An order type decomposition theorem

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Introduction

The work which has been done on countable linear order types (more generally, the class $\mathfrak{M}$ of countable unions of scattered types) has shown that these order types, both individually and as a class, have properties which are reminiscent of the ordinals and which imply the known facts about ordinals as a special case. For example, consider the following properties of ordinals:

(a) the class of nonzero ordinals is obtained by closing the set $\{1\}$ under sums indexed by regular nonzero ordinals,

(b) the class of ordinals is well ordered under embeddability,

(c) every ordinal is a finite sum of additively indecomposable ordinals (an order type φ is said to be AI if $\varphi = (\varphi_1 + \varphi_2) \rightarrow \varphi \leq \varphi_1$ or $\varphi \leq \varphi_2$), and

(d) the AI ordinals are those of the form ω^ρ.

The ordinals may be viewed as being generated in a one-dimensional manner; the generalization to order types comes when we allow other generating operations (such as, in the case of scattered types, the operations of taking converse well ordered sums). The theory needed to handle this wider situation depends on Nash-Williams' theory of better-quasi-orderings, as developed in his paper [10] on infinite trees. The results[1] which correspond to the above properties of ordinals are:

(a) the class of nonzero members of $\mathfrak{M}$ is obtained by closing the set $\{1\}$ under sums indexed by regular nonzero members of $\mathfrak{M}$,

(b) $\mathfrak{M}$ is better-quasi-ordered under embeddability,

(c) every type in $\mathfrak{M}$ is a finite sum of AI types, and

(d) the AI members of $\mathfrak{M}$ are those types generated from $\{0, 1\}$ by closure under "regular unbounded" sums.

The additively indecomposable ordinals coincide of course with the ordinals which are strongly indecomposable (an order type φ is said to be SI if $\varphi \rightarrow (\varphi, \varphi)^1$, i.e., whenever $\mathrm{tp}(L) = \varphi$, $L = L_1 \cup L_2$, then $\varphi \leq \mathrm{tp}(L_1)$ or $\varphi \leq \mathrm{tp}(L_2)$). In this paper we will prove a decomposition theorem for the AI members of $\mathfrak{M}$ which can be viewed as the fact which corresponds to this

[1] See [9]. Terminology is in Sections 2 and 3 below.

property of AI ordinals. The theorem involves a representation of the AI types in $\mathfrak{M}$ by finite labelled trees. This tree representation gives each such type, up to equivalence under embeddability, as a certain kind of shuffle of a finite number of SI types.

Various properties of the order types in $\mathfrak{M}$ are obtained as a corollary, in particular the theorem has the following combinatorial consequence:

for each $\varphi \in \mathfrak{M}$ there exists an $n < \omega$ such that

$$\varphi \longrightarrow (\varphi)^1_{<\omega/n} ,$$

that is, for any partition of an ordered set of type φ into finitely many subsets, the union of some $\leqq n$ of the subsets contains a set of type φ.

This answers affirmatively a question of A. Hajnal as to whether for every countable φ, $\varphi \rightarrow [\varphi]^1_{\omega}$.[1]

A characterization, for each n, of the types $\varphi \in \mathfrak{M}$ such that $\varphi \rightarrow (\varphi)^1_{<\omega/n}$ can be read off of the tree representation of AI types; in particular the case $n = 1$ is where φ is strongly indecomposable, and the SI types are characterized by the theorem as the types which are equivalent under embeddability to "hereditarily increasing" types, where, e.g., the class of hereditarily increasing scattered types is the least class containing 0, 1, and closed under sums

$$\sum\nolimits_{\alpha<\kappa} \varphi_\alpha , \ \sum\nolimits_{\alpha<\kappa^*} \varphi_\alpha ,$$

where κ is an infinite regular cardinal and $\alpha < \beta \rightarrow \varphi_\alpha \leq \varphi_\beta$. This characterization answers affirmatively R. Fraïssé's Conjecture IV of [4].

In Section 1, a notion of many-one embedding is considered which is weaker than the usual notion of homeomorphic embedding for trees, and a combinatorial theorem is proved which shows that if Q is a well-quasi-ordered set, then every set of finite Q-labelled trees has a "bounded cover" under this notion of embeddability.

In Sections 2 and 3, a class of order types having certain tree representations is defined. It is shown that these types are well behaved, and, using the results in [9] and the tree theorem of Section 1, it is proved that every AI type in $\mathfrak{M}$ is equivalent to one of these types. The part of the theorem which deals with the scattered types is singled out and proved in Section 2, and in Section 3 it is indicated how to obtain the analogous results for types in $\mathfrak{M}$ and for Q-labelled types, Q better-quasi-ordered.

In Section 4 some combinatorial consequences of the results in Sections 2 and 3 will be proved. The idea is that in classical combinatorial theorems

[1] I would like to express thanks to F. Galvin for bringing Hajnal's question to my attention.

about ordinals, the strong one-dimensional properties of ordinals are seldom used; rather, weaker properties such as the $\varphi \to (\varphi)^1_{<\omega/n}$ relation above are often seen to be what are really needed in the proofs[1]. This leads to the expectation that any partition, decomposition, or mapping theorem about ordinals, suitably stated, will also be provable for the class $\mathfrak{M}$ with the aid of results in Sections 2 and 3. Section 4 has been written in the spirit of giving a nonexhaustive set of examples of this. Included are a mapping property, some partition relations which for ordinals are due to Erdös, Hajnal, and Milner, and a method for piecing together order types using a notion of reduced complement.

In Section 5 it is shown that for any infinite cardinal κ, the partial order of scattered types of power $\leq \kappa$ has dimension κ^+.

Standard set theoretic notation is used. For Section 1 familiarity is assumed with the basic methods and results of wqo theory (see [7], [8], [12]). The results on bqo's and order types which are used in the remaining sections can be found in [6], [9], and [10].

1. Finite wqo-labelled trees

Q is always assumed to be an arbitrary set on which a quasi-order (transitive, reflexive relation) $\leq$ is defined. If $q_1, q_2 \in Q$, define $q_1 < q_2 \leftrightarrow q_1 \leq q_2$ and $q_2 \not\leq q_1$, and $q_1 \equiv q_2 \leftrightarrow q_1 \leq q_2$ and $q_2 \leq q_1$. Recall that Q is well-quasi-ordered (wqo) just in case Q satisfies any of the following equivalent conditions:

(i) for every $f\colon \omega \to Q$ there are $i < j < \omega$ such that $f(i) \leq f(j)$,

(ii) for every $f\colon \omega \to Q$ there is an infinite $X \subseteq \omega$ such that $i, j \in X$ and $i \leq j \to f(i) \leq f(j)$,

(iii) Q is well founded under $<$ and every set of mutually incomparable elements of Q is finite,

(iv) every extension of the partial order $Q/\equiv$ to a linear order is a well order.

Our aim in this section is to prove that any set of finite Q-trees, Q wqo, has a certain kind of "bounded cover". We first give some definitions and lemmas.

If $X, Y \subseteq Q$, recall that $X \leq_1 Y$ ($X \leq_m Y$) means that there is a 1–1 (many–one) $f\colon X \to Y$ such that for all $x \in X$, $x \leq f(x)$. For a set A, let $\mathcal{P}(A)$ be the power set of A, and let $[A]^{<\omega}$ be the set of finite subsets of A.

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[1] In this regard A. Tarski has mentioned some theorems on Boolean algebras with well ordered bases, due to him and A. Mostowski, which involve the $\alpha \to (\alpha)^1_{<\omega/n}$ principle for ordinals α. For a summary of these results and related results on countable BA's and BA's with scattered bases, see [14], pp. 364–369.

THEOREM 1.1. [7]. *If* Q *is* wqo *then* $[Q]^{<\omega}$ *is* wqo *under* $\leq_1$, $\leq_m$.

THEOREM 1.2. ([10], [12]). *If* Q *is* wqo *then* $\mathcal{P}(Q)$ *is well founded under* $<_1$, $<_m$.

A pair (T, l) is a Q-tree if T is a tree (considered here to be a partially ordered, rooted set such that for each $x \in T$, $\{y : y <_T x\}$ is well ordered) and l: $T \to Q$. Let $(\mathcal{FT})_Q$ be the collection of finite Q-trees. Define quasi-orders on $(\mathcal{FT})_Q$ by:

$$(T_1, l_1) \leq_m (T_2, l_2) \leftrightarrow_{\mathrm{df}} \text{there is an } f\colon T_1 \to T_2$$
$$\text{such that } x <_{T_1} y \to f(x) <_{T_2} f(y), \text{ and } l_1(x) \leq l_2(f(x)),$$
$$(T_1, l_1) \leq_1 (T_2, l_2) \leftrightarrow_{\mathrm{df}} \text{in addition to the above,}$$
$$f(x \wedge y) = f(x) \wedge f(y) \text{ for all } x, y \in T_1,$$

where $x \wedge y$ is the greatest lower bound of x and y.

