Ramsey Theory Is Needed for Solving Definability Problems of Generalized Quantifiers ⋆

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1 Introduction

In recent years, generalized quantifiers (see [H3]) have received quite a lot of novel interest because of their applications to computer science and linguistics. Their definability theory has made considerable progress during the last decade, which will be the subject of the next section. The proofs of many of these results often use results of Ramsey theory, such as theorems of van der Waerden and Folkman, and yet, the answers to some of the definability problems seem obvious from the outset. This raises the natural question whether Ramsey theory is really needed in the proofs (cf. [vBW]) or whether easier ways of proof might be discovered. The purpose of this paper is to argue in favour of the former and to convince the reader of the cruciality of Ramsey theory for quantifier definability theory.

The task of showing the necessity of Ramsey theory for quantifier theory does not translate easily to a rigorous mathematical problem, so the formulation of the framework is itself a problem. Roughly, we want to find an undefinability result $\mathcal{P}$ concerning quantifiers and a combinatorial theorem $\mathcal{K}$ and then show that $\mathcal{P} \Rightarrow \mathcal{K}$. We may as well fix that $\mathcal{K}$ is van der Waerden's theorem, whose variants have occured frequently in the recent quantifier literature. However, since $\mathcal{P}$ and $\mathcal{K}$ are both theorems of ZFC, the statement $\mathcal{P} \Rightarrow \mathcal{K}$ holds trivially, which is not really what we meant. This kind of difficulties are usually overcome in reverse mathematics by the metamathematical change to weaker axiom systems. However, I have abandoned this approach in order to preserve the clarity of the exposition. The reason is that generalized quantifiers are complex objects and the formulation of the basics of their theory in a weak axiom system would require a lot of tedious technical work.

Instead, a more combinatorial way of arguing is used. At first, a simple proof of $\mathcal{P} \Rightarrow \mathcal{K}$ is presented (in Section 4), which should already convince a full-fledged pragmatic. In essence, this proof shows that certain function related to quantifiers grow faster than van der Waerden's function W, which will be elaborated in Section 5. What exactly is meant, becomes more apparent as we proceed.

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We have not yet paid any attention to the choice of the undefinability result $\mathcal{P}$. Clearly it is not reasonable to allow $\mathcal{P}$ to be any quantifier result whatsoever, since it may be possible to code extremely difficult mathematical problems as quantifier problems; we should rather try to find a result $\mathcal{P}$ relevant for the field. As a result, we are not ready for the choice of $\mathcal{P}$ until in Section 4. Before that, we shall have a look at the problematics of quantifier definability theory in Section 2. We shall find out that we may as well restrict our considerations to monadic quantifiers, whose theory is sketched in the Section 3. The point is that, in the case of monadic quantifiers, the definability questions can be reduced to colouring problems about relations. These, in turn, give rise to certain fast-growing functions which give an upper bound for the van der Waerden's function.

2 Quantifier Definability Theory

Although 40 years has passed since Andrzej Mostowski presented his notion of a generalized quantifier, systematic treatment of definability problems of quantifiers is a fresh research subject. Up to the mid-1980's, the emphasis of the research in generalized quantifier theory was on finding logics with good model-theoretic properties. Definability and undefinability results were merely by-products of this study, e.g., Keisler's proof [K] that $\mathcal{L}_{\omega\omega}(Q_1)$ does not have the Δ-interpolation property was based on the following hidden result:

Theorem 1. (Keisler) *Q_1^E is not definable in $\mathcal{L}_{\omega\omega}(Q_1)$.* □

To explain the quantifiers involved, let us introduce some notation. For a class of cardinals S, let C_S be the quantifier with defining class

$$K_{C_S} = \{\, \mathfrak{M} \in \mathrm{Str}(\{U\}) \mid \left|U^{\mathfrak{M}}\right| \in S \,\}$$

where U is a unary relation symbol. In other words, for every $\{U\}$-structure $\mathfrak{M}$ we have

$$\mathfrak{M} \models C_S x\, U(x) \iff \left|U^{\mathfrak{M}}\right| \in S.$$

Quantifiers of form C_S are called *cardinality quantifiers*. Similarly, E_S is the quantifier whose vocabulary is $\{E\}$ with E binary, and defining class K_{E_S} is the class of all structures $\mathfrak{M} \in \mathrm{Str}(\{E\})$ such that $E^{\mathfrak{M}}$ is an equivalence relation with the number of equivalence classes κ belonging to S. Clearly C_S is definable in $\mathcal{L}_{\omega\omega}(E_S)$ by the sentence

$$E_S xy\ (U(x) \wedge x = y).$$

Now $Q_1 = C_S$ and $Q_1^E = E_S$ where S is the class of all uncountable cardinals. Hence, Keisler's result shows that E_S is not necessarily definable in $\mathcal{L}_{\omega\omega}(C_S)$. His counterexample to Δ-interpolation of $\mathcal{L}_{\omega\omega}(Q_1)$ was later generalized by Xavier Caicedo [Ca2]. This, in turn, inspired some quantifier definability theory in finite model theory related to quantifiers similar to E_S by Flum, Schielen and Väänänen [FSV].

As it has become clear, the simplest form of a definability problem is the following:

$$(*) \qquad \text{Is } Q \text{ definable in the logic } \mathcal{L}_{\omega\omega}(\mathcal{Q})?$$

where Q is a generalized quantifier and $\mathcal{Q}$ is a set of such. Most of the papers on generalized quantifiers where this is the main motivation are less than fifteen years old. One of the earliest is the paper by Luis Jaime Corredor [Co] solving completely the definability problems amongst cardinality quantifiers. Denote by $\oplus$ the common extension of integer and cardinal addition such that $\kappa \oplus n = n \oplus \kappa = \kappa$ for every infinite cardinal κ and $n \in \mathbb{Z}$. For S a class of cardinals and $n \in \mathbb{Z}$, let $S \oplus n = \{\, \kappa \oplus n \mid \kappa \in S \,\}$.

