Parameterized Complexity of Finding Subgraphs with Hereditary Properties*

Subhash Khot[1] and Venkatesh Raman[2]

[1] Department of Computer Science, Princeton University, New Jersey, USA
khot@cs.princeton.edu

[2] The Institute of Mathematical Sciences, Chennai-600113, INDIA
vraman@imsc.ernet.in

Abstract. We consider the parameterized complexity of the following problem under the framework introduced by Downey and Fellows[4]: Given a graph G, an integer parameter k and a non-trivial hereditary property Π, are there k vertices of G that induce a subgraph with property Π? This problem has been proved NP-hard by Lewis and Yannakakis[9]. We show that if Π includes all independent sets but not all cliques or vice versa, then the problem is hard for the parameterized class $W[1]$ and is fixed parameter tractable otherwise. In the former case, if the forbidden set of the property is finite, we show, in fact, that the problem is $W[1]$-complete (see [4] for definitions). Our proofs, both of the tractability as well as the hardness ones, involve clever use of Ramsey numbers.

1 Introduction

Many computational problems typically involve two parameters n and k, e.g. finding a vertex cover or a clique of size k in a graph G on n vertices. The parameter k contributes to the complexity of the problem in two qualitatively different ways. The parameterized versions of Vertex Cover and Undirected Feedback Vertex Set problems can be solved in $O(f(k)n^{\alpha})$ time where n is the input size, α is a constant independent of k and f is an arbitrary function of k (against a naive $\Theta(n^{ck})$ algorithm for some constant c). This "good behavior", which is extremely useful in practice for small values of k, is termed *fixed parameter tractability* in the theory introduced by Downey and Fellows[2,3,4].

On the other hand, for problems like Clique and Dominating Set, the best known algorithms for the parameterized versions have complexity $\Theta(n^{ck})$ for some constant c. These problems are known to be hard for the parameterized complexity classes $W[1]$ and $W[2]$ respectively and are considered unlikely to be fixed parameter tractable (denoted by FPT) (see [4] for the definitions and more on the parameterized complexity theory). In this paper, we investigate the parameterized complexity of finding induced subgraphs of any non-trivial hereditary property in a given graph.

* Part of this work was done while the first author was at IIT Bombay and visited the Institute of Mathematical Sciences, Chennai

D.-Z. Du et al. (Eds.): COCOON 2000, LNCS 1858, pp. 137–147, 2000.

A graph property Π is a collection of graphs. A graph property Π is non-trivial if it holds for at least one graph and does not include all graphs. A non-trivial graph property is said to be *hereditary* if a graph G is in property Π implies that every *induced subgraph* of G is also in Π. A graph property is said to be *interesting* [9] if the property is true (as well as false) for infinite families of graphs. Lewis and Yannakakis[9] (see also [6]) showed that if Π is a non-trivial and interesting hereditary property, then it is NP-hard to decide whether in a given graph, k vertices can be deleted to obtain a graph which satisfies Π.

For a hereditary property Π, let $\mathcal{F}$ be the family of graphs not having the property. The set of minimal members (minimal with respect to the operation of taking induced subgraphs) of $\mathcal{F}$ is called the forbidden set for the property Π. For example, the collection of all bipartite graphs is a hereditary property whose forbidden set consists of all odd cycles. Conversely, given any family $\mathcal{F}$ of graphs, we can define a hereditary property by declaring its forbidden set to be the set of all minimal members of $\mathcal{F}$.

Cai[1] has shown that the graph modification problem for a non-trivial hereditary property Π with a finite forbidden set is fixed parameter tractable (FPT). This problem includes the node deletion problem addressed by Lewis and Yannakakis (mentioned above). While the parameterized complexity of the question, when Π is a hereditary property with an infinite forbidden set, is open, we address the parametric dual problem in this paper. Given any property Π, let $P(G, k, \Pi)$ be the problem defined below.

Given: A simple undirected graph $G(V, E)$

Parameter: An integer $k \leq |V|$

Question: Is there a subset $V' \subseteq V$ with $|V'| = k$ such that the subgraph of G induced by V' has property Π ?