THEOREM 1.3. [8]. *If* Q *is* wqo *then* $(\mathcal{FT})_Q$ *is* wqo *under* $\leq_1$, $\leq_m$.

Note: We have added on the $\leq_m$ version of Theorem 1.3 (it follows from the $\leq_1$ version in [8]) because Theorem 1.6 below only goes through using the weaker $\leq_m$ relation. The reader is reminded that the case of infinite trees having no paths of length $> \omega$ was settled in [10] (see also [9], Theorem 2.2, for the extension to bqo labelled trees).

For $q \in Q$, let $Q_q = \{r \in Q\colon q \not\leq r\}$. We will use the following form of wqo induction principle: a statement $S(Q)$ is true for all wqo sets Q provided that

(1) $$S(\varnothing)\ , \text{ and}$$

(2) $$\forall \text{ wqo } Q(\forall\, q \in Q\, S(Q_q) \to S(Q))\ .$$

$X \subseteq Q$ is called a chain if X is linearly ordered under $\leq$, an antichain if the members of X are pairwise incomparable. $q_1, q_2 \in Q$ are said to be compatible if $\exists\, q \in Q$ $q_1, q_2 \leq q$.

LEMMA 1.4. *If* Q *is countable and has no infinite antichain then there is an* $n < \omega$ *and chains* $X_i \subseteq Q$, $i \leq n$ *such that* $\bigcup_{i \leq n} X_i \equiv_m Q$.

Proof. Q has a well founded (and hence wqo) cofinal subset, so it suffices to prove the theorem for countable wqo sets Q. Assume, by the wqo induction hypothesis, that the lemma holds for Q_q, all $q \in Q$. Let $Q = \{q_i\colon i < \omega\}$.

Case 1. $\forall\, i, j\ \exists\, k q_i$, $q_j \leq q_k$. Then build up in ω stages a chain $X \subseteq Q$, $X \equiv_m Q$.

Case 2. $\exists\, i, j\ \forall\, k q_i \not\leq q_k$ or $q_j \not\leq q_k$. Then $Q = Q_{q_i} \cup Q_{q_j}$. By the induction hypothesis Q_{q_i}, Q_{q_j} are $\equiv_m$ finite unions of chains, and thus Q is.

We state without proof a more general proposition (observed by the author for regular κ, by Hajnal for singular κ): if κ is an infinite cardinal, Card $Q = \kappa$, and Q has no infinite antichains (if $\kappa = \omega$ this may be weakened to: there is no infinite set of mutually incompatible members of Q) then there is a collection of fewer than κ chains in Q whose union is $\equiv_m Q$.

Convention. If R is a quasi-ordered space built up from Q by means of the operations listed above (for instance, $R = \mathcal{P}([Q]^{<\omega})$, $R = \mathcal{P}(\mathcal{F}\mathcal{T})_Q)$, then the $\leq_m$ quasi-order on R is understood to be the ordering inherited from the ordering on Q and the $\leq_m$ orderings associated with the operations.

If X, $Y \subseteq [Q]^{<\omega}$, $\forall\, y \in Y\ \exists\, x \in X\ \ y \subseteq x$, and $Y \equiv_m X$, then call Y a cover for X. For $n < \omega$, say that X is n-coverable if there is a cover Y for X such that $y \in Y \rightarrow \text{Card}\,(y) \leqq n$.

LEMMA 1.5. *If Q is* wqo *and $X \subseteq [Q]^{<\omega}$ then for some $n < \omega$, X is n-coverable.*

Proof. Assume the lemma holds for Q_q, all $q \in Q$. If $X \subseteq [Q]^{<\omega}$ is a counterexample to the lemma then there is, for each $i < \omega$, an $x_i \in X$ such that

$$\forall\, x \in X\ \ \forall\, y \subseteq x\quad (\text{Card}\ y \leqq i \rightarrow x_i \not\leq_m y)\ ;$$

we may as well assume then that

$$X = \{x_i \colon i < \omega\}\ .$$

By Theorem 1.1 and Lemma 1.4, X is $\equiv_m$ a finite union of chains $X_r \subseteq X$; if the lemma holds for each X_r then it holds for X, so we may assume X itself is a chain and that $i < j \rightarrow x_i \leq_m x_j$.

Case 1. $\forall\, i\ \exists\, j\ \exists\, q \in x_j\ \ x_i \leq_m \{q\}$. Then, since X is a chain, X is 1-coverable.

Case 2. $\exists\, i\ \forall\, j\ \forall\, q \in x_j\ \ x_i \not\leq_m \{q\}$. Let $x_i = \{q_1, q_2, \cdots, q_k\}$. By the assumption, for each $j < \omega$ we can write

$$x_j = x_{j1} \cup x_{j2} \cup \cdots \cup x_{jk}\ ,\ \text{ where }\ \{q_r\} \not\leq_m x_{jr}\ ,\ \text{ all }\ r \leqq k\ .$$

Apply version (ii) in the definition of wqo k times to obtain an infinite $A \subseteq \omega$ such that

$$j_1, j_2 \in A \text{ and } j_1 \leqq j_2 \longrightarrow \text{for all } r \leqq k\quad x_{j_1 r} \leq_m x_{j_2 r}\ ,$$

where note $\{x_j \colon j \in A\} \equiv_m X$. Let

$$X(r) = \{x_{jr} \colon j \in A\}\ .$$

We have that $X(r) \subseteq [Q_{q_r}]^{<\omega}$, so, by applying the wqo induction hypothesis to each $X(r)$, an $n_r < \omega$ can be chosen so that for all $r \leqq k$, $X(r)$ is n_r-

coverable. But now, using the fact that each $X(r)$ is a chain, it can be seen that X is $\sum_{r\leqq k} n_r$-coverable, which gives the lemma.

A subtree of a tree T is a rooted subset of T. If $t \in T$, then $\mathrm{br}(t) = \{t' \in T\colon t \leqq_T t'\}$ is the branch of T with root t. A branch $\mathrm{br}(t)$ of T is called proper if $t \neq$ the root of T. These definitions apply also to Q-trees, taking the restricted labelling function. A treetop of T is a maximal node of T.

If $W \subseteq (\mathcal{F}\mathcal{T})_Q$, $n < \omega$, a cover for W is a $W' \subseteq (\mathcal{F}\mathcal{T})_Q$ such that $(T', l) \in W' \to (T', l)$ is a subtree of some $(T, l) \in W$, and $W \equiv_m W'$. W is n-coverable if there is a cover W' for W such that $(T', l) \in W' \to T'$ has $\leqq n$ treetops.

THEOREM 1.6. *If Q is* wqo *and $W \subseteq (\mathcal{F}\mathcal{T})_Q$ then for some $n < \omega$, W is n-coverable.*

Proof. By Theorems 1.2 and 1.3, $\mathcal{P}\big((\mathcal{F}\mathcal{T})_Q\big)$ is well founded, so we may suppose the theorem holds for all $W' <_m W$. Suppose the theorem is false for W; then, as in the proof of Lemma 1.5, we may assume that $W = \{(T_i, l_i)\colon i < \omega\}$, where if $j < \omega$ and (T'_j, l_j) is a subtree of (T_j, l_j) which has $\leqq i$ treetops then $(T_i, l_i) \not\leq_m (T'_j, l_j)$. Using Lemma 1.4 we may reduce as in Lemma 1.5 to the case where W is a chain and assume that

$$i < j \longrightarrow (T_i, l_i) \leq_m (T_j, l_j)\ , \ \text{ and each } \ \mathrm{Card}\,(T_i) > 1\ .$$

Since now every infinite $W' \subseteq W$ is $\equiv_m W$, we may assume by Ramsey's theorem that either

(a) $\forall\, i, j\ \big(i < j \to (T_i, l_i)$ is not $\leq_m$ embeddable into any proper branch of $(T_j, l_j)\big)$ or

(b) $\forall\, i, j\ \big(i < j \to (T_i, l_i)$ is $\leq_m$ embeddable into some proper branch of $(T_j, l_j)\big)$.

First, suppose (a) is true. Let S_i be the set of branches $\big(\mathrm{br}(x), l_i\big)$ of (T_i, l_i) such that x is an immediate successor of the root of T_i. Applying Lemma 1.5 to the wqo space $(\mathcal{F}\mathcal{T})_Q$ we may find an $S'_i \subseteq S_i$, all i, and a $k < \omega$ such that

(1) each $\mathrm{Card}\, S'_i \leqq k$ and $\forall\, i\, \exists\, j\ \ S_i \leq_m S'_j$.