Theorem 2. (Corredor) *Let S and T_i, $i \in I$, be classes of cardinals. Then the following are equivalent:*
1) C_S is definable in $\mathcal{L}_{\omega\omega}(\{\, C_{T_i} \mid i \in I \,\})$.
2) There is a (finite) Boolean combination T of classes $T_i \oplus n$, $i \in I$ and $n \in \mathbb{Z}$, such that $|S \Delta T| < \omega$, i.e., the symmetric difference of S and T is finite. □

Observing that $\exists = C_E$ where E is the class of non-zero cardinals, case 2 can actually be replaced by
2') S is a Boolean combination of classes $T_i \oplus n$ and $E \oplus n$ where $i \in I$ and $n \in \mathbb{Z}$.
In the hindsight we may say that the neat formulation of the preceding theorem is possible because cardinality quantifiers enhance the expressive power of the first order logic by expressing things about extremely simple structures: the defining class of a cardinality quantifier consists of structures whose vocabulary contains only one predicate that is unary. In the same vein, it is reasonable to ask if theorem 1 is only an instance of a more general result, i.e., since the vocabulary of Q_1^E is $\{E\}$ with E binary and that of Q_1 is U with U unary, could it be possible that Q_1^E is not definable by any set $\mathcal{Q}$ of cardinality quantifiers. As it happens, this is exactly the case, and this is not accidental. For Q a quantifier with vocabulary τ, let the *arity* of Q be

$$\operatorname{ar}(Q) = \sup\{\, n_R \mid R \in \tau \,\}$$

where n_R is the arity of the relation symbol $R \in \tau$. A quantifier Q is *monadic*, if $\operatorname{ar}(Q) = 1$, *binary*, if $\operatorname{ar}(Q) = 2$, and *ternary*, if $\operatorname{ar}(Q) = 3$. The collection of all quantifiers Q with arity at most n is denoted by $\mathbf{Q}_n$. Then we have the following result due to Hella [H1] and implicitly by Caicedo [Ca1].

Theorem 3. *(Caicedo, Hella) Q_1^E is not definable by monadic quantifiers, i.e., in the logic $\mathcal{L}_{\omega\omega}(\mathbf{Q}_1)$, nor even in $\mathcal{L}_{\infty\omega}(\mathbf{Q}_1)$.* □

Even better, after the preliminary results of the other people (e.g., [G], [V1]) Hella was able to establish the following hierarchy result, among other results.

Theorem 4. *(Hella) For every $n \in \omega$ and non-zero ordinal α, the Magidor-Malitz quantifier Q_α^{n+1} is not definable in the logic $\mathcal{L}_{\infty\omega}(\mathbf{Q}_n)$.* □

The definition of Q^n_α is omitted; an interested reader may consult the original paper [H1] or the survey [HL].

At that time in the late 1980's, the focus in the research of generalized quantifiers was rapidly shifting towards finite structures. Hella's methods, certain model-theoretic games, were easily adaptable in the new context, as shown in the paper [H2]. Kolaitis and Väänänen [KV] did some systematic study in the realm of monadic simple quantifiers. A quantifier Q is *simple*, if its vocabulary consists of a single relation symbol. The *Härtig quantifier*, or the equicardinality quantifier, I, is an example of a monadic quantifier that is not simple; it is the monadic quantifier binding two formulas with the defining class

$$K_I = \{\, \mathfrak{M} \in \mathrm{Str}(\{U, V\}) \mid \left|U^{\mathfrak{M}}\right| = \left|V^{\mathfrak{M}}\right| \,\}.$$

The *Rescher quantifier* R has the same vocabulary as I, but its defining class is

$$K_R = \{\, \mathfrak{M} \in \mathrm{Str}(\{U, V\}) \mid \left|U^{\mathfrak{M}}\right| \le \left|V^{\mathfrak{M}}\right| \,\}.$$

I is easily definable in $\mathcal{L}_{\omega\omega}(R)$ by the sentence

$$Rxy(U(x), V(y)) \wedge Rxy(V(x), U(y)).$$

Kolaitis and Väänänen proved the following:

Theorem 5. *(Kolaitis and Väänänen) Let $\mathcal{Q}$ be a finite set of simple monadic quantifiers. Then:*
a) I is not definable in $\mathcal{L}_{\omega\omega}(\mathcal{Q})$.
b) R is not definable in $\mathcal{L}_{\omega\omega}(\mathcal{Q} \cup \{I\})$.
c) $E_{2\mathbb{N}}$ is not definable in $\mathcal{L}_{\omega\omega}(\mathcal{Q} \cup \{I\})$ where $2\mathbb{N}$ is the set of even natural numbers.
These statements hold even if restricted to finite structures. □

A notable feature in the proof is that it rests on the following results of Ramsey theory:

van der Waerden's Theorem. [vW] *For every $k, t \in \mathbb{N}$, there is $w \in \mathbb{N}$ such that if the set $\{0, \ldots, w-1\}$ is coloured with at most t colours, say, by the colouring $\chi\colon \{0, \ldots, w-1\} \to F$ with $|F| \le t$, then there is a monochromatic arithmetic progression of length k, i.e., there are $a, d \in \mathbb{N}$, $d \ne 0$, such that $a + (k-1)d < w$ and $\chi(a) = \chi(a + id)$, for every $i = 0, \ldots, k-1$.* □

Folkman's Theorem. *For every $k, t \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that for every colouring $\chi\colon \{0, \ldots, w-1\} \to F$ with at most t colours (i.e., $|F| \le t$), there exist $K \subset \mathbb{N}$ of size $|K| = k$ such that $\sum_{i \in K} i < n$ and for all non-empty $I, J \subset K$, we have $\chi(\sum_{i \in I} i) = \chi(\sum_{j \in J} j)$.* □

These theorems give rise to combinatorial functions, e.g., van der Waerden's function $W\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where for every $k, t \in \mathbb{N}$, $W(k, t)$ is the least $w \in \mathbb{N}$ satisfying the condition in the theorem. We also denote $W_2\colon \mathbb{N} \to \mathbb{N}$, $W_2(n) = W(n, 2)$.