This problem is the same as '$|V| - k$' node deletion problem (i.e. can we remove all but k vertices of G to get a graph with property Π) and hence NP-hard. However the parameterized complexity of this problem doesn't follow from Cai's result. We prove that if Π includes all independent sets, but not all cliques or vice versa, then the problem $P(G, k, \Pi)$ is $W[1]$-complete when the forbidden set of Π is finite and $W[1]$-hard when the forbidden set is infinite. The proof is by a parametric reduction from the INDEPENDENT SET problem. If Π includes all independent sets and all cliques, or excludes some independent sets and some cliques then we show that the problem is fixed parameter tractable.

Note, from our and Cai's result, that the parameterized dual problems dealt with, have complimentary parameterized complexity. This phenomenon has been observed in a few other parameterized problems as well. In a graph $G(V, E)$, finding a vertex cover of size k is FPT whereas finding an independent set of size k (or a vertex cover of size $|V| - k$) is $W[1]$-complete; In a given boolean 3-CNF formula with m clauses, finding an assignment to the boolean variables that satisfies at least k clauses is FPT whereas finding an assignment that satisfies at least $(m - k)$ clauses (i.e. all but at most k clauses) is known to be $W[P]$-hard [2] (k is the parameter in both these problems). The k-IRREDUNDANT SET problem is $W[1]$-hard whereas CO-IRREDUNDANT set or $(n - k)$ IRREDUNDANT

SET problem is FPT [5]. Our result adds one other (general) problem to this list.

Throughout the paper, by a graph we mean an undirected graph with no loops or multiple edges. By a non-trivial graph, we mean a graph with at least one edge. Given a graph G and $A \subseteq V(G)$, by $I_G(A)$ we mean the subgraph of G induced by vertices in A. For two graphs H and G, we use the notation $H \subseteq G$ to mean that H is isomorphic to an induced subgraph of G. For the graph properties Π we will be concerned with in this paper, we assume that Π is recursive; i.e. given a graph G on n vertices, one can decide whether or not G has property Π in $f(n)$ time for some function of n.

We had already defined the notion of fixed parameter tractable problems. To understand the hardness result, we give below some definitions. See [4] for more details.

A parameterized language L is a subset of $\Sigma^* \times N$ where Σ is some finite alphabet and N is the set of all natural numbers. For $(x, k) \in L$, k is the parameter. We say that a parameterized problem A reduces to a parameterized problem B, if there is an algorithm Φ which transforms (x, k) into $(x', g(k))$ in time $f(k)|x|^\alpha$ where $f, g : N \to N$ are arbitrary functions and α is a constant independent of k, so that $(x, k) \in A$ if and only if $(x', g(k)) \in B$. The essential property of parametric reductions is that if A reduces to B and if B is FPT, then so is A.

Let F be a family of boolean circuits with *and*, *or* and *not* gates; We allow that F may have many different circuits with a given number of inputs. Let the weight of a boolean vector be the number of 1's in the vector. To F we associate the parameterized circuit problem $L_F = \{(C, k) : C$ accepts an input vector of weight $k\}$. Let the weft of a circuit be the maximum number of gates with fan-in more than two, on an input-output path in the circuit.

A parameterized problem L belongs to $W[t]$ if L reduces to the parameterized circuit problem $L_{F(t,h)}$ for the family $F(t, h)$ of boolean circuits with the weft of the circuits in the family bounded by t, and the depth of the circuits in the family bounded by a constant h. This naturally leads to a completeness program based on a hierarchy of parameterized problem classes:

$$FPT \subseteq W[1] \subseteq W[2] \subseteq \ldots$$

The parameterized analog of NP is $W[1]$, and $W[1]$-hardness is the basic evidence that a parameterized problem is unlikely to be FPT.

The next section deals with the hereditary properties for which the problem is fixed parameter tractable, and Section 3 proves the $W[1]$-hardness result for the remaining hereditary properties. Section 4 concludes with some remarks and open problems.