Setting $S'_i = \{(T_{i1}, l_i) \cdots (T_{ik}, l_i)\}$, we may cut down k times using the wqo property (ii) and assume that:

(2) $\forall\, i < j\ \ (T_{ir}, l_i) \leq_m (T_{jr}, l_j)$ for each $r \leqq k$ and

(3) $\forall\, i < j\ \ l_i(\text{root } T_i) \leq l_j(\text{root } T_j)$.

From (a) and the fact that W is a chain, it follows that for each $r \leqq k$ the set $W_r = \{(T_{ir}, l_i)\colon i < \omega\}$ is $<_m W$. We may apply the theorem, then, to each W_r, giving numbers $n_r < \omega$ such that W_r is n_r-coverable. Using this and (1), (2), and (3), the reader may now check that W is $\sum_{r\leqq k} n_r$-coverable (recall that the $\leq_m$ embedding for members of $(\mathcal{F}\mathcal{T})_Q$ need not be 1–1), as desired.

Suppose then that (b) is true. For each finite $a \subseteq Q$ and $(T_i, l_i) \in W$, split (T_i, l_i) into an upper and lower part determined by a, by

$$\begin{aligned} U(a, T_i) &= \{(\mathrm{br}(x), l_i): x \in T_i,\ l_i(x) \geq q \text{ for some } q \in a\ , \\ &\quad \text{and } \forall\, y <_{T_i} x\ \ l_i(y) \not\geq q \text{ for all } q \in a\}\ , \\ L(a, T_i) &= (\{x \in T_i: \forall\, y \leqq_{T_i} x\ \ l_i(y) \not\geq q \text{ for all } q \in a\}, l_i)\ . \end{aligned}$$

Let

$$U(a, W) = \bigcup\nolimits_{i<\omega} U(a, T_i)\ ,$$

and

$$L(a, W) = \{L(a, T_i): i < \omega\}\ .$$

Claim. For some $a \subseteq Q$, $U(a, W) <_m W$ and for each i, if *x is a node of* $L(a, T_i)$ then $U(\{l_i(x)\}, W) \equiv_m W$.

Proof of claim. We have $U(\varnothing, W) = \varnothing <_m W$, and $L(\varnothing, W) = W$, so if $a = \varnothing$ fails to satisfy the claim there will be a $q_0 \in Q$ such that $U(\{q_0\}, W) <_m W$. Suppose, continuing in this way, we have picked a set $\{q_0, q_1, \cdots, q_k\}$ such that

$$i < j \leqq k \longrightarrow q_i \not\leq q_j \ \text{ and } \ U(\{q_0, \cdots, q_k\}, W) <_m W\ .$$

If $a = \{q_0, \cdots, q_k\}$ doesn't satisfy the claim then for some $i < \omega$ and node $x \in L(\{q_0, \cdots, q_k\}, T_i)$, $U(\{l_i(x)\}, W) <_m W$. Set

$$q_{k+1} = l_i(x)\ .$$

We have

$$r \leqq k \longrightarrow q_r \not\leq q_{k+1}\ ,$$

since $x \in L(\{q_0, \cdots, q_k\}, T_i)$. We have that

$$U(\{q_0, \cdots, q_{k+1}\}, W) \subseteq (U(\{q_0, \cdots, q_k\}, W) \cup U(\{q_{k+1}\}, W))\ .$$

Each of the two sets on the right hand side is $<_m W$ by hypothesis, but W is a chain, from which it follows that

$$(U(\{q_0, \cdots, q_k\}, W) \cup U(\{q_{k+1}\}, W)) <_m W$$

and hence

$$U(\{q_0, \cdots, q_{k+1}\}, W) <_m W\ ,$$

completing the induction step. Thus, for some k, $a = \{q_0, \cdots, q_k\}$ must satisfy the claim lest an infinite sequence $\{q_i: i < \omega\}$ be obtained with $i < j \rightarrow q_i \not\leq q_j$, which would contradict Q wqo.

Given $a \subseteq Q$ as in the claim, then, we want to build up an n-cover of Q, for some $n < \omega$. As in the first part of the theorem, find by Lemma 1.5 a $u < \omega$ and sets $U'(a, T_i) \subseteq U(a, T_i)$ all of cardinality $\leqq u$ such that $\{U'(a, T_i): i < \omega\} \equiv_m \{U(a, T_i);\ i < \omega\}$. Write

$$U'(a, T_i) = \{(T_{i1}, l_i), \cdots, (T_{iu}, l_i)\} ,$$

and cut down to an infinite $A \subseteq \omega$ such that for each $r \leqq u$, $\{(T_{ir}, l_i): i \in A\}$ is a $\leq_m$-chain. Now, using the fact that $U(a, W) <_m W$, pick n_r: $r \leqq u$ such that $\{(T_{ir}, l_i): i < \omega\}$ is n_r-coverable.

To handle the set $L(a, W)$, let $Q(a) \subseteq Q$ be $\{l(x): x \in L(a, T_i)$, some $i\}$. If $p_1, \cdots, p_n \in Q(a)$, $i < \omega$, let

$$[p_1, \cdots, p_n ;\ (T_i, l_i)]$$

be the Q-tree which is for its first n levels a single chain $x_1, \cdots, x_n$, where x_j is labelled by p_j, and which above the chain is a copy of (T_i, l_i).

Claim. For all $p_1, \cdots, p_n \in Q(a)$, $i < \omega$, there is a $j < \omega$ with $[p_1, \cdots, p_n; (T_i, l_i)] \leq_m (T_j, l_j)$.

Proof of claim. By induction on n. The case $n = 1$ is like the case $n > 1$ considered here. By the induction hypothesis choose a j such that

$$[p_2, \cdots, p_n; (T_i, l_i)] \leq_m (T_j, l_j) .$$

By the assumption (b) above we may $\leq_m$ embed (T_j, l_j) into a proper branch of some (T_k, l_k). By the previous claim,

$$U(\{p_1\}, W) \equiv_m W$$

so there will be a branch of some tree in W, having root node labelled $\geq p_1$, which is $\geq_m (T_k, l_k)$. This branch is $\geq_m [p_1, p_2, \cdots, p_n; (T_i, l_i)]$ as desired, giving the claim.

It now follows that W is $\sum_{r \leqq u} n_r$-coverable, as follows. Given $(T_i, l_i) \in W$, to cover it, enumerate $L(a, T_i)$ in a sequence $x_1, x_2, \cdots, x_n$ compatible with the order on T_i. Now consider the tree

$$[l_i(x_1), l_i(x_2), \cdots, l_i(x_n); (T_j, l_j)] ,$$

where by the construction it is possible to choose (T_j, l_j) to contain a subtree with at most $\sum_{r \leqq u} n_r$ treetops which is $\geq_m$ every member of $U(a, T_i)$. We have that

$$(T_i, l_i) \leq_m [l_i(x_1), l_i(x_2), \cdots, l_i(x_n); (T_j, l_j)] ,$$

but by the claim, $[l_i(x_1), \cdots, l_i(x_n); (T_j, l_j)]$ is $\leq_m$ some (T_k, l_k). Thus there is a subtree of (T_k, l_k) having $\leqq \sum_{r < u} n_r$ treetops which is $\geq_m (T_i, l_i)$. This completes the proof.

2. The main theorem for scattered types

Quasi-order the class of order types by the relation

$$\varphi \leq \psi \leftrightarrow_{\mathrm{df}} \varphi \text{ is order embeddable in } \psi.$$

Let η be the order type of the rationals. An order type φ is defined to be scattered just in case $\eta \not\leq \varphi$. Let $\mathcal{S}$ be the class of all scattered types. In this section the version of our main result which applies to scattered types will be proved; for the extensions to Q-types (which are occasionally needed in applications) and to $\mathfrak{M}$ types, see the next section.

We assume familiarity with the basic definitions involved. Results from [9] will be used; Theorems 2.1–2.5, below, extracted from the more general versions in [9], summarize the basic properties of $\mathcal{S}$ and its members which will be needed.

THEOREM 2.1. [6]. *$\mathcal{S}$ is the closure of the set of order types* $\{0, 1\}$ *under well ordered and converse well ordered sums.*

THEOREM 2.2. [9]. *$\mathcal{S}$ is better-quasi-ordered under embeddability.*

A type φ is called additively indecomposable (AI) if $\varphi = (\psi + \theta) \rightarrow \varphi \leq \psi$ or $\varphi \leq \theta$.