The reason why finite model theory pushed combinatorics in the front-line lies in the nature of the model constructions. While infinite structures can be made closed under different kinds of conditions and combinatorics remains hidden in the structures, this is usually not possible when constructing finite structures. This means that explicit use of combinatorial principles is required. In the sequel, Ramsey theory became a common tool in quantifier definability theory. The hierarchy result 4 inspired search for refinements of the hierarchy. In particular, linguistic interest (see [W]) seems to have spurred research on monadic quantifiers. The *width* of the quantifier Q is just the cardinality of its vocabulary τ, in symbols, $\mathrm{wd}(Q) = |\tau|$. Various researchers ([Li],[NV],[Lu]) obtained independently the following monadic hierarchy result:

Theorem 6. (Lindström, Nešetřil and Väänänen, Luosto) *For every $n \in \mathbb{N}$, there is a monadic quantifier of width $n+1$ which is not definable by means of monadic quantifiers of width at most n, i.e., in any $\mathcal{L}_{\omega\omega}(\mathcal{Q})$ where $\mathcal{Q}$ is a set of monadic quantifiers of width at most n.* □

The explained involvement in combinatorics caught the eye of Johan van Benthem and Dag Westerståhl. They end the section 3 of their paper [vBW] by a short speculation of the need of the Ramsey theory. After a short description of the results and the methods in [KV], they write:

> "This leads to the question whether every proof of the definability result requires Ramsey theory?"

And later:

> "Given the very general nature of generalized quantifiers it may be worthwhile to do some 'reverse mathematics' in the field of finite combinatorics and definability questions, and thus to assess the *combinatorial content* of certain results about generalized quantifiers."

In order to be able to answer to these demands, we need to sketch the basics of monadic quantifier definability theory from [Lu] in the next section.

3 Reduction to Combinatorics

In this section some basics of the monadic quantifier definability theory are described, which is a prerequisite for the analysis of the combinatorial contents of undefinability results. The technicalities are kept aside, so that the presentation is necessarily sketchy and appeals a lot on the intuition of the reader. (The results and exact definitions can be found in [Lu].) Something concrete is needed for the further treatment, though, and this core of the theory is condensed in the notion of the irreducibility of a relation in the end of the section.

To start with, we note that monadic structures are easily describable. A structure with three unary predicates, say $\mathfrak{M} \in \mathrm{Str}(\{P, Q, R\})$, is drawn in the Figure 1 below.

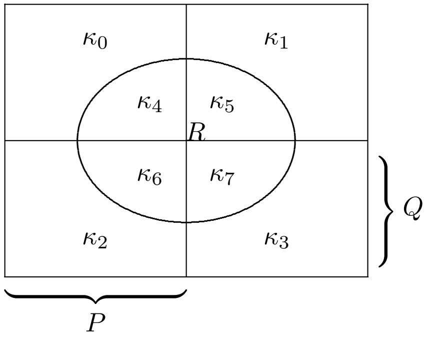

Figure 1

The structure $\mathfrak{M}$ is uniquely determined if we count the number of elements in each part, i.e., by the tuple $(\kappa_0, \ldots, \kappa_7)$. In general, a monadic structure $\mathfrak{M}$ for a finite vocabulary τ, say $|\tau| = l$, can be identified by a tuple $\kappa_{\mathfrak{M}}$ of length 2^l of cardinal invariants. In other words, if $\mathfrak{M}, \mathfrak{N} \in \mathrm{Str}(\tau)$, then $\mathfrak{M}$ and $\mathfrak{N}$ are isomorphic iff $\bar{\kappa}_{\mathfrak{M}} = \bar{\kappa}_{\mathfrak{N}}$.

Recall that for each cardinality quantifier Q there is a class S of cardinals such that $Q = C_S$. Analogously, each monadic quantifier Q with finite vocabulary can be reduced to a relation

$$\mathcal{R}(Q) = \{\, \bar{\kappa}_{\mathfrak{M}} \mid \mathfrak{M} \in K_Q \,\}.$$

Example 1. The vocabulary of the Härtig quantifier is $\{U, V\}$ and for $\mathfrak{M} \in \mathrm{Str}(\{U, V\})$, we have

$$\mathfrak{M} \in K_I \iff \left|U^{\mathfrak{M}}\right| = \left|V^{\mathfrak{M}}\right|.$$

To describe $\mathfrak{M} \in \mathrm{Str}(\{U, V\})$ up to isomorphism, we count that there are $\kappa_{0,\mathfrak{M}}$ elements in the intersection of the predicates, $U^{\mathfrak{M}} \cap V^{\mathfrak{M}}$, $\kappa_{1,\mathfrak{M}}$ elements in $U^{\mathfrak{M}} \smallsetminus V^{\mathfrak{M}}$, $\kappa_{2,\mathfrak{M}}$ elements in $V^{\mathfrak{M}} \smallsetminus U^{\mathfrak{M}}$ and $\kappa_{3,\mathfrak{M}}$ elements outside the predicates $U^{\mathfrak{M}}$ and $V^{\mathfrak{M}}$ (note that we have to fix some order in which to enumerate the invariants, but it is immaterial which particular order, as soon as it is the same for all structures). Therefore,

$$\bar{\kappa}_{\mathfrak{M}} = \left(\left|U^{\mathfrak{M}} \cap V^{\mathfrak{M}}\right|, \left|U^{\mathfrak{M}} \smallsetminus V^{\mathfrak{M}}\right|, \left|V^{\mathfrak{M}} \smallsetminus U^{\mathfrak{M}}\right|, \left|\mathrm{Dom}(\mathfrak{M}) \smallsetminus U^{\mathfrak{M}} \cup V^{\mathfrak{M}}\right|\right)$$

and

$$\mathfrak{M} \in K_I \iff \kappa_{0,\mathfrak{M}} \oplus \kappa_{1,\mathfrak{M}} = \kappa_{0,\mathfrak{M}} \oplus \kappa_{2,\mathfrak{M}}.$$

Hence,

$$\mathcal{R}(I) = \{\, (\kappa_0, \kappa_1, \kappa_2, \kappa_3) \mid \kappa_0 \oplus \kappa_1 = \kappa_0 \oplus \kappa_2 \,\}.$$

