(\vec{x}, x)]$$

iff

there is an $a \in L_1$ such that

$$(\mathcal{L}_1; \vec{a}, a) \models \psi(\vec{x}, x).$$

KEY: We call a *WITNESS TO THE TRUTH!* of $(\exists x)[\psi(\vec{x}, x)]$. We need to find an analogous witness in L_2 . We use the game to find *WITNESS TO THE TRUTH!*. Duplicator's winning move is an analog of a and is used to locate the *WITNESS TO THE TRUTH!*.

Recall that $(\mathcal{L}_1; \vec{a}) \equiv_{m+1}^G (\mathcal{L}_2; \vec{b})$. Hence Duplicator has a winning strategy. If Spoiler were to make the move $a \in L_1$ as his first move then Duplicator has a response. Let b be that response. Note that by Fact 3.7 $(\mathcal{L}_1; \vec{a}, a) \equiv_{m+1-1}^G (\mathcal{L}_2; \vec{b}, b)$. By the induction hypothesis applied to $\psi(\vec{x}, x)$

$$(\mathcal{L}_1; \vec{a}, a) \models \psi(\vec{x}, x) \text{ iff } (\mathcal{L}_2; \vec{b}, b) \models \psi(\vec{x}, x).$$

Hence

$$(\mathcal{L}_1; \vec{a}, a) \models \psi(\vec{x}, x)$$

iff

$$(\mathcal{L}_2; \vec{b}, b) \models \psi(\vec{x}, x)$$

iff

$$(\mathcal{L}_2; \vec{b}) \models (\exists x)[\psi(\vec{x}, x)].$$

iff

$$(\mathcal{L}_2; \vec{b}) \models \phi(\vec{x}).$$

Therefore we have

$$(\mathcal{L}_1; \vec{a}) \models \phi(\vec{x}) \text{ iff } (\mathcal{L}_2; \vec{b}) \models \phi(\vec{x}).$$

CASE 3: $\phi(\vec{x})$ is a boolean combination of m -simple formulas. This is an easy induction of formation.

By Lemma 3.11 we have covered all of the cases.

Second Implication:

$$(\mathcal{L}_1; \vec{a}) \equiv_{m+1}^T (\mathcal{L}_2; \vec{b}) \implies (\mathcal{L}_1; \vec{a}) \equiv_{m+1}^G (\mathcal{L}_2; \vec{b})$$

We need to find a winning strategy for Duplicator.

Assume that Spoiler plays $a \in L_1$ (the case where he plays $b \in L_2$ is similar). Duplicator needs to find a $b \in L_2$ that is analogous to $a \in L_1$.

Let F be the set of all formulas $\psi(\vec{x}, x)$ such that

- $\text{qd}(\psi) \leq m$.
- $|\vec{x}| = |\vec{a}|$.
- $(\mathcal{L}_1; \vec{a}, a) \models \psi(\vec{x}, x)$.

KEY: There are only a finite number of formulas in F .
Note that

$$(\mathcal{L}_1; \vec{a}, a) \models \bigwedge_{\psi \in F} \psi(\vec{x}, x).$$

Hence

$$(\mathcal{L}_1; \vec{a}) \models (\exists x) \left[\bigwedge_{\psi \in F} \psi(\vec{x}, x) \right].$$

KEY: $\text{qd}((\exists x) [\bigwedge_{\psi \in F} \psi(\vec{x}, x)]) = m + 1$. Hence, since $(\mathcal{L}_1; \vec{a}) \equiv_{m+1}^T (\mathcal{L}_2; \vec{b})$

$$(\mathcal{L}_2; \vec{b}) \models (\exists x) \left[\bigwedge_{\psi \in F} \psi(\vec{x}, x) \right].$$

Let b be the witness. Hence

$$(\mathcal{L}_2; \vec{b}, b) \models \bigwedge_{\psi \in F} \psi(\vec{x}, x).$$

KEY: We use the logical equivalence to get the witness which will give us the winning move for Duplicator.

We claim that if Duplicator plays b then he can win the game. Note that we now need to show

$$(\mathcal{L}_1; \vec{a}, a) \equiv_m^G (\mathcal{L}_2; \vec{b}, b).$$

By the induction hypothesis it will suffice to show

$$(\mathcal{L}_1; \vec{a}, a) \equiv_m^T (\mathcal{L}_2; \vec{b}, b).$$

Let $\phi(\vec{x}, x)$ be a formula such that $\text{qd}(\phi(\vec{x}, x)) = m$ and $|\vec{x}| = |\vec{a}|$. Assume

$$(\mathcal{L}_1; \vec{a}, a) \models \phi(\vec{x}, x).$$

Then $\phi \in F$.

Since

$$(\mathcal{L}_2; \vec{b}, b) \models \bigwedge_{\psi \in F} \psi(\vec{x}, x).$$

We have that

$$(\mathcal{L}_2; \vec{b}, b) \models \phi(\vec{x}, x).$$

■

4 Generalization

Section 2 was about the Duplicator-Spoiler Game on a linear order. However, this can be generalized.

Exercise 1 Define Duplicator-Spoiler games on pairs of graphs. State and prove an analog of Theorem 3.4.

Exercise 2 Define Duplicator-Spoiler games on pairs of hypergraphs. State and prove an analog of Theorem 3.4.

References

- [1] A. Ehrenfeucht. An application of games to the completeness theorem for formalized theories. *Fundamenta Mathematicae*, 49:129–141, 1969.
- [2] R. Fraïssé. Sur une nouvelle classification des systèmes de relations. *Association Française pour l'Avancement des Sciences; Comptes Rendus*, 230:1022–1024, 1950.
- [3] R. Fraïssé. Sur quelques classifications des systèmes de relations, thesis. *Publications Scientifiques de l'Université d'Alger, series A*, 1:35–182, 1954.
- [4] R. Fraïssé. *Cours de Logique Mathématique*. Tome 2, Paris, 1972.
- [5] J. G. Rosenstein. *Linear Orderings*. Academic Press, New York, 1981. Pure and Applied Math Series 98.