THEOREM 2.3. [9]. *Every $\varphi \in \mathcal{S}$ is a finite sum of* AI *types.*

Define a regular unbounded sum of order types φ_α to be a sum $\sum_{\alpha<\kappa} \varphi_\alpha$ or a sum $\sum_{\alpha<\kappa^*} \varphi_\alpha \big(=_{\mathrm{df}}(\varphi_0 + \varphi_1 + \cdots + \varphi_\alpha + \cdots)$ and $(\cdots + \varphi_\alpha + \cdots + \varphi_1 + \varphi_0)$, resp.$\big)$, where κ is an infinite regular cardinal and

$$\forall\, \alpha < \kappa \ \ \mathrm{Card}\, \{\beta < \kappa \colon \varphi_\alpha \leq \varphi_\beta\} = \kappa\ .$$

THEOREM 2.4. [9]. *The class of* AI *members of $\mathcal{S}$ is the closure of* $\{0, 1\}$ *under regular unbounded sums.*

Let

$$\mathcal{S}_{(<\kappa)} = \{\varphi \in \mathcal{S} \colon \mathrm{Card}\, \varphi < \kappa\}\ .$$

THEOREM 2.5 [9]. *For any cardinal κ,* $\mathrm{Card}\, (\mathcal{S}_{(<\kappa)}/\equiv) = \kappa$.

(For instance, while there are $2^{\aleph_0}$ countable order types, there are only $\aleph_1$ when mutually embeddable types are identified.)

As a beginning in the proof of the representation theorem, define a regular increasing sum of order types φ_α to be a sum $\sum_{\alpha<\kappa} \varphi_\alpha$ or a sum $\sum_{\alpha<\kappa^*} \varphi_\alpha$, where κ is an infinite regular cardinal and $\forall\, \alpha, \beta < \kappa$ $(\alpha < \beta \rightarrow \varphi_\alpha \leq \varphi_\beta)$. Define $\mathcal{HI} \subset \mathcal{S}$ to be the closure of the set of order types $\{0, 1\}$ under the operations of taking regular increasing sums. Members of the class $\mathcal{HI}$ will be called hereditarily increasing types.

Recall that a type φ is called strongly indecomposable (SI) if whenever L is a linearly ordered set, $\mathrm{tp}(L) = \varphi$, and $L = L_1 \cup L_2$, then $\varphi \leq \mathrm{tp}(L_1)$ or $\varphi \leq \mathrm{tp}(L_2)$. From the construction of $\mathcal{HI}$ it is seen by induction that

LEMMA 2.6. *Every member of $\mathcal{HI}$ is strongly indecomposable.*

We will define a class

$$\mathfrak{U} \subset (\mathcal{FT})_{\mathcal{HI}}$$

(i.e., a member of $\mathfrak{U}$ is a finite (T, l) where $l: T \to \mathcal{HI}$, $\mathcal{HI}$ quasi-ordered by embeddability). Suppose T is a tree and for each $\alpha < \kappa$, $l_\alpha: T \to \mathbb{S}$. Then define

$$\textstyle\sum_{\alpha<\kappa} (T, l_\alpha) \ \left(\sum_{\alpha<\kappa^*} (T, l_\alpha), \text{ resp.}\right)$$

to be (T, l), where for each $x \in T$, $l(x) = \sum_{\alpha<\kappa} l_\alpha(x)$ (resp., for each $x \in T$, $l(x) = \sum_{\alpha<\kappa^*} l_\alpha(x)$). If $(T_1, l_1), \cdots, (T_n, l_n)$ are $\mathbb{S}$-trees, $\varphi \in \mathbb{S}$, define

$$[\varphi; (T_1, l_1), \cdots, (T_n, l_n)]$$

to be the $\mathbb{S}$-tree whose root is labelled by φ and whose branches which begin at the immediate successors of the root are $(T_1, l_1), \cdots, (T_n, l_n)$. If $l_\alpha, l_\beta: T \to \mathbb{S}$, write

$$(T, l_\alpha) \leq_I (T, l_\beta)$$

just in case $\forall\, x \in T \ \ l_\alpha(x) \leq l_\beta(x)$.

$\mathfrak{U}$, then, is defined to be the smallest class of $\mathbb{S}$-trees containing $\varnothing$ (the empty tree) and 1_1 (the one point tree labelled by the order type 1) such that

(i) if κ is an infinite regular cardinal and for some T, $l_0, \cdots, l_\alpha, \cdots$, $(T, l_\alpha) \in \mathfrak{U}$ for all $\alpha < \kappa$, and $\forall\, \alpha, \beta < \kappa$, $\alpha < \beta \to (T, l_\alpha) \leq_I (T, l_\beta)$, then $\sum_{\alpha<\kappa} (T, l_\alpha) \in \mathfrak{U}$ and $\sum_{\alpha<\kappa^*} (T, l_\alpha) \in \mathfrak{U}$,

(ii) (a) If $(T_1, l_1), \cdots, (T_n, l_n) \in \mathfrak{U}$ for each $i \leqq n$, $(T_i, l_i) = \sum_{j<\omega} (T_i, l_{ij})$, $i < k \to (T_i, l_{ij}) \leq_I (T_k, l_{kj})$, then $[\omega; (T_i, l_i), \cdots, (T_n, l_n)] \in \mathfrak{U}$, and

(b) if as in (a), except each (T_i, l_i) is $\sum_{i<\omega^*} (T_i, l_{ij})$, then $[\omega^*; (T_1, l_1), \cdots, (T_n, l_n)] \in \mathfrak{U}$.

Convention. If a sum is written $\sum^I_{\alpha<\kappa} (T, l_\alpha)$ or $\sum^I_{\alpha<\kappa^*} (T, l_\alpha)$, it is assumed that $\alpha < \beta \to (T, l_\alpha) \leq_I (T, l_\beta)$, as above.

By the construction we have that all the members of $\mathfrak{U}$ are in fact $\mathcal{HI}$-trees. To each $(T, l) \in \mathfrak{U}$, now, we associate an order type $\overline{(T, l)} \in \mathbb{S}$. $\overline{(T, l)}$ can be viewed as a "shuffle" of the types $l(x)$, x a treetop of T, where the various depths at which the shuffle is being carried out are given by the bottom part of (T, l). Let

$$\bar{\varnothing} = 0\,, \ \ \bar{1}_1 = 1\,.$$

If $(T, l) \in \mathfrak{A}$ is a sum $\sum^{I}_{\alpha<\kappa} (T, l_\alpha)$, $\left(\sum^{I}_{\alpha<\kappa^*} (T, l_\alpha)\right.$, resp.$\left.\right)$ obtained as in (i), let

$$\overline{(T, l)} = \sum_{\alpha<\kappa} \overline{(T, l_\alpha)}, \ \left(\sum_{\alpha<\kappa^*} \overline{(T, l_\alpha)}, \ \text{resp.}\right) .$$

If $(T, l) = [\omega; (T_1, l_1), \cdots, (T_n, l_n)]$ is obtained by (ii) (a), where thus each $\overline{(T_i, l_i)} = \sum_{j<\omega} \overline{(T_{ij}, l_i)}$, let

$$\overline{(T, l)} = \sum_{j<\omega} \left(\overline{(T_{1j}, l_1)} + \overline{(T_{2j}, l_2)} + \cdots + \overline{(T_{nj}, l_n)}\right) ;$$

if $\overline{(T, l)}$ falls under case (ii) (b) take the symmetric ω^* sum. (Note: strictly speaking, $\overline{(T, l)}$ should be taken as the set of all types which can be associated to (T, l) by these rules, but we don't do this, citing the proof of Lemma 2.7 below, which shows that all such types are $\equiv$.)

LEMMA 2.7. *If* $(S, l), (T, m) \in \mathfrak{A}$ *and* $(S, l) \leq_m (T, m)$, *then* $\overline{(S, l)} \leq \overline{(T, m)}$.

Proof. Assume the lemma holds for all $(T', m) <_1 (T, m)$, and that with (T, m) it holds for all $(S', l) <_m (S, l)$. If (T, m) is $\varnothing$ or 1_1, we are done, likewise we can assume that $\operatorname{Card} S > 1$.

Case 1. $(T, m) = [\omega; (T_1, m_1), \cdots, (T_n, m_n)]$. We look at the possible cases for (S, l).

If $(S, l) = [\omega: (S_1, l_1), \cdots, (S_r, l_r)]$ then $\forall\, i \;\exists\, j\; (S_i, l_i) \leq_m (T_j, m_j)$; using the induction hypothesis and the fact that the $\overline{(S_i, l_i)}$ and $\overline{(T_j, m_j)}$'s are ω-chains, we can construct an embedding of $\overline{(S, l)}$ into $\overline{(T, m)}$.

If $(S, l) = [\omega^*; (S_1, l_1), \cdots, (S_r, l_r)$ then $\exists\, j\; (S, l) \leq_m (T_j, m_j)$, so by induction $\overline{(S, l)} \leq \overline{(T_j, m_j)} \leq \overline{(T, m)}$.