2 Hereditary Properties That Are FPT to Find

Lemma 1. *If a hereditary property Π includes all independent sets and all cliques, or excludes some independent sets as well as some cliques, then the problem $P(G, k, \Pi)$ is fixed parameter tractable.*

Proof. For any positive integers p and q, there exists a minimum number $R(p, q)$ (the Ramsey number) such that any graph on at least $R(p, q)$ vertices contains either a clique of size p or an independent set of size q. It is well-known that $R(p, q) \leq \binom{p+q-2}{q-1}$[8].

Assume that Π includes all cliques and independent sets. For any graph G with $|V(G)| \geq R(k, k)$, G contains either a clique of size k or an independent set of size k. Since all independent sets and all cliques have property Π, the answer to the problem $P(G, k, \Pi)$ in this case is "yes".

When $|V(G)| \leq R(k, k)$, we can use brute force by picking all k-elements subsets of $V(G)$ and checking whether the induced subgraph on the subset has property Π. This will take time $\binom{R(k,k)}{k} f(k)$ where $f(k)$ is the time to decide whether a given graph on k vertices has property Π. Thus the problem $P(G, k, \Pi)$ is fixed parameter tractable.

If Π excludes some cliques and some independent sets, let s and t respectively be the sizes of the smallest clique and independent set which do not have property Π. Since any graph with at least $R(s, t)$ vertices has either a clique of size s or an independent set of size t, no graph with at least $R(s, t)$ vertices can have property Π (since Π is hereditary). Hence any graph in Π has at most $R(s, t)$ vertices and hence Π contains only finitely many graphs. So if $k > R(s, t)$, then the answer to the $P(G, k, \Pi)$ problem is NO for any graph G. If $k \leq R(s, t)$, then check, for each k subset of the given vertex set, whether the induced subgraph on the subset has property Π. This will take time $\binom{n}{k} f(k) \leq Cn^{R(s,t)}$ for an n vertex graph, where C is time taken to check whether a graph of size at most $R(s, t)$ has property Π. Since s and t depend only on the property Π, and not on k or n, and $k \leq R(s, t)$, the problem $P(G, k, \Pi)$ is fixed parameter tractable in this case also. □

We list below a number of hereditary properties Π (dealt with in [11]) each of which includes all independent sets and cliques, and hence for which the problem $P(G, k, \Pi)$ is fixed parameter tractable.

Corollary 1. *Given any simple undirected graph G, and an integer k, it is fixed parameter tractable to decide whether there is a set of k vertices in G that induces (a) a perfect graph, (b) an interval graph (c) a chordal graph, (d) a split graph, (e) an asteroidal triple free (AT-free) graph, (f) a comparability graph, or (g) a permutation graph. (See [11] or [7] for the definitions of these graphs.)*

3 Hereditary Properties That Are W-Hard to Find

In this section, we show that the problem $P(G, k, \Pi)$ is $W[1]$-hard if Π includes all independent sets but not all cliques or vice versa.

For a graph G, let $\overline{G}$ denote the edge complement of G. For a property Π, let $\overline{\Pi} = \{\overline{G} \mid G \textit{ has property } \Pi\}$. We note that Π is hereditary if and only if $\overline{\Pi}$ is hereditary, and Π includes all independent sets but not all cliques if and only if $\overline{\Pi}$ includes all cliques but not all independent sets. Thus it suffices to prove $W[1]$-hardness when Π includes all independent sets, but not all cliques.

First we will show that, the problem is in $W[1]$ if the forbidden set for Π is finite.

Lemma 2. *Let Π be a non-trivial hereditary property having a finite forbidden set $\mathcal{F}$ =$\{H_1, H_2, \ldots, H_s\}$. Then the problem $P(G, k, \Pi)$ is in $W[1]$.*

Proof. Let $\nu = max(|H_i|)$. Let $A_1, \ldots, A_q$ be all the subsets of $V(G)$ such that for every $1 \leq j \leq q$, $I_G(A_j)$ is isomorphic to some H_i. The sets A_j's can be determined in $O(f(\nu)n^{\nu})$ time by trying every subset of $V(G)$ of size at most ν. Here f is some function of ν alone.