Note that for finite $\mathfrak{M} \in \mathrm{Str}(\{U, V\})$ we have the simple relation

$$\mathfrak{M} \in K_I \iff \kappa_{1,\mathfrak{M}} = \kappa_{2,\mathfrak{M}}.$$

Similarly, for the Rescher quantifier we get

$$\mathcal{R}(R) = \{ (\kappa_0, \kappa_1, \kappa_2, \kappa_3) \mid \kappa_0 \oplus \kappa_1 \leq \kappa_0 \oplus \kappa_2 \}$$

and

$$\mathcal{R}(R) \cap \mathbb{N}^4 = \{ (m_0, m_1, m_2, m_3) \in \mathbb{N}^4 \mid m_1 \leq m_2 \}.$$

There is a result analogous to Theorem 2 for cardinality quantifiers that characterizes the definability among monadic quantifiers. In order to explain the notation, let us consider the structure $\mathfrak{A}$ from Figure 1 and the sentence

$$\varphi = Ixy(P(x) \vee Q(x), R(y)).$$

Then $\mathfrak{A} \models \varphi$ iff $\left|P^{\mathfrak{A}} \cup Q^{\mathfrak{A}}\right| = \left|R^{\mathfrak{A}}\right|$ iff the interpreted structure $\mathfrak{M}$ is in K_I for which $U^{\mathfrak{M}} = P^{\mathfrak{A}} \cup Q^{\mathfrak{A}}$ and $V^{\mathfrak{M}} = R^{\mathfrak{A}}$.

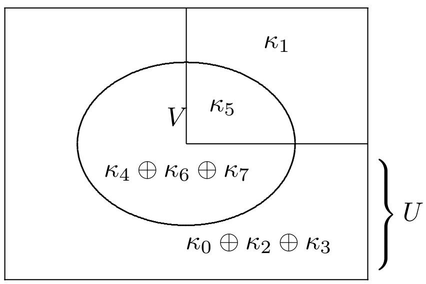

Figure 2

It is easily seen that $\bar{\kappa}_{\mathfrak{M}} = (\kappa_4 \oplus \kappa_6 \oplus \kappa_7, \kappa_0 \oplus \kappa_2 \oplus \kappa_3, \kappa_5, \kappa_1)$. Schematically we get $\bar{\kappa}_{\mathfrak{M}}$ from the components of $\bar{\kappa}_{\mathfrak{A}}$ when we sum over the set of indices $\{4,6,7\}$, $\{0,2,3\}$, $\{5\}$ and $\{1\}$, each in turn. In symbols, we write

$$\bar{\kappa}_{\mathfrak{M}} = \bar{s}(\bar{\kappa}_{\mathfrak{A}}, \overline{U})$$

where $\overline{U} = (\{4,6,7\}, \{0,2,3\}, \{5\}, \{1\})$. Then we have $\mathfrak{A} \models \varphi$ iff $\bar{s}(\bar{\kappa}, \overline{U}) \in \mathcal{R}(I)$.

Similarly to Theorem 2 we have to take translates into account. Below $J_{n,l}$ stands roughly for the set of all meaningful pairs $(\overline{U}, \bar{n})$ when we reduce the size of the tuple from n to l. Reformulating a result of Väänänen [V2] in our notation, we get:

Theorem 7. (Väänänen) *Let Q be a monadic quantifier and $\mathcal{Q}$ a set of monadic quantifiers, all with finite vocabularies. Let $n = 2^{\mathrm{wd}(Q)}$ and $l_q = 2^{\mathrm{wd}(q)}$, for each $q \in \mathcal{Q} \cup \{\exists\}$. Then the following are equivalent:*
1) Q is definable in $\mathcal{L}_{\omega\omega}(\mathcal{Q})$.

2) $\mathcal{R}(Q)$ is a Boolean combination of relations of form

$$\{\, \bar{\kappa} \mid \bar{s}(\bar{\kappa}, \overline{U}) \oplus \bar{n} \in \mathcal{R}(q) \,\}$$

where $q \in \mathcal{Q} \cup \{\exists\}$ and $(\overline{U}, \bar{n}) \in J_{n,l_q}$. □

It is noteworthy that if we replace $\mathcal{R}(Q)$ by R and $\mathcal{R}(Q)$'s by some S_i's in case 2, what is left is a totally combinatorial condition on relations. This kind of consideration led to the notions of the rank $r(R)$ of a relation R and the relative rank $r_+(R)$ of a relation relative to a monoid $\langle M, +\rangle$ in [Lu]. The former is simpler, but the latter corresponds better to the intended application.

A quantifier Q with vocabulary τ is called *universe-independent*, if we have $\mathfrak{A} \in K_Q$ iff $\mathfrak{B} \in K_Q$ whenever $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$ are such that for every $R \in \tau$, it holds that $R^{\mathfrak{A}} = R^{\mathfrak{B}}$. Then the following hold:

Theorem 8. *Let $l \in \mathbb{N}$ and κ be an infinite cardinal and C be the set of cardinals below κ. Let Q be a monadic quantifier such that $n = 2^{\mathrm{wd}(Q)} \in \mathbb{N}$. Then the following are equivalent:*
1) Q is definable by monadic universe-independent quantifiers of width at most l on structures of cardinality less than κ.
2) $r_\oplus(\mathcal{R}(Q) \cap C^n) \leq 2^l - 1$. □

Proposition 1. *Let $R \subset C^n$ be a relation where C is a set of cardinals. Then $r_\oplus(R) \leq r(R)$. If C is infinite, $C \cap \mathbb{N} = \{0\}$ and $r(R) > 3$, then $r_\oplus(R) = r(R)$.* □

Let me sketch one application of these combinatorial results. Every quantifier has liftings which are called resumptions. For example, the second resumption of the Härtig quantifier is the quantifier $I^{(2)}$ with the vocabulary $\{R, S\}$ where R and S are binary and such that for every $\mathfrak{M} \in \mathrm{Str}(\{R, S\})$,

$$\mathfrak{M} \models I^{(2)}xy, tu(R(x, y), S(t, u))$$

iff

$$|R^{\mathfrak{M}}| = |S^{\mathfrak{M}}|,$$

i.e., if there are as many *pairs* in $R^{\mathfrak{M}}$ as there are pairs in $S^{\mathfrak{M}}$. Note that even if the definitions of I and $I^{(2)}$ have the same appearance, $I^{(2)}$ is a binary quantifier rather than monadic, since it binds two variables in both formulas.