Finally, if (S, l) is $\sum^{I}_{\alpha<\kappa} (S, l_\alpha)$ or $\sum^{I}_{\alpha<\kappa^*} (S, l_\alpha)$, then, since $\operatorname{Card} S > 1$, the construction of $\mathfrak{A}$ is such that we must have $l(\text{root } S) \not\leq \omega$, whence $(S, l) \leq_m (T_j, m_j)$ for some j, and we are done by induction.

The case where $(T, m) = [\omega^*;(T_1, m_1), \cdots, (T_n, m_n)]$ is symmetric, so suppose now that

Case 2. $(T, m) = \sum^{I}_{\alpha<\kappa} (T, m_\alpha)$.

First suppose that $(S, l) = \sum^{I}_{\beta<\lambda} (S, l_\beta)$. Lest $(S, l) \leq_m (T, m_\alpha)$ for some α (where we'd be done by induction), we may assume $\lambda \leqq \kappa$. Since the (S, l_β) and (T, m_α)'s are finite and $\mathcal{HI}$ labelled, and the types $l(x)$: $x \in S$, are AI, we have: if $\lambda = \kappa = \omega$ then $\forall\, r < \omega \;\; \exists\, s < \omega \;\; (S, l_r) \leq (T, m_s)$, and if $\kappa > \omega$ then

$$\forall\, \beta < \lambda \;\; \exists\, \alpha < \kappa \;\; (S, l_\beta) \leq_m \sum^{I}_{\alpha'<\omega^\alpha} (T, m_{\alpha'}) .$$

The latter sum, being $\leq_I$-increasing and of length ω^α, is verifiably in $\mathfrak{A}$. By the induction hypothesis and regularity of κ, an embedding of $\overline{(S, l)}$ into

$\overline{(T, m)}$ may be constructed.

Suppose now that $(S, l) = \sum_{\beta<\lambda^*} (S, l_\beta)$. We must have $(S, l) \leq_m (T, m_\alpha)$ for some $\alpha < \kappa$, done by induction.

If $(S, l) = [\omega; (S_1, l_1), \cdots, (S_n, l_n)]$ where each $(S_i, l_i) = \sum^I_{j<\omega} (S_i, l_{ij})$ then if $\kappa = \omega$, we will have that $\forall j < \omega \ \exists r < \omega \ \forall i \leqq n \ (S_i, l_{ij}) \leq_m (T_r, m_r)$, and if $\kappa > \omega$ then

$$\forall j < \omega \ \exists \alpha < \kappa \ \forall i \leqq n \ (S_i, l_{ij}) \leq_m \sum^I_{\alpha'<\omega^\alpha} (T_{\alpha'}, m_{\alpha'}) \ .$$

We can then use the induction hypothesis to embed $\overline{(S, l)}$ into $\overline{(T, m)}$.

Finally, if $(S, l) = [\omega^*; (S_1, l_1), \cdots, (S_n, l_n)]$, then for some $\alpha < \kappa$ $(S, l) \leq_m (T_\alpha, m_\alpha)$, done by the induction hypothesis.

The case $(T, m) = \sum^I_{\alpha<\kappa^*} (T, m_\alpha)$ being symmetric, this completes the proof.

Using Lemma 2.7 it is verified by induction that

LEMMA 2.8. *If* $(T, l) \in \mathfrak{U}$, *then* $\overline{(T, l)}$ *is additively indecomposable.*

LEMMA 2.9. (a) *If* $(T, l) \in \mathfrak{U}$, S *a subtree of* T *such that* $y \in S$ *and* $x <_T y \rightarrow x \in S$, *then* $(S, l) \in \mathfrak{U}$.

(b) *If* $(T, l) \in \mathfrak{U}$, $x \in T$ *a node with exactly one immediate successor, then the collapsed tree* $(T - \{x\}, l) \in \mathfrak{U}$ *and* $\overline{(T - \{x\}, l)} \equiv \overline{(T, l)}$.

Proof. (a) If $(T, l) = \sum^I_{\alpha<\kappa} (T, l_\alpha)$, then $(S, l) = \sum^I_{\alpha<\kappa} (S, l_\alpha)$, where each $(S, l_\alpha) \in \mathfrak{U}$ by the induction hypothesis, and thus $(S, l) \in \mathfrak{U}$. If $(T, l) = [\omega; (T_1, l_1), \cdots, (T_n, l_n)]$, then $(S, l) = [\omega; (S_{i_1}, l_{i_1}), \cdots, (S_{i_r}, l_{i_r})]$ where S_{i_j} is a downward closed subtree of T_{i_j}; by induction as in the first part, each (S_{i_j}, l_{i_j}) is in $\mathfrak{U}$ and is an ω-increasing sum, so $(S, l) \in \mathfrak{U}$. The converse well ordered cases are symmetric.

(b) If $(T, l) = \sum^I_{\alpha<\kappa} (T, l_\alpha)$, then $(T - \{x\}, l) = \sum^I_{\alpha<\kappa} (T - \{x\}, l_\alpha)$, and we are done by induction. If $(T, l) = [\omega; (T_1, l_1), \cdots, (T_n, l_n)]$, and x is the root of T, then $n = 1$ and the case is clear by construction. If $x \in T_i$ for some i, then $\overline{(T_i - \{x\}, l_i)}$ is by induction an ω increasing sum $\equiv$ $\overline{(T_i, l_i)}$, so $\overline{(T - \{x\}, l)} \equiv \overline{(T, l)}$ as desired. Again, converse well ordered sums are done symmetrically.

LEMMA 2.10. *If* $\varphi \in \mathfrak{S}$ *is a regular unbounded sum* $\sum_{\alpha<\kappa} \varphi_\alpha$ *with* $\kappa > \omega$, *then* $\varphi \equiv \sum_{\alpha<\kappa} \psi_\alpha$, *where each* ψ_α *is* AI *and* $\alpha < \beta \rightarrow \psi_\alpha \leq \psi_\beta$.

Proof. Using Theorem 2.3, write each initial segment $\sum_{\alpha<\gamma} \varphi_\alpha$ of φ as a finite sum

$$(\psi_{\gamma 1} + \cdots + \psi_{\gamma n_\gamma})$$

of AI types. Since $\kappa > \omega$ is regular and $\mathfrak{S}$ is wqo we may cut down to a κ

powered subset of κ where each $n_\gamma = n$ and for each $r \leqq n$, $\gamma < \delta \rightarrow \psi_{\gamma r} \leqq \psi_{\delta r}$. The reader may verify that for the least $r \leqq n$ such that the sequence $\{\psi_{\alpha r}\}$ is unbounded in φ we must have

$$\sum_\alpha \psi_{\alpha r} \equiv \varphi ,$$

as desired.

THEOREM 2.11. *If* $\varphi \in \mathfrak{S}$ *is additively indecomposable then for some* $(T, l) \in \mathfrak{U}$, $\varphi \equiv (\overline{T, l})$.

Proof. Assume the theorem holds for all types $\psi < \varphi$. By the inductive hierarchy for AI types given by Theorem 2.4, assume first that φ is a regular unbounded well ordered sum of smaller AI types. We consider separately the cases whether or not this sum can be taken to be increasing.

Case 1. $\varphi \equiv \sum_{\alpha<\kappa} \varphi_\alpha$, where κ is an infinite regular cardinal, each φ_α is AI, and $\alpha < \beta \rightarrow \varphi_\alpha \leq \varphi_\beta$. For each $\alpha < \kappa$, pick $(T_\alpha, l_\alpha) \in \mathfrak{U}$ with $(\overline{T_\alpha, l_\alpha}) \equiv \varphi_\alpha$, by the induction hypothesis. We obtain a tree (T, l) as follows.

If $\kappa > \omega$ cut down to a κ powered subset of κ on which $\alpha < \beta \rightarrow T_\alpha = T_\beta = T$, then cut down further (Card T times) to assume $\alpha < \beta \rightarrow (T, l_\alpha) \leq_I (T, l_\beta)$; take $(T, l) = \sum_\alpha^I (T, l_\alpha)$.