Consider the boolean formula

$$\bigwedge_{j=1}^{q}(\bigvee_{u\in A_j} \overline{x_u})$$

If this formula has a satisfying assignment with weight (the number of true variables in the assignment) k, then the subset X of $V(G)$ defined by $X = \{u \in V | x_u = 1\}$ is a subset of $V(G)$ of cardinality k such that $A_j \nsubseteq X$ for any j. This implies that G has an induced subgraph of size k with property Π.

Conversely if G has an induced subgraph of size k with property Π, then setting $x_u = 1$ for vertices u in the induced subgraph gives a satisfying assignment with weight k. Since $q \leq n^{\nu+1}$ and $|A_i| \leq \nu$ (a constant) for all i, it follows from the definitions that the problem is in $W[1]$. □

We now show that the problem is $W[1]$-hard when one of the graphs in the forbidden set of Π is a complete bipartite graph.

Lemma 3. *Let Π be a hereditary property that includes all independent sets but not all cliques, having a finite forbidden set $\mathcal{F}$ =$\{H_1, H_2, \ldots, H_s\}$. Assume that some H_i, say H_1 is a complete bipartite graph. Then the problem $P(G, k, \Pi)$ is $W[1]$-complete.*

Proof. In Lemma 2 we have shown that the problem is in $W[1]$.

Let Π be as specified in the Lemma. Let $t = max(|V_1|, |V_2|)$ where $V_1 \bigcup V_2$ is bipartition of H_1. If $t = 1$, $H_1 = K_2$, and the given problem P is identical to the k-independent set problem, hence $W[1]$-hard. So assume $t \geq 2$. Note that $H_1 \subseteq K_{t,t}$. Let H_s be the clique of smallest size that is not in Π, hence in the forbidden set $\mathcal{F}$.

Now we will show that the problem is $W[1]$-hard by a reduction from the Independent Set Problem. Let G_1 be a graph in which we are interested in finding an independent set of size k_1. For every vertex $u \in G_1$ we take r independent vertices (r to be specified later) $u^1, \ldots, u^r$ in G. If (u, v) is an edge in G_1, we add all r^2 edges (u^i, v^j) in G. G has no other edges.

We claim that that G_1 has an independent set of size k_1 if and only if G has rk_1 vertices that induce a subgraph with property Π.

Suppose G_1 has an independent set $\{u_i | 1 \leq i \leq k_1\}$ of size k_1. Then the set of rk_1 vertices $\{u_i^j \mid 1 \leq i \leq k_1, 1 \leq j \leq r\}$ is an independent set in G and hence has property Π.

Conversely let S be a set of rk_1 vertices in G which induces a subgraph with property Π. This means $I_G(S)$ does not contain any H_i, in particular it does not contain H_1. Group the rk_1 vertices according to whether they correspond to the same vertex in G_1 or not. Let $X_1, \ldots, X_h, Y_1, \ldots, Y_p$ be the groups and $u_1, \ldots, u_h, v_1, \ldots, v_p$ be the corresponding vertices in G_1 such that $|X_i| \geq t \; \forall \; i$ and $|Y_j| < t \; \forall \; j$. Observe that $\{u_1, \ldots, u_h\}$ must be independent in G_1 because if we have an edge (u_i, u_j), $H_1 \subseteq K_{t,t} \subseteq I_G(X_i, X_j) \subseteq I_G(S)$, a contradiction. If $h \geq k_1$ we have found an independent set of size at least k_1 in G_1. Therefore assume that $h \leq k_1 - 1$. Then $\sum_{i=1}^{h} |X_i| \leq r(k_1 - 1)$ which implies that $\sum_{j=1}^{p} |Y_j| \geq r$ or $p \geq r/(t-1)$. Since vertices in distinct groups (one vertex per group) in G and the corresponding vertices in G_1 induce isomorphic subgraphs, the vertices $v_1, \ldots, v_p$ induce a subgraph of G_1 with property Π (since Π is hereditary). Since this subgraph has property Π, it does not contain H_s as an induced subgraph. We choose r large enough so that any graph on $r/(t-1)$ vertices that does not contain a clique of size $|H_s|$ has an independent set of size k_1. With this choice of r, it follows that G_1 does contain an independent set of size k_1. The number r depends only on $|H_s|$ and the parameter k_1 and not on $n_1 = |V(G_1)|$. So the reduction is achieved in $O(f(k_1){n_1}^{\alpha})$ time where f is some function of k_1 and α is some fixed constant independent of k_1. □