Westerståhl posed, among other problems, the question if $I^{(2)}$ is definable in terms of monadic quantifiers of finite width. The question was answered to the negative in [Lu]. Indeed, suppose to the contrary that there is a finite set $\mathcal{Q}$ of monadic quantifiers of finite width such that $I^{(2)}$ is definable in $\mathcal{L}_{\omega\omega}(\mathcal{Q})$. Without loss of generality, these quantifiers are universe-independent. Pick $m \in \mathbb{N}$ such that $m \geq \mathrm{wd}(Q)$, for every $Q \in \mathcal{Q}$. Let $n = 2^m$. For $\bar{q} \in \mathbb{Z}^n$, consider the relation

$$R_{\bar{q}} = \{\, \bar{x} \in \mathbb{N}^n \mid \bar{x} \neq \bar{0},\ \bar{q} \text{ and } \bar{x} \text{ orthogonal} \,\}.$$

(Here, orthogonality is just the usual notion of linear algebra, i.e., $\bar{q}$ and $\bar{x}$ are orthogonal iff $\bar{q} \cdot \bar{x} = 0$.) Then it can be shown that if $Q_{\bar{q}}$ is the quantifier with $\mathcal{R}(Q_{\bar{q}}) = R_{\bar{q}}$, then Q is definable in $\mathcal{L}_{\omega\omega}(I^{(2)})$ and therefore also in $\mathcal{L}_{\omega\omega}(\mathcal{Q})$. To demonstrate this by an example, consider $\bar{q} = (0, 3, -2, 0)$. Supposing that the vocabulary of the quantifier is $\{U, V\}$ and the indices as in Example 1, put

$$\begin{aligned} \exists y_0, y_1, y_2 \quad \exists t_0, t_1 \quad \Big(&\neg y_0 = y_1 \wedge \neg y_0 = y_2 \wedge \neg y_1 = y_2 \wedge \neg t_0 = t_1 \wedge \\ &I^{(2)} xy, zt \,\big(U(x) \wedge \neg V(x) \wedge (x = x_0 \vee x = x_1 \vee x = x_2), \\ &\qquad V(x) \wedge \neg U(x) \wedge (t = t_0 \vee t = t_1)\big)\Big). \end{aligned}$$

Then for every $\mathfrak{M} \in \mathrm{Str}(\{U, V\})$, $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \in K_{Q_{\bar{q}}}$ provided that there are at least three elements in $\mathfrak{M}$.

Now by the previous theorem, we have $r_{\oplus}(R_{\bar{q}}) \leq 2^m - 1 < n$. On the other hand, it may be shown that there are vectors $\bar{q}$ such that $r_{\oplus}(R_{\bar{q}}) = n$, which is a contradiction.

The notion of irreducibility is derived from the concept of the rank, not of that of relative rank. The preceding results show that the rank works well if we consider infinite structures. Since the emphasis of today's research is on finite structures, though, we might have a problem. The explanation why irreducibility is based on rank rather than the relative rank is postponed later.

Definition 1. *Let k and n be positive integers. The relation $R \subset A^n$ is k-reducible, if there are colourings $\xi_i \colon A^{n-1} \to F_i$ of A^{n-1} with at most k colours ($|F_i| \leq k$), for $i = 0, \dots, n-1$, such that the following holds: Let $\xi \colon A^n \to F_0 \times \cdots F_{n-1}, \xi(a_0, \dots, a_{n-1}) = (c_0, \dots, c_{n-1})$ where*

$$c_i = \xi_i(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_{n-1}).$$

(We call ξ the merge of ξ_i, $i = 0, \dots, n-1$. We use this notion even if the sequence ξ_i is not complete.) Then for $\bar{a}, \bar{b} \in A^n$, if $\xi(\bar{a}) = \xi(\bar{b})$ and $\bar{a} \in R$, then also $\bar{b} \in R$. R is reducible if it is k-reducible for some $k \in \mathbb{N}^*$, otherwise R is irreducible.

Since understanding the definition of irreducibility is absolutely crucial for the purpose of this paper, let me repeat the definition in a different form: A relation R is reducible if there is a finite alphabet Σ and a language L of words of length n such that, given $\bar{a} \in A^n$, you can decide if $\bar{a} \in R$ by the following kind of a procedure. You first write down the tuple

$$\bar{a} = (a_0, \dots, a_i, \dots, a_{n-1})$$

on the paper. Then, for each $i = 0, \dots, n-1$ consecutively, you first hide the component a_i, then take a look at the remaining tuple

$$\bar{a}' = (a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_{n-1}),$$

write down your notes about the tuple, or actually just one letter $c_i \in \Sigma$, and then forget about the whole thing. After the process, you have written a word

$$w = c_0 c_1 \cdots c_{n-1}.$$

If $w \in L$, then $\bar{a} \in R$, otherwise $\bar{a} \notin R$.

Example 2. Let X be the set of all people and R the relation of all triples $(a, b, c) \in X^3$ such that a, b and c are sisters. Then R is reducible, even 2-reducible (actually all finite relations are reducible, so k-reducible for some k, but the parameter k there may vary). Indeed, first hide c and take a look at the pair (a, b). If a and b are sisters, write down Y(es), otherwise N(o). Proceed similarly, and if in the end you have the word YYY on the sheat, a and b and c are sisters.

The notion of irreducibility of n-ary R has been tailored equivalent to the statement $r(R) = n$. Let us then present a special case of Theorem 8. A monadic quantifier of finite width Q is called a *set quantifier* if $\mathcal{R}(Q)$ is a set, or equivalently $\mathcal{R}(Q) \subset C^n$ for some set C of cardinals and $n \in \mathbb{N}^*$. A monadic set quantifier Q of finite width is *biassed towards infinite*, if this C contains only infinite cardinals and possibly 0. Another way to put this is to say that there is an infinite κ such that for every $\mathfrak{M} \in K_Q$, the cardinality of $\mathfrak{M}$ is less than κ and every element a of $\mathfrak{M}$ has an infinite orbit under the automorhism group of $\mathfrak{M}$. Theorem 8, with the aid of Proposition 1 and the fact that irreducible relations are just those whose rank is the same as arity, implies the following theorem.