If $\kappa = \omega$, then obtain, by Theorem 1.6, an $n < \omega$ and a subtree (T'_α, l_α) of each (T_α, l_α) such that each T'_α has $\leqq n$ treetops and $\forall \alpha \exists \beta$ $(T_\alpha, l_\alpha) \leq_m (T'_\beta, l_\beta)$. We may assume that each (T'_α, l_α) is closed downward in (T_α, l_α), which implies, by Lemma 2.9(a), that each $(T'_\alpha, l_\alpha) \in \mathfrak{U}$. Also, it can be seen that the (T'_α, l_α)'s can be chosen so that $\alpha < \beta \rightarrow (T'_\alpha, l_\alpha) \leq_m (T'_\beta, l_\beta)$. Hence, by Lemma 2.7, $\sum_\alpha (\overline{T'_\alpha, l_\alpha}) \equiv \sum_\alpha (\overline{T_\alpha, l_\alpha})$. Using repeated applications of Lemma 2.9(b), collapse each (T'_α, l_α) to a subtree $(T''_\alpha, l_\alpha) \in \mathfrak{U}$ with the same number of treetops such that each (T''_α, l_α) has height $\leqq n$ and $(\overline{T''_\alpha, l_\alpha}) \equiv (\overline{T'_\alpha, l_\alpha})$. Since there are now only finitely many different T''_α's, we may cut down to an infinite subset of ω and assume they are the same tree T, and cut down further to assume $\alpha < \beta \rightarrow (T, l_\alpha) \leq_I (T, l_\beta)$. Take $(T, l) = \sum_\alpha^I (T, l_\alpha)$.

In either case, $(T, l) \in \mathfrak{U}$ and $(\overline{T, l}) \equiv \varphi$.

Case 2. Case 1 fails. Then in view of Lemma 2.10, we must have $\kappa = \omega$. By Lemma 1.4, we can write

$$\varphi \equiv \sum_{n<\omega} (\varphi_{n1} + \varphi_{n2} + \cdots + \varphi_{nr}) ,$$

where each φ_{ni} is AI, and $m < n \rightarrow \varphi_{mi} \leq \varphi_{ni}$ for all $i \leqq r$. Let $\varphi_i = \sum_n \varphi_{ni}$. Apply the methods of Case 1 to each φ_i to obtain $(T_i, l_i) \in \mathfrak{U}$, $(T_i, l_i) = \sum_{j<\omega}^I (T_{ij}, l_j)$, $(\overline{T_i, l_i}) \equiv \varphi_i$. Letting $(T, l) = [\omega; (T_1, l_1), \cdots, (T_r, l_r)]$, we have that $(T, l) \in \mathfrak{U}$ and $(\overline{T, l}) \equiv \varphi$.

The cases where φ is a converse well ordered sum are dealt with symmetrically, so the proof is complete.

The partition relation $\varphi \rightarrow (\varphi)^1_{<\omega/n}$ means: if $(L, <_L)$ is a linearly ordered set of type φ and $L = \bigcup_{i \in I} L_i$ where Card $I < \omega$, then for some $\leqq n$ element $I' \subseteq I$, $\varphi \equiv \mathrm{tp}\,(\bigcup_{i \in I'} L_i, <_L)$.

THEOREM 2.12. (a) *If* $(T, l) \in \mathfrak{U}$, *T has n treetops, and* $\overline{(T, l)} = \varphi$, *then* $\varphi \rightarrow (\varphi)^1_{<\omega/n}$.

(b) *If* $\varphi \in \mathcal{S}$ *then for some* $n < \omega$, $\varphi \rightarrow (\varphi)^1_{<\omega/n}$.

Proof. (a). If $(T, l) = \sum^I_{\alpha<\kappa} (T, l_\alpha)$, then, letting $\overline{(T, l_\alpha)} = \varphi_\alpha$, we have $\varphi \equiv \sum_{\alpha<\kappa} \varphi_\alpha$, where the φ_α's are increasing and $\varphi_\alpha \rightarrow (\varphi_\alpha)^1_{<\omega/n}$ by induction. It follows that $\varphi \rightarrow (\varphi)^1_{<\omega/n}$. If $(T, l) = [\omega; (T_1, l_1), \cdots, (T_r, l_r)]$ then, letting $\overline{(T_i, l_i)} = \varphi_i$, we have $\varphi_i \rightarrow (\varphi_i)^1_{<\omega/n_i}$, n_i the number of treetops of T_i, by induction, and it follows that $\varphi \rightarrow (\varphi)^1_{<\omega/\sum_{i \leqq r} n_i}$, as desired. Converse well ordered sums are symmetrically handled.

(b) Let $\varphi = (\varphi_1 + \varphi_2 + \cdots + \varphi_r)$ be the decomposition of φ into a finite sum of AI's given by Theorem 2.3. By Theorems 2.11 and 2.12(a), there are $n_i < \omega$ such that $\varphi_i \rightarrow (\varphi_i)^1_{<\omega/n_i}$, $i \leqq r$. Hence $\varphi \rightarrow (\varphi)_{<\omega/\sum_{i \leqq r} n_i}$.

THEOREM 2.13. $\varphi \in \mathcal{S}$ *is strongly indecomposable* (*i.e.*, $\varphi \rightarrow (\varphi)^1_{<\omega/1}$) *just in case* $\varphi \equiv \psi$ *for some* $\psi \in \mathcal{HI}$.

Proof. Given φ SI, by Theorem 2.11, pick $(T, l) \in \mathfrak{U}$, $\overline{(T, l)} \equiv \varphi$, where subject to those conditions, (T, l) has the least possible number n of treetops. By construction, (T, l) gives a canonical decomposition of $\overline{(T, l)}$ into subsets of type $l(x)$, x a treetop of T. We must then have $n = 1$, otherwise $\overline{(T, l)}$ could be partitioned into a finite number of smaller parts. But now $\varphi \equiv l$ (the treetop of T), an $\mathcal{HI}$ type.

Note that Theorem 2.13 can be generalized to characterize, for each $r < \omega$, the class $\{\varphi \in \mathcal{S}\colon \varphi \rightarrow (\varphi)^1_{<\omega/r+1}$ and $\varphi \nrightarrow (\varphi)^1_{<\omega/r}\}$; the characterization comes out of Theorems 2.3, 2.11, and 2.12.

3. Extension to the class $\mathfrak{M}_Q$

Let $\mathfrak{M}$ be the class of all order types φ such that a linearly ordered set of type φ can be partitioned into $\leqq \aleph_0$ subsets, each of which is scattered under the inherited ordering. The class $\mathfrak{M}$ was introduced and given a Hausdorff-like classification by F. Galvin; we assume familiarity with this work (see [9], Section 3). In this section we will indicate how to prove the results of Section 2 for this larger class of types; since the proofs, though more complicated, should be accessible to those familiar with [9] and with the scattered case above, we leave them to the reader.

Recall that for Q a quasi-ordered set, a Q-type Φ with base φ is a labelled order type $\mathrm{tp}(L, l)$, $\mathrm{tp}(L) = \varphi$, $l\colon L \to Q$. Writing $\mathrm{bs}(\Phi)$ for the base type of Φ we let $\mathfrak{M}_Q$ be the class of all Q-types Φ with $\mathrm{bs}(\Phi) \in \mathfrak{M}$. For $q \in Q$, let 1_q be the one point order type, labelled by q. All the basic definitions for order types (embeddability, sums, decomposability) carry over naturally to Q-types.

We state our results in terminology which can be specialized to the scattered case. Call a type φ regular if whenever $\mathrm{tp}(L) < \varphi$ and for each $x \in L$, $\varphi_x < \varphi$, then $\sum_{x \in L} \varphi_x < \varphi$. Call a Q-type Φ regular unbounded just in case Φ is of the form $\mathrm{tp}(L, l)$, where $\mathrm{tp}(L)$ is infinite and regular, $l\colon L \to Q$, and if $L' \subseteq L$ is an open interval of L with $\mathrm{tp}(L') \equiv \mathrm{tp}(L)$, then $\mathrm{tp}(L', l) \equiv \mathrm{tp}(L, l)$. If $\Phi \in \mathfrak{M}_{\mathfrak{M}_Q}$, i.e. $\Phi = \mathrm{tp}(L, l)$, $l\colon L \to \mathfrak{M}_Q$, let $\bar{\Phi} \in \mathfrak{M}_Q$ be $\sum_{x \in L} l(x)$. A regular unbounded sum of members of $\mathfrak{M}_Q$ is a type of the form $\bar{\Phi}$, where $\Phi \in \mathfrak{M}_{\mathfrak{M}_Q}$ is regular unbounded.

The following theorem summarizes the results from [9]; Galvin's classification, stated in the present terminology, is part (a) and (b), the direction left to right of (a) being provable by those methods.

THEOREM 3.1. [9]. (a) *An infinite* $\varphi \in \mathfrak{M}$ *is regular* $\leftrightarrow$ *either* φ *is* κ *or* κ^*, *for some infinite regular cardinal* κ, *or* φ *is* $\equiv$ *an* $\eta_{\alpha\beta}$.

(b) *The class of nonzero members of* $\mathfrak{M}$ *is the closure of* $\{1\}$ *under nonzero regular* $\mathfrak{M}$ *sums.*

(c) Q bqo $\to \mathfrak{M}_Q$ bqo.