Next, we will show that the problem is $W[1]$-hard even if none of the graphs in the forbidden set is complete-bipartite.

Theorem 1. *Let Π be a hereditary property that includes all independent sets but not all cliques, having a finite forbidden set $\mathcal{F} = \{H_1, H_2, \ldots, H_s\}$. Then the problem $P(G, k, \Pi)$ is $W[1]$-complete.*

Proof. The fact that the problem is in $W[1]$ has already been proved in Lemma 2. Assume that none of the graphs H_i in the forbidden set of Π is complete-bipartite. Let H_s be the clique of smallest size that is not in Π, hence in the forbidden set $\mathcal{F}$.

For a graph H_i in $\mathcal{F}$, select (if possible) a subset of vertices Z such that the vertices in Z are independent and every vertex in Z is connected to every vertex in $H_i \backslash Z$. Let $\{H_{ij} | 1 \leq j \leq s_i\}$ be the set of graphs obtained from H_i by removing such a set Z for every possible choice of Z. Since H_i is not complete-bipartite, every H_{ij} is a non-trivial graph. Let $\mathcal{F}_1 = \mathcal{F} \bigcup \{H_{ij} | 1 \leq i \leq s, 1 \leq j \leq s_i\}$. Note that $\mathcal{F}_1$ contains a clique of size $|H_s| - 1$ because a set Z, consisting of a single vertex, can be removed from the clique H_s. Let Π_1 be the hereditary property defined by the forbidden set $\mathcal{F}_1$. Observe that Π_1 also includes all independent sets but not all cliques. Let P_1 be the problem $P(G_1, k_1, \Pi_1)$.

We will prove that P_1 is $W[1]$-hard later. Now, we will reduce P_1 to the problem $P(G, k, \Pi)$ at hand.

Given G_1, we construct a graph G as follows. Let $V(G) = V(G_1) \bigcup D$ where D is a set of r independent vertices (r to be specified later). Every vertex in $V(G_1)$ is connected to every vertex in D. Let $\nu = max_i(|H_i|)$.

We claim that G_1 has an induced subgraph of size k_1 with property Π_1 if and only if G has $k_1 + r$ vertices that induce a subgraph with property Π.

Let A be a subset of $V(G_1), |A| = k_1$ such that $I_{G_1}(A) \in \Pi_1$. Let $S = A \bigcup D$. Suppose on the contrary that $I_G(S)$ contains some H_i as a subgraph. If this H_i contains some vertices from D, we throw away these independent vertices. The remaining portion of H_i, which is some $H_{ij}, 1 \leq j \leq s_i$, must lie in $I_G(A)$. But this is a contradiction because $I_G(A) = I_{G_1}(A)$ and by hypothesis, $I_{G_1}(A)$ has property Π_1 and it cannot contain any H_{ij}. Similarly H_i cannot lie entirely in $I_G(A)$ because $\mathcal{F} \subseteq \mathcal{F}_1$, so $I_G(A)$ does not contain any H_i as induced subgraph. Therefore $I_G(S)$ does not contain any H_i, hence it has property Π and $|S| = k_1 + r$.