Theorem 9. *Let Q be a monadic set quantifier biassed to infinite with* $\mathrm{wd}(Q) \geq 2$. *Then the following are equivalent:*
1) Q is not definable in terms of monadic universe-independent quantifiers of width at most $\mathrm{wd}(Q)$.
2) $\mathcal{R}(Q)$ is irreducible. □

In the sequel, we need some technical results on irreducibility.

Lemma 1. *A non-unary relation $R \subset \mathbb{N}^n$ is irreducible iff for every $k \in \mathbb{N}^*$ there is $l \in \mathbb{N}$ such that $R \subset \{0, \ldots, l-1\} \subset \mathbb{N}^n$ is k-irreducible.*

Proof. Let us prove the equivalence of negations. Suppose first R is reducible. By definition, R is then k-reducible for some $k \in \mathbb{N}^*$ and for $i = 0, \ldots, n-1$, there are colourings $\xi_i \colon \mathbb{N}^{n-1} \to F_i$ such that for every $\bar{a}, \bar{b} \in \mathbb{N}^n$, if $\bar{a} \in R$ and $\xi(a) = \xi(b)$, then $\bar{b} \in R$, where ξ is the merge of the colourings ξ_i, $i = 0, \ldots, n-1$. Let $l \in \mathbb{N}$ be arbitrary. Pick a new colour c^* and put

$$\xi_i' : \mathbb{N}^{n-1} \to F_i \cup \{c^*\}, \xi_i'(\bar{a}) = \begin{cases} \xi_i(\bar{a}), \text{ for } \bar{a} \in \{0, \ldots, l-1\}^n \\ c^*, \quad \text{otherwise.} \end{cases}$$

Then clearly colourings ξ_i', $i = 0, \ldots, n-1$ show that $R \subset \{0, \ldots, l-1\} \subset \mathbb{N}^n$ is k+1-reducible.

To the other direction, suppose that there is $k \in \mathbb{N}^*$ such that for every $l \in \mathbb{N}$, the relation $R \subset \{0, \dots, l-1\}^n$ is k-reducible. We apply techniques from [LT]. Endowing the space $\mathcal{P}(\mathbb{N}^n)$ with its natural topology it can be shown that the set K ($K = K_n^n - 1, k, \mathbb{N}$ in the notation of [LT]) of k-reducible relations of $\mathbb{N}$ is a closed subset of $\mathcal{P}(\mathbb{N}^n)$. Now either for some $l \in \mathbb{N}$ it holds that $R = R \cap \{0, \dots, l-1\}^n$ or R is an accumulation point of $\{ R \cap \{0, \dots, l-1\}^n \mid l \in \mathbb{N} \} \subset K$. In both of the cases, $R \in K$, i.e., R is k-irreducible. $\square$

Definition 2. *Let $R \subset \mathbb{N}^n$ be a non-unary irreducible relation. Then the complexity of irreducibility of R is $i_R \colon \mathbb{N}^* \to \mathbb{N}$,*

$$i_R(k) = \min\{ l \in \mathbb{N} \mid R \cap \{0, \dots, l-1\}^n \subset \mathbb{N}^n \text{ is } k\text{-irreducible.} \}$$

4 Reverse Combinatorics: Proving van der Waerden's Theorem

We are finally in the position to choose the undefinability result $\mathcal{P}$ on which to base our analysis. For any ordinal α, denote $\operatorname{ind}(\aleph_\alpha) = \alpha$. For each $l \in \mathbb{N}^*$, let $n = 2^l$ and let S_l be the monadic quantifier with the vocabulary $\tau_n = \{U_0, \dots, U_{n-1}\}$ and with the defining class K_{S_l} consisting of $\mathfrak{M} \in \mathrm{Str}(\tau_n)$ such that all $U_i^{\mathfrak{M}}$, $i = 0, \dots, n-1$ are disjoint and infinite, but $\left|U_i^{\mathfrak{M}}\right| < \aleph_\omega$, and

$$\operatorname{ind}(\left|U_{n-1}^{\mathfrak{M}}\right|) = \sum_{i=0}^{n-2} \operatorname{ind}(\left|U_i^{\mathfrak{M}}\right|).$$

Let $\mathcal{P}_l$ be the following undefinability result (cf. [Lu, Example 4.6]).

$$\begin{gathered}\mathcal{P}_l : \text{The quantifier } S_l \text{ is not definable by means}\\ \text{of any universe-independent quantifiers of width } l.\end{gathered}$$

Moreover, let $\mathcal{P}$ be the statement: For every $l \in \mathbb{N}$, $\mathcal{P}_l$ holds.

Let us analyze quantifiers S_l in the spirit of the previous section. Let $\mathfrak{M} \in K_{S_l}$. If we enumerate the invariants in a suitable order, then

$$\bar{\kappa}_{\mathfrak{M}} = (\aleph_{m_0}, \dots, \aleph_{m_{n-1}}, 0, \dots, 0, \lambda)$$

with

$$m_{n-1} = \sum_{i=0}^{n-2} m_i$$

where for $i = 0, \dots, n-1$, $\aleph_{m_i} < \aleph_\omega$ is the number of elements in $U_i^{\mathfrak{M}}$, but outside other predicates, zeros refer to the empty intersections, and λ is the number of elements outside the predicates. Conversely, any $\mathfrak{M} \in \mathrm{Str}(\tau_l)$ with an invariant of this form is in K_{S_n}, so that

$$\begin{aligned}\mathcal{R}(S_n) = \{(\aleph_{m_0}, \dots, \aleph_{m_{n-1}}, 0, \dots, 0, \lambda) \mid\ & m_0, \dots, m_{n-1} \in \mathbb{N},\\ & \lambda \neq 0 \text{ or } m_{n-1} \neq 0, \text{ and } m_{n-1} = \textstyle\sum_{i=0}^{n-2} m_i\}.\end{aligned}$$

Constant components and vacuous variables (such as λ) do not have any effect on the ranks of relations (cf. [Lu]), nor does the replacement of $\aleph_j$ by j. So for every $l \in \mathbb{N}^*$ and $n = 2^l$, $\mathcal{P}_n$ is equivalent to the following statement:

$\mathcal{P}'_n$: The relation

$$T_n = \{ (m_0, \dots, m_{n-1}) \in \mathbb{N}^n \mid m_{n-1} = \sum_{i=0}^{n-2} m_i \}$$

is irreducible.

