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Note

On the Diagonal Queens Domination Problem

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It is shown that the problem of covering an $n \times n$ chessboard with a minimum number of queens on a major diagonal is related to the number-theoretic function $r_3(n)$, the smallest number of integers in a subset of $\{1, \ldots, n\}$ which must contain three terms in arithmetic progression. © 1986 Academic Press, Inc.

Several problems concerning the covering of chessboards by queens have been studied in the literature [2]. In this note we are interested in determining the minimum number of queens which can be placed on the major diagonal of an $n \times n$ chesboard and dominate (cover) all squares. Suppose that the squares are labelled (i, j), so that black and white squares have $(i+j)$ even and odd, respectively. A subset K of $N = \{1, \ldots, n\}$ is called a *diagonal dominating set* if queens placed in positions $\{(k, k): k \in K\}$ on the black major diagonal dominate the entire board. Let

$$\text{diag}(n) = \min\{|K|; K \text{ is a diagonal dominating set}\}.$$

A subset X of N is called *midpoint-free* if for all $\{i, j\} \subseteq X$, $(i+j)/2 \notin X$ and X is called an *even-sum subset* if the sum of each pair of elements of X is even, i.e., its elements are either all odd or all even.

THEOREM 1. *K is a diagonal dominating set if and only if $N - K$ is a midpoint-free, even-sum set.*

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Proof. Suppose K is a diagonal dominating set and $\{i, j\} \subseteq N - K$. Then square (i, j) is not covered by a queen along a row or column. Since, only black squares are covered diagonally, square (i, j) must be black, which implies that $(i+j)$ is even, i.e., $N-K$ is an even-sum set. Since square (i, j) is covered, by a queen at position (k, k) for some $k \in K$, we have $i+j=2k$. Hence $(i+j)/2 \notin N-K$ and $N-K$ is midpoint-free.

Conversely, suppose $N-K$ is a midpoint-free, even-sum set. Place queens at $\{(k, k) \mid k \in K\}$. If (i, j) is a white square, i.e., $i+j$ is odd, then by the even-sum property, either i or j is in K and (i, j) is covered by a queen along a row or a column. If (i, j) is a black square and not covered by row or column, then $\{i, j\} \subseteq N-K$ and $i+j=2l$ for some $l \in N$. Since $N-K$ is midpoint-free, $l \notin N-K$. Therefore, $l \in K$ and (i, j) is dominated by the queen at position (l, l). This completes the proof.

COROLLARY 1. $\operatorname{diag}(n) = n - \max\{|X| \mid X$ *is midpoint-free, even-sum subset of* $N\}$.

The following elementary result simplifies the computation of midpoint-free, even-sum sets and enables us to relate $\operatorname{diag}(n)$ to a well-studied, number-theoretic function. We omit the proof.

PROPOSITION. $\{2a_1, 2a_2, \ldots, 2a_k\}$ *or* $\{2a_1 - 1, 2a_2 - 1, \ldots, 2a_k - 1\}$ *is a midpoint-free, even-sum set of* $\{1, \ldots, n\}$ *if and only if* $\{a_1, a_2, \ldots, a_k\}$ *is a midpoint-free subset of* $\{1, \ldots, \lceil n/2 \rceil\}$.

Let $r_3(n)$ denote the smallest k such that any k-subset of N contains a 3-term arithmetic progression. We note that $r_3(n) - 1$ is the largest cardinality of a midpoint-free subset of N. The above Proposition and Corollary 1 imply

THEOREM 2. $\operatorname{diag}(n) = n + 1 - r_3(\lceil n/2 \rceil)$.

COROLLARY 2. $\lim_{n \to \infty} (\operatorname{diag}(n)/n) = 1$.

Proof. Immediate from the theorem and the result of Roth [5] stating that $\lim_{n \to \infty} (r_3(n)/n) = 0$.

Several estimates for $r_3(n)$ have appeared in the literature [1–6] and these, together with Theorem 2, yield estimates for $\operatorname{diag}(n)$. It is a simple matter to obtain the first few values of $\operatorname{diag}(n)$ and minimum diagonal domination sets using a computer and Theorem 2. Some values are given in Table I.

It is interesting to note that minimum queens dominating sets in which queens may be placed anywhere on the chessboard, cannot, in general, be achieved by placing queens only on the major diagonal. A counter example

TABLE I

Values of diag(n) and Minimum Diagonal Dominating Sets

n	diag(n)	Minimum diagonal dominating set
7	4	$\{2, 4, 5, 6\}$
8	5	$\{2, 4, 5, 6, 8\}$
11	7	$\{1, 3, 5, 6, 7, 9, 11\}$
15	11	$N - \{2, 4, 8, 10\}$
20	15	$N - \{2, 4, 8, 10, 20\}$
24	18	$N - \{2, 4, 8, 10, 20, 22\}$
25	18	$N - \{1, 3, 7, 9, 19, 21, 25\}$
30	22	$N - \{1, 3, 7, 9, 19, 21, 25, 27\}$
40	31	$N - \{2, 4, 12, 14, 18, 28, 30, 36, 40\}$

occurs at $n = 11$. See [2, p. 76], where it is shows that 5 queens suffice to dominate an 11×11 chessboard, yet $\text{diag}(11) = 7$.

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References

1. F. A. Behrend, On sets of integers which contain no 3 terms in arithmetic progression, *Proc. Nat. Acad. Sci. U.S.A.* **32** (1946), 331–332.
2. R. K. Guy, "Unsolved Problems in Number Theory," Vol. 1. Springer-Verlag, New York, 1981.
3. L. Moser, On non-averaging sets of integers, *Canad. J. Math.* **5** (1953), 245–252.
4. J. Riddell, On sets of numbers containing no l terms in arithmetic progression, *Nieuw Arch. Wisk. (3)* **17** (1969), 204–209.
5. K. F. Roth, Sur quelques ensembles d'entiers, *C. R. Acad. Sci. Paris* **234** (1952), 388–390.
6. K. F. Roth, On certain sets of integers, *J. London Math. Soc.* **28** (1953), 104–109.