Conversely, suppose we can choose a set S, $|S| = k_1 + r$ such that $I_G(S)$ does not contain any H_i. Since $|D| = r$ we must choose at least k_1 vertices from $V(G_1)$. Let $A \subseteq S \bigcap V(G_1)$ with cardinality k_1. If $I_G(A)$ does not contain any H_{ij}, we are through. Otherwise let $H_{i_0 j_0} \subseteq I_G(A)$ for some i_0, j_0. Now $H_{i_0 j_0}$ is obtained from H_{i_0} by deleting an independent set of size at most ν. Hence S can contain at most $\nu - 1$ vertices from D, otherwise we could add sufficient number of vertices from D to the graph $H_{i_0 j_0}$ to get a copy of H_{i_0} which is not possible. Hence $|S \bigcap D| < \nu$ which implies that $|S \bigcap V(G_1)| > k_1 + r - \nu$. Thus $I_{G_1}(S \bigcap V(G_1))$ is an induced subgraph of G_1 of size at least $k_1 + r - \nu$ that does not contain any H_i, in particular it does not contain H_s which is a clique of size say μ. We can select r (as before, by Ramsey Theorem) such that any graph on $k_1 + r - \nu$ vertices that does not contain a μ-clique has an independent set of size k_1. Hence G_1 has an independent set of size k_1 which has property Π_1. The number r depends only on the family $\mathcal{F}$ and parameter k_1 and not on $n_1 = |V(G_1)|$. So the reduction is achieved in $O(g(k_1){n_1}^{\beta})$ time where g is some function of k_1 and β is a constant. Also $|V(G)| = |V(G_1)| + r$, so the size of the input problem increases only by a constant.

We will be through provided the problem P_1 is $W[1]-$hard. If any of the H_{ij} is complete-bipartite, then it follows from Lemma 3. Otherwise, we repeatedly apply the construction, given at the beginning of the proof, of removing set Z of vertices from each graph in the forbidden set, to get families $\mathcal{F}_2$, $\mathcal{F}_3, \ldots$ and corresponding problems $P_2, P_3, \ldots$ such that there is a parametric reduction from P_{m+1} to to P_m. Since $\mathcal{F}_{m+1}$ contains a smaller clique than a clique in $\mathcal{F}_m$, eventually some family $\mathcal{F}_{m_0}$ contains a clique of size 2 (the graph K_2) or a complete-bipartite graph. In the former case, the problem P_{m_0} is same as parameterized independent set problem, so $W[1]$-hard. In the latter case P_{m_0} is $W[1]$-hard by Lemma 3. □

We can extend Theorem 1 to the case when the forbidden set is infinite. However we could prove the problem only $W[1]$-hard; we don't know the precise class in the W-hierarchy the problem belongs to.

Theorem 2. *Let Π be a hereditary property that includes all independent sets but not all cliques (or vice versa). Then the problem $P(G, k, \Pi)$ is $W[1]$-hard.*

Proof. Every hereditary property is defined by a (possibly infinite) forbidden set [9] and so let the forbidden family for Π be $\mathcal{F} = \{H_1, H_2, \ldots\}$. The proof is almost the same as in Theorem 1. Note that Lemma 3 does not depend on finiteness of the forbidden family. Also the only point where the finiteness of $\mathcal{F}$ is used in Theorem 1 is in the argument that if $I_G(A)$ does contain some $H_{i_0 j_0}$ then S can contain at most $\nu - 1$ vertices from the set D. This argument can be modified as follows. Since $I_G(A)$ contains some $H_{i_0 j_0}$, $|V(H_{i_0 j_0})| \leq |A| = k_1$. Also $H_{i_0 j_0}$ is obtained from some H_i by removing an independent set adjacent to all other vertices of H_i. (If there are more than one such H_is from which $H_{i_0 j_0}$ is obtained, we choose an arbitrary H_i.) Let $\nu_1 = \; max \; (|V(H_i)| - |V(H_{ij})|)$ where the maximum is taken over all H_{ij} such that $|V(H_{ij})| \leq k_1$. Hence if $I_G(A)$ does contain some $H_{i_0 j_0}$, we can add at most ν_1 vertices from D to get H_{i_0}. So S must contain less than ν_1 vertices from D. The choice of r will have to be modified accordingly. □

Corollary 2 follows from Theorem 2 since the collection of forests is a hereditary property with the forbidden set as the set of all cycles. This collection includes all independent sets and does not include any clique of size ≥ 3.