Where does the relevance of this result lie? Firstly, $\mathcal{P}$ gives an easy way to prove the monadic hierarchy theorem, i.e., the fact that extending the width has the capacity of enhancing the expressive power of monadic quantifiers. Since this motive comes from inside the field rather than from applications, let me point out that in the notation of the previous section, $T_n = R_{\bar{q}}$ with $\bar{q} = (1, 1, , \dots, 1, -1) \in \mathbb{Z}^n$, for every $n \in \mathbb{N}$, $n \geq 2$, so that this problematics is related to the proof that $I^{(2)}$ is not definable by monadic quantifiers. Unfortunately, irreducibility does not mean that the simple relations T_n were useful for that particular proof. As it happens, $r_\oplus(T_n) = 2$ albeit $r(T_n) = n$, which illustrates well the difference between finite and infinite arithmetics. In spite of this, we may view it as a mere pragmatic simplification that we use the result $\mathcal{P}$ rather than undefinability of $I^{(2)}$ as the basis of our analysis. Indeed, we could have introduced the notion of irreducibility for the relative rank, and then we would have got functions $i_R^\oplus$ satisfying $i_R^\oplus(n) \geq i_R(n)$, for every $n \in \mathbb{N}^*$, and we could have gone through an analysis in the similar vein for relations of form $R_{\bar{q}}$. The trade-off would have been bad in the sense that we ought to have dealt with more complicated notions, with linear-algebraic technicalities (cf. [Lu, Section 5]) and in the end, the results would have been almost the same as we shall get now.

Abandoning this side-track, we first prove a technical lemma concerning partial functions. For tuples $\bar{a}$ and $\bar{b}$, let $\bar{a}\hat{\ }\bar{b}$ be the concatenation of $\bar{a}$ and $\bar{b}$.

Lemma 2. *Let $k, n \in \mathbb{N}^*$ and let f be an n-ary partial function on A which is $k^n + 1$-irreducible. Let $\xi_i \colon A^n \to F_i$, $i = 0, \dots, n-1$, be finite colourings with at most k colours. Let ξ be the merge of ξ_i, $i = 0, \dots, n-1$. Then there are $\bar{a} = (a_0, \dots, a_{n-1}) \in \operatorname{dom}(f)$ and $c \in A$ such that $c \neq f(\bar{a})$ and $\xi(\bar{a}\hat{\ }(f(a))) = \xi(\bar{a}\hat{\ }(c))$, i.e., $\xi(a_0, \dots, a_{n-1}, f(a)) = \xi(a_0, \dots, a_{n-1}, c)$.*

Proof. Pick a new colour c^* and put $F_n = (F_0 \times \cdots \times F_{n-1}) \cup \{c^*\}$ and

$$\xi_n \colon A^n \to F_n,\ \xi_n(\bar{a}) = \begin{cases} \xi(\bar{a}\hat{\ }(f(\bar{a}))), & \text{for } \bar{a} \in \operatorname{dom}(f) \\ c^*, & \text{otherwise} \end{cases}$$

Then $|F_n| \leq k^n + 1$. Let ξ^* be the merge of ξ_i, $i = 0, \dots, n$. Since f is $k^n + 1$-irreducible, ξ^* cannot carry all the information to determine if a tuple is in f, so there are $\bar{a}, \bar{b} \in A^n$, $c, d \in A$ such that 1) $\xi^*(\bar{a}\hat{\ }(c)) = \xi^*(\bar{b}\hat{\ }(d))$ and 2) $d = f(\bar{b})$ and 3) either $\bar{a} \notin \operatorname{dom}(f)$ or $c \neq f(\bar{a})$. As $\bar{b} \in \operatorname{dom}(f)$, we have $\xi_n(\bar{b}) = \xi_n(\bar{a})$, whence the former possibility in case 3 is outruled. But $\xi_n(\bar{a}) = \xi_n(\bar{b})$ says even more, namely that $\xi(\bar{a}\hat{\ }(f(\bar{a}))) = \xi(\bar{b}\hat{\ }(f(\bar{b})))$. On the other hand, $\xi^*(\bar{a}\hat{\ }(c)) = \xi^*(\bar{b}\hat{\ }(d))$ implies $\xi(\bar{a}\hat{\ }(c)) = \xi(\bar{b}\hat{\ }(d))$. Hence, $\xi(\bar{a}\hat{\ }(c)) = \xi(\bar{a}\hat{\ }(f(a)))$. □

Proposition 2. *Let $n \in \mathbb{N}$ with $n \geq 2$ and $\varrho\colon \mathbb{N}^n \to F$ be a finite colouring. Then the irreducibility of T_{n+1} implies that there are $\bar{a} \in \mathbb{N}^n$ and $d \in \mathbb{Z} \setminus \{0\}$ such that $\varrho(\bar{a}) = \varrho(\bar{a} + d\bar{e}_k)$, for $k = 0, \ldots, n-1$, where $\bar{e}_k$ is the unit vector having k^{th} component* 1. □

Proof. Pick a new colour c^*. For each $i = 0, \ldots, n-1$, colour

$$\bar{a}'_i = (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$$

by c^* if $b = a_n - \sum_{j=0,\, j\neq i}^{n-1} a_j < 0$, otherwise colour $\bar{a}'_i$ by

$$\varrho(a_0, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_{n-1}).$$

Let the resulting colouring be $\xi_i\colon \mathbb{N}^n \to F \cup \{c^*\}$. We can think of this colouring as an attempt to first recover the hole (i^{th} component) in the tuple and then to use colouring ϱ. Now, by the previous lemma, the merge ξ of ξ_i, $i = 0, \ldots, n-1$, does not carry all the information about T_n. Consequently, there are $\bar{a} = (a_0, \ldots, a_{n-1}) \in \mathbb{N}^n$ and $d \in \mathbb{Z}$, $d \neq 0$, such that $\xi(\bar{a}\hat{\ }(s)) = \xi(\bar{a}\hat{\ }(s+d))$ where $s = \sum_{i=0}^{n-1} a_i$. Unfolding the definition of the colourings, we see that for every $i = 0, \ldots, n-1$,

$$\begin{aligned} \varrho(\bar{a}) &= \xi_i(a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n-1}, s) \\ &= \xi_i(a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n-1}, s+d) = \varrho(\bar{a} + d\bar{e}_i). \end{aligned} \qquad \square$$