(d) Q bqo $\to$ *each* $\Phi \in \mathfrak{M}_Q$ *is a finite sum of additively indecomposable members of* $\mathfrak{M}_Q$.

(e) *If* Q wqo, $\Phi \in \mathfrak{M}_Q$, $\mathrm{bs}(\Phi) = \varphi$, φ *infinite, then* Φ *is a* $< \varphi$ *sum of types* 1_q: $q \in Q$ *and regular unbounded members of* $\mathfrak{M}_Q$.

(f) Q bqo $\to$ *the class of* AI *members of* $\mathfrak{M}_Q$ *is the closure of* $\{0\} \cup \{1_q\colon q \in Q\}$ *under regular unbounded sums.*

(g) Q bqo, κ *an infinite cardinal,* $\mathrm{Card}\,(Q/_\equiv) \leq \kappa \to \mathrm{Card}\big((\mathfrak{M}_{(<\kappa)})_{Q/\equiv}\big) = \kappa$.

Define a regular increasing sum of members of $\mathfrak{M}_Q$ to be a sum $\sum_{\alpha<\kappa} \Phi_\alpha$ or $\sum_{\alpha<\kappa^*} \Phi_\alpha$, where κ is an infinite regular cardinal and

$$\alpha < \beta \longrightarrow \Phi_\alpha \leq \Phi_\beta \ ,$$

or a sum $\sum_{x \in \eta_{\alpha\beta}} \Phi_x$, where

$$x,\, y \in \eta_{\alpha\beta} \longrightarrow (\Phi_x \leq \Phi_y \ \text{ or } \ \Phi_y \leq \Phi_x) \ ,$$

and for every interval (x, y) of $\eta_{\alpha\beta}$,

$$\forall\, z \in \eta_{\alpha\beta} \ \ \exists\, w \in (x, y) \ \ \Phi_z \leq \Phi_w \ .$$

Let $\mathcal{HI}_Q$, the class of hereditarily increasing members of $\mathfrak{M}_Q$, be the closure of $\{0\} \cup \{1_q \colon q \in Q\}$ under regular increasing sums.

We define a class $\mathfrak{U}_Q$ of finite labelled trees; $\mathfrak{U}_Q$ is the smallest class containing $\varnothing$ and the one point tree labelled by 1_q, each $q \in Q$, which is closed under the operations (i) and (ii) used to define the class $\mathfrak{U}$ of Section 2 and

(iii) if for each $z \in \eta_{\alpha\beta}$, $(T, l_z) \in \mathfrak{U}_Q$ and $\forall\, z, z' \in \eta_{\alpha\beta}$, $(T, l_z) \leq_I (T, l_{z'})$ or $(T, l_{z'}) \leq_I (T, l_z)$, and for all intervals (u, v) of $\eta_{\alpha\beta}$,

$$\forall\, z \in \eta_{\alpha\beta},\ \exists\, z' \in (u, v),\ (T, l_z) \leq_I (T, l_{z'})\ ,$$

then $(T, l) \in \mathfrak{U}_Q$, where for $x \in T$, $l(x) = \sum_{z \in \eta_{\alpha\beta}} l_z(x)$.

We will have that if $(T, l) \in \mathfrak{U}_Q$, then the treetops of T are $\mathcal{HI}_Q$ labelled, and the nodes below the treetops are labelled by $\mathcal{HI}$ members of $\mathfrak{M}$. To each $(T, l) \in \mathfrak{U}_Q$ we may now assign a type $\overline{(T, l)} \in \mathfrak{M}_Q$, in a manner unique up to $\equiv$, by extending the rules of Section 2 to include a clause for $\eta_{\alpha\beta}$-increasing sums corresponding to (iii) above. The analogs of Lemmas 2.6–2.9 may then be proved. The proposition which we add to Lemma 2.10 is the following:

If Q is bqo *and $\Phi \in \mathfrak{M}_Q$ is an $\eta_{\alpha\beta}$-unbounded sum $\sum_{x \in \eta_{\alpha\beta}} \Phi_x$, each $\Phi_x < \Phi$, then $\Phi \equiv \sum_{x \in \eta_{\alpha\beta}} \Psi_x$, where each $\Psi_x < \Phi$ and the sum $\sum_{x \in \eta_{\alpha\beta}} \Psi_x$ is regular increasing.*

As an indication of the proof, assume without loss of generality that $\alpha \leq \beta$, and thus Card $\eta_{\alpha\beta} = \delta$ where $\beta = \delta^+$. Let $\{\Phi_x \colon x \in \eta_{\alpha\beta}\} = \{\Phi_\gamma \colon \gamma < \delta\}$. Using various parts of Theorem 3.1, apply the method of Lemma 2.10 to the type $\sum_{\gamma<\delta} \Phi_\gamma$ to obtain increasing sequence $\Psi_\rho \colon \rho < \sigma$, $\sigma \leq \delta$ of types $< \sum_{\gamma<\delta} \Phi_\gamma$ such that $\forall\, \gamma < \delta\ \exists\, \rho < \sigma$, $\Phi_\gamma \leq \Psi_\rho$ (the cases δ regular and δ singular are considered separately). Since $\delta' < \delta \rightarrow \delta' < \eta_{\alpha\beta}$, it can be seen that each $\Psi_\rho < \Phi$. Finally, it is verified that an $\eta_{\alpha\beta}$-increasing sum of the Ψ_ρ's will be $\equiv \Phi$, as desired.

The analog of Theorem 2.11 may then be proved; in the case where Φ is an $\eta_{\alpha\beta}$-unbounded sum of smaller AI Q-types, the proposition above is used.

The relation $\Phi \longrightarrow (\Phi)^1_{<\omega/n}$ for Q-types means: whenever the base of Φ is partitioned into $< \omega$ subsets, the union of some $\leqq n$ of those subsets will induce a Q-type $\equiv \Phi$. The main results which can now be verified are given by

THEOREM 3.2. *Assuming Q is* bqo, *then*

(a) *if $\Phi \in \mathfrak{M}_Q$ is* AI, *there is a $(T, l) \in \mathfrak{U}_Q$, $\overline{(T, l)} \equiv \Phi$*

(b) *if $(T, l) \in \mathfrak{U}_Q$ has $\leqq n$ treetops, $\overline{(T, l)} \equiv \Phi$, then $\Phi \longrightarrow (\Phi)^1_{<\omega/n}$.*

(c) *if $\Phi \in \mathfrak{M}_Q$ then for some $n < \omega$, $\Phi \longrightarrow (\Phi)^1_{<\omega/n}$.*

(d) *if $\Phi \in \mathfrak{M}_Q$ is* SI *then for some $\Psi \in \mathcal{HI}_Q$, $\Phi \equiv \Psi$.*

Theorem 3.2(d) should be compared with Theorem 3.1(b) and (f). Although in 3.2(d) the SI types are only generated up to $\equiv$, they can in fact all be generated by taking regular increasing sums where the set $\{\Phi_\gamma: \gamma < \delta\}$ of summands, instead of being a chain, only satisfies $\forall \gamma < \delta$, $\exists \beta < \delta$, $\forall \alpha < \gamma$, $\Phi_\alpha \leq \Phi_\beta$.

4. Mappings and partitions

In this section the results of Section 2 are used to give some combinatorial properties of order types. For simplicity we consider only countable types (though at the end a result is sketched for scattered types). If a classical theorem holds for only countable ordinals then of course its analog is only expected to hold for countable types, but if it holds for arbitrary ordinals a generalization to the entire class $\mathfrak{M}$ would be expected.

The analog to the notion of limit ordinal for the class $\mathfrak{M}$ is the following: $\varphi \in \mathfrak{M}$ is a limit type if no 1 appears in the expression of φ as a minimal sum of AI's, that is, φ has no points which are left fixed under every order embedding of φ into itself. From Theorem 3.1, it is seen that the limit types are those which can be written $\equiv$ a type of the form $\sum_{x \in L} \varphi_x$ where each φ_x is ω or ω^*.

If φ is a countable limit type, $\varphi = \mathrm{tp}(L)$, $L = \sum_{y \in L'} L'_y$, each $\mathrm{tp}(L'_y)$ either ω or ω^*, then let

$$L = x_1, x_2, \cdots, x_n, \cdots \; (n < \omega)$$

where, for each y, this ordering restricted to L'_y is the natural ω-ordering of L'_y. Call $\langle x_n : n < \omega \rangle$ a standard enumeration of L; it possesses the property that if $x_{i_1} <_L x_{i_2} <_L \cdots <_L x_{i_n}$ and $\{x_{i_1}, \cdots, x_{i_n}\} = \{x_1, \cdots, x_n\}$, then each of the open L-intervals $(-\infty, x_{i_1})$, (x_{i_1}, x_{i_2}), $\cdots$, $(x_{i_{n-1}}, x_{i_n})$, $(x_{i_n}, +\infty)$ has limit order type.