Corollary 2. *The following problem is $W[1]$-hard:*
Given (G, k), does G have k vertices that induce a forest?

This problem is the parametric dual of the UNDIRECTED FEEDBACK VERTEX SET problem which is known to be fixed parameter tractable [4].

Corollary 3. *Following problem is $W[1]$-complete:*
Given (G, k), does there exist an induced subgraph of G with k vertices that is bipartite ?

Proof. Hardness follows from Theorem 2 since all independent sets are bipartite and no clique of size at least 3 is bipartite.

To show that the problem is in $W[1]$, given the graph G, consider the boolean formula

$$\bigwedge_{u \in V(G)} (\overline{x_u} \vee \overline{y_u}) \;\; \bigwedge_{(u,v) \in E(G)} ((\overline{x_u} \vee \overline{x_v}) \bigwedge (\overline{y_u} \vee \overline{y_v}))$$

We claim that G has an induced bipartite subgraph of size k if and only if the above formula has a satisfying assignment with weight k. Suppose G has an induced bipartite subgraph with k vertices with partition V_1 and V_2. Now for each vertex in V_1 assign $x_u = 1, y_u = 0$, for each vertex in V_2 assign $x_u = 0, y_u = 1$ and assign $x_u = y_u = 0$ for the remaining vertices. It is easy to see that this assignment is a weight k satisfying assignment for the above formula.

Conversely, if the above formula has a weight k satisfying assignment, the vertices u such that $x_u = 1, y_u = 0$ or $x_u = 0, y_u = 1$ induce a bipartite subgraph of G with k vertices.

The corollary follows as the above formula can be simulated by a $W[1]$-circuit (i.e. a circuit with bounded depth, and weft 1). □

Corollary 4 can be proved along similar lines of Corollary 3.

Corollary 4. *Following problem is $W[1]$-complete: Given (G, k) and a constant l, does there exist an l-colorable induced subgraph of size k?*

Finally we address the parametric dual of the problem addressed in Corollary 3. Given a graph G, and an integer k, are there k vertices in G whose removal makes the graph bipartite? We will call this problem '$n-k$ bipartite'.

The precise parameterized complexity of this problem is unknown since though bipartiteness is a hereditary property, it has an infinite forbidden set, and so the problem is not covered by Cai's result [1].

The 'edge' counterpart of the problem, given a graph G with m edges, and an integer k, are there k edges whose removal makes the graph bipartite, is the same as asking for a cut in the graph of size $m-k$. It is known[10] that there exists a parameterized reduction from this problem to the following problem, which we call 'all but k 2-SAT'.

Given: A Boolean 2 CNF formula F

Parameter: An integer k

Question: Is there an assignment to the variables of F that satisfies all but at most k clauses of F?

We show that there is also a parameterized reduction from the '$n-k$ bipartite problem' to the 'all but k 2-SAT' problem.

Theorem 3. *There is a parameterized reduction from the '$n-k$ bipartite problem' to the 'all but k 2-SAT' problem.*

Proof. Given a graph G, for every vertex, we set two variables (x_u, y_u) and construct clauses in the same manner as in the proof of Corollary 3. The clauses are as follows:

Set 1:

$$\overline{x_u} \vee \overline{y_u} \quad \forall\ u \in V(G)$$
$$\overline{x_u} \vee \overline{x_v}\ ;\ \overline{y_u} \vee \overline{y_v} \quad \forall (u,v) \in E(G)$$

Each clause in Set 1 is repeated $k+1$ times.

Set 2: $x_u \vee y_u \quad \forall\ u \in V(G)$.