Now the goal is at hand:

Theorem 10. *(van der Waerden) Let $\chi\colon \mathbb{N} \to F$ be a finite colouring and $k \in \mathbb{N}$. Then there are $a, d \in \mathbb{N}$ such that $d \neq 0$ and $\chi(a) = \chi(a + id)$, for every $i = 0, \ldots, k-1$.*

Proof. We may assume that $k \geq 2$. Let $n = k-1 \in \mathbb{N}^*$ and consider the coloring

$$\varrho\colon \mathbb{N}^n \to F,\ \varrho(x_0, \ldots, x_{n-1}) = \chi(\sum_{j=0}^{n-1}(j+1)x_j).$$

The previous proposition implies that there are $\bar{a} = (a_0, \ldots, a_{n-1}) \in \mathbb{N}^n$ and $d \in \mathbb{Z} \setminus \{0\}$ such that $\varrho(\bar{a}) = \varrho(\bar{a} + d\bar{e}_i)$, for every $i = 0, \ldots, n-1$. Consequently, for $a = \sum_{j=0}^{n-1}(j+1)a_j$ and $i = 0, \ldots, n-1$ we have

$$\chi(a) = \varrho(\bar{a}) = \varrho(\bar{a} + d\bar{e}_i) = \chi(a + (i+1)d).$$

Equivalently, for every $i = 0, \ldots, k-1$, it holds that $\chi(a) = \chi(a + id)$. If it happens that $d < 0$, the numbers $a' = a + (k-1)d$ and $d' = -d > 0$ fulfil the claim instead of a and d. □

5 Fast-Growing Functions

As it was mentioned in the introduction, the ideas of the previous section can be converted to results on combinatorial functions. One of the most interesting of such is the van der Waerden's function $W\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$. For some time there was speculation over whether W is even a primitive-recursive function until Saharon Shelah [S] proved that W is actually quite low in the Grzegorczyk (or Ackermann) hierarchy. Still, it is possible that W_2 grows faster than the exponential tower, i.e., for every $n\in\mathbb{N}$,

$$W_2(n) \geq \left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}n \text{ times}.$$

The last step of the paper is to link complexities of irreducibilities to the discussion by proving an upper bound for W. It is worth mentioning here that a fast-growing complexity of irreducibility i_R of R means intuitively that the irreducibility of R is difficult to discover, R is nearly reducible.

Theorem 11. *For every $k,n\in\mathbb{N}$, $k,n\geq 2$, we have that*

$$W(n,k) \leq (n-1)i_{T_n}((k+1)^{n-1}+1).$$

Proof. Denote $t=i_{T_n}((k+1)^{n-1}+1)$ and $w=(n-1)t$. Let $\chi\colon\{0,\dots,w-1\}\to F$ be an arbitrary colouring with $|F|\leq k$. We need to show that $\{0,\dots,w-1\}$ includes a monochromatic progression of length k. For that purpose, we do the same tricks as in the previous section and consider the colouring $\varrho\colon\{0,\dots,t-1\}\to F\cup\{c^*\}$,

$$\varrho(a_0,\dots,a_{n-2}) = \begin{cases} \chi(\sum_{l=0}^{n-2}(l+1)a_l), & \text{if defined} \\ c^*, & \text{otherwise} \end{cases}$$

where c^* is a new colour. For each $i=0,\dots,n-2$ and

$$\bar{a}' = (a_0,\dots,a_{i+1},a_{i+1},\dots,a_{n-1})\in\mathbb{N}^n,$$

let $\bar{b}_i(\bar{a}') = (a_0,\dots,a_{i-1},a_{n-1}-\sum\limits_{l=0,\,l\neq i}^{n-2}(l+1)a_l,\dots,a_{n-2})$ be the tuple we can recover from $\bar{a}$, and let

$$\xi_i\colon\{0,\dots,t-1\}^{n-1}\to F\cup\{c^*\},\ \xi_i(\bar{a}') = \begin{cases} \varrho(\bar{b}_i(\bar{a}')), & \text{if defined} \\ c^* & \text{otherwise} \end{cases}$$

Note that $T'=T_n\cap\{0,\dots,t-1\}^n$ is a partial function from $\{0,\dots,t-1\}^{n-1}$ to $\{0,\dots,t-1\}$. By the choice of t, T' is $(k+1)^{n-1}+1$-irreducible, so applying lemma 2 to T' and the merge ξ of ξ_i, $i=0,\dots,n-2$, we find $\bar{a}=(a_0,\dots,a_{n-2})\in\{0,\dots,t-1\}^{n-1}$ and $c\in\{0,\dots,t-1\}$ such that for $s=\sum_{i=0}^{n-2}a_i$, we have that

$c \neq s < t$ and $\xi(\bar{a}\hat{\ }(s)) = \xi(\bar{a}\hat{\ }(c))$. Observe that $u = \sum_{l=0}^{n-2}(l+1)a_l \leq (n-1)s < (n-1)t = w$. Hence, $\varrho(\bar{a}) = \chi(u)$ and for every $i = 0, \dots, n-2$,

$$\begin{aligned} &\xi_i(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_{n-2}, s) \\ &= \varrho(a_0, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{n-2}) \\ &= \varrho(\bar{a}) = \chi(u). \end{aligned}$$

On the other hand, $\chi(\bar{a}\hat{\ }(s)) = \chi(\bar{a}\hat{\ }(c))$ implies that

$$\begin{aligned} &\xi_i(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_{n-2}, c) \\ &= \varrho(a_0, \dots, a_{i-1}, a_i + d, a_{i+1}, \dots, a_{n-2}) \\ &= \varrho(\bar{a} + d\bar{e}_i) = \varrho(\bar{a}) \end{aligned}$$

where $d = c - s \neq 0$. Hence $\chi(u) = \chi(u + id)$, for every $i = 0, \dots, n-1$, so that $\{0, \dots, w-1\}$ includes a monochromatic aritmetic progression of length n. Consequently, $W(n,k) \leq w$. □

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