We show that it is possible to remove k vertices to make the given graph bipartite if and only if there is an assignment to the variables in the above formula that makes all but at most k clauses true.

If there is an assignment that makes all but at most k clauses true, then the clauses in Set 1 must be true because each of them occurs $k+1$ times. This ensures that the variables x_u, y_u corresponding to the vertices are assigned respectively 0,0 or 0,1 or 1,0 and each edge $e = (u, v)$ has $x_u = x_v = y_u = y_v = 0$ or $x_u = 0, y_u = 1$ and $x_v = 1, y_u = 0$ or vice versa. The vertices s for which

$x_s = y_s = 0$ are removed to get a bipartite graph. At most k clauses in Set 2 are false. This ensures that at most k vertices are removed.

Conversely if there exist k vertices whose removal results in a bipartite graph with partition $V_1 \bigcup V_2$, consider the assignment corresponding to each vertex u in the graph, $x_u = y_u = 0$ if the vertex u is removed, $x_u = 1, y_u = 0$ if $u \in V_1$ and $x_u = 0, y_u = 1$ if $u \in V_2$.

It is easy to see that this assignment makes all but at most k clauses of the formula true.

Note that the reduction is actually a polynomial time reduction. □

4 Concluding Remarks

We have characterized the hereditary properties for which finding an induced subgraph with k vertices having the property in a given graph is $W[1]$- hard. In particular, using Ramsey Theorem, we have shown that if the property includes all independent sets and all cliques or if it excludes some independent sets as well as cliques, then the problem is fixed parameter tractable. However, for some of these specific properties, we believe that a more efficient fixed parameter algorithms (not based on Ramsey numbers) is possible.

It remains an open problem to determine the parameterized complexity of both the problems stated in Theorem 3 (the '$n - k$ bipartite problem' and the 'all but k 2-SAT' problem). More generally, the parameterized complexity of the node-deletion problem for a hereditary property with an infinite forbidden set is open.

Finally we remark that our results prove that the parametric dual of the problem considered by Cai[1] (and was proved FPT) is $W[1]$-hard. This observation adds weight to the conjecture (first made in [10]) that typically parametric dual problems have complimentary parameterized complexity. It would be interesting to explore this in a more general setting.

References

1. L. Cai, Fixed Parameter Tractability of Graph Modification Problem for Hereditary Properties, *Information Processing Letters*, **58** (1996), 171-176.
2. R.G. Downey, M.R. Fellows, Fixed Parameter Tractability and Completeness I: Basic theory, *SIAM Journal on Computing* **24** (1995) 873-921.
3. R.G. Downey, M.R. Fellows, Fixed Parameter Tractability and Completeness II: Completeness for W[1], *Theoretical Computer Science* **141** (1995) 109-131.
4. R.G. Downey, M.R. Fellows, *Parameterized Complexity*, Springer Verlag New York, 1999.
5. R.G. Downey, M.R. Fellows, V. Raman, The Complexity of Irredundant sets Parameterized by Size, *Discrete Applied Mathematics* **100** (2000) 155-167.
6. M. R. Garey and D. S. Johnson, Problem [GT21], *Computers and Intractability, A Guide to The Theory of NP-Completeness*, Freeman and Company, New York, 1979.
7. M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs.* Academic Press, New york, 1980.

8. Harary, *Graph Theory*, Addison-Wesley Publishing Company, 1969.
9. J. M. Lewis and M. Yannakakis, The Node-Deletion Problem for Hereditary Properties is NP-complete, *Journal of Computer and System Sciences* **20(2)** (1980) 219-230.
10. M. Mahajan and V. Raman, Parameterizing above Guaranteed Values: MaxSat and MaxCut, *Journal of Algorithms* **31** (1999) 335-354.
11. A. Natanzon, R. Shamir and R. Sharan, Complexity Classification of Some Edge Modification Problems, in *Proceedings of the 25th Workshop on Graph-Theoretic Concepts in Computer Science* (WG'99), Ascona, Switzerland (1999); Springer Verlag Lecture Notes in Computer Science.