THEOREM 4.1(a). *Let φ be a countable limit type,* $\mathrm{tp}(L) = \varphi$, $f\colon L \to L$ (*not necessarily order preserving*), $fx \neq x$, *all* $x \in L$. *Then* $\exists\, M \subset L$ $\mathrm{tp}(M) = \varphi$, $M \cap f[M] = \varnothing$.

(b) *If, additionally, f is onto L, then*

$$\exists\, M \subset L \;\; \mathrm{tp}(M) \equiv \mathrm{tp}(f[M]) \equiv \varphi \,, \quad M \cap f[M] = \varnothing \,.$$

Proof. (a). Let $x_1, x_2, \cdots, x_n, \cdots$ $(n < \omega)$ be a standard enumeration of L. Suppose $n < \omega$ and we have found $y_1, y_2, \cdots, y_n$, $\{y_1, \cdots, y_n\} \subset Y_n \subset L$ such that

$$(Y_n, <_L, y_1, \cdots, y_n) \cong (L, <_L, x_1, \cdots, x_n)$$

and

$$i \leqq n \longrightarrow \left(f(y_i) \notin Y_n, \left(f^{-1}(\{y_i\}) \cap Y_n\right) = \varnothing\right) .$$

We show that y_{n+1} and $Y_{n+1} \subset Y_n$ can be found satisfying like conditions; the set $M = \{y_1, y_2, \cdots, y_n, \cdots\}$ which is obtained by this process will then satisfy the conditions of the theorem. Let $\{x_1, \cdots, x_n\} = \{x_{i_1}, \cdots, x_{i_n}\}$, $x_{i_1} <_L x_{i_2} <_L \cdots <_L x_{i_n}$. Suppose, for instance, $x_{i_j} <_L x_{n+1} <_L x_{i_{j+1}}$. Since $\mathrm{tp}(x_{i_j}, x_{i_{j+1}})$ is a limit type, there is an ω or ω^* sequence $A \subseteq \left(Y_n \cap (y_{i_j}, y_{i_{j+1}})\right)$ of images of x_{n+1} under order embeddings from $(x_{i_j}, x_{i_{j+1}})$ into $\left(Y_n \cap (y_{i_j}, y_{i_{j+1}})\right)$. Let $\varphi_0, \varphi_1, \cdots, \varphi_{n+2}$ be the types of the open intervals determined by $-\infty$, $x_{i_1}, \cdots, x_{i_j}, x_{n+1}, x_{i_{j+1}}, \cdots, x_{i_n}, +\infty$. By Theorem 2.12(b) pick $r_i < \omega$ so that $\varphi_i \longrightarrow (\varphi_i)^1_{<\omega/r_i}$. Let $p = \sum_{i \leqq n+2} r_i$. Choose an $a \subseteq A$, $\mathrm{Card}\, a = p + 1$. $\left(Y - \{y_1, \cdots, y_n\}\right) -$ (the closed interval containing a) is divided into sets B_i of type φ_i, $i \leqq n + 2$. Write each B_i as a union of disjoint sets:

$$B_i = \bar{B}_i \cup \bigcup_{x \in a} \left(B_i \cap f^{-1}(x)\right) .$$

Applying successively the $n + 2$ partition relations given above gives sets $B_i' \subseteq B_i$ of type φ_i and a $y_{n+1} \in a$ such that

$$f^{-1}\left(\{y_{n+1}\}\right) \cap B_i' = \varnothing ,$$

for all i. Take

$$Y_{n+1} = \left(\{y_1, y_2, \cdots, y_{n+1}\} \cup \bigcup_{i \leqq n+2} B_i'\right) - \{f(y_{n+1})\} .$$

Since the removal of $f(y_{n+1})$ does not affect the order types of the B_i''s — they are limit types — Y_{n+1} is as desired, completing the induction step.

Note that (a) could be strengthened by only requiring that, for some fixed $j < \omega$, each $f(x) \subset L$, $x \notin f(x)$ and $\mathrm{Card}\, f(x) \leqq j$. For the proof of (b), which is left to the reader, the induction step is as in (a), with the addition that we have constructed also $z_1, \cdots, z_n, w_1, \cdots, w_n$, where $f(w_i) = z_i$, $y_i <_L y_j \leftrightarrow z_i <_L z_j$, and $\left(f^{-1}(\{w_i\}) \cap Y_n\right) = \varnothing$, with the intention of taking $M = \{y_1, y_2, \cdots, w_1, w_2, \cdots\}$.

We would like to mention two other applications of Theorem 2.12(b). Erdös and Milner [3], and independently, Galvin, have generalized Milner's $\omega^{1+\alpha n} \longrightarrow (2^n, \omega^{1+\alpha})^2$ to a theorem which holds for countable order types. Galvin [5] has shown that if $\mathrm{tp}(L) \leq \eta$ has at most one fixed point, and two players, White and Black, play an infinite game consisting of alternately picking previously unchosen members of $[L]^2$, then White has a strategy which guarantees that after ω steps he will have obtained all the 2-element subsets from an $L' \subseteq L$, $\mathrm{tp}(L') = \mathrm{tp}(L)$.

We will give two partition theorems for order types whose proofs involve the representation theorem rather than just Theorem 2.12(b). For the rest of this section, all types will be in $\mathfrak{S}$.

Since the properties of order types we are interested in are invariant under $\equiv$, we may always assume that we are working with types $\varphi \in \mathfrak{S}$ which have associated with them a canonical representation, that is, $\varphi = \overline{(T_1, l_1)} + \overline{(T_2, l_2)} + \cdots + \overline{(T_n, l_n)}$, $(T_i, l_i) \in \mathfrak{U}$, n minimal. In places then, definitions and theorems about order types are tacitly understood to proceed inductively on a given tree representation. We use the notation $\sum_{\alpha<\kappa}^{I} L_\alpha$, $\sum_{\alpha<\kappa^*}^{I} L_\alpha$, $[\omega; L_1, \cdots, L_n]$, and $[\omega^*; L_1, \cdots, L_n]$ to denote linear orderings obtained in a canonical way from subsets via a given tree representation.

Let $\varphi \in \mathfrak{S}$, $\operatorname{tp}(L) = \varphi$. We define the notion $L' \subseteq_c L$ (L' is a complete subset of L) by induction:

(i) if $\operatorname{tp}(L) = 0$ or 1, $L' \subseteq_c L \leftrightarrow L' = L$,

(ii) if $L = \sum_{\alpha<\kappa}^{I} L_\alpha$ or $\sum_{\alpha<\kappa^*}^{I} L_\alpha$, $L' \subseteq_c L \leftrightarrow \operatorname{Card}\{\alpha: L' \cap L_\alpha \subseteq_c L_\alpha\} = \kappa$,

(iii) if $L = [\omega; L_1, \cdots, L_n]$ or $[\omega^*; L_1, \cdots, L_n]$, $L' \subseteq_c L \leftrightarrow \forall\, i \leqq n$ $L' \cap L_i \subseteq_c L_i$,

(iv) if $L = (L_1 + L_2 + \cdots + L_n)$, a minimal sum of AI's, $L' \subseteq_c L \leftrightarrow \forall\, i \leqq n$ $L' \cap L_i \subseteq_c L_i$.

Define the notion $L' \subseteq_a L$ (L' is almost all of L) by replacing $\subseteq_c$ by $\subseteq_a$ everywhere in the above definition, and replacing the cardinality condition in (iii) by

$$\operatorname{Card}\{\alpha: L' \cap L_\alpha \not\subseteq_a L_\alpha\} < \kappa\ .$$

By induction it can be shown that

LEMMA 4.2. (a) $\subseteq_c$, $\subseteq_a$ *are transitive.*

(b) $L' \subseteq_a L \rightarrow L' \subseteq_c L' \rightarrow \operatorname{tp} L' = \operatorname{tp} L$.

(c) $\{L': L' \subseteq_a L\}$ *is a filter.*

(d) *If* $\operatorname{tp}(L)$ *is* SI *and* $L = L' \cup L''$, *then either* $L' \subseteq_c L$ *or* $L'' \subseteq_a L$.

If $(T, l) \in \mathfrak{U}$, $\{x_1, \cdots, x_n\}$ a set of treetops of T, then let $(T_{\vec{x}}, l)$ be the subtree of (T, l) obtained by closing $x_1, \cdots, x_n$ downwards in T. If $\operatorname{tp}(L) = \overline{(T, l)}$ and the treetops of T, written without repetition, are $x_1, \cdots, x_m$, $y_1, \cdots, y_n$, then L *is* canonically decomposed into $L_{\vec{x}} \cup L_{\vec{y}}$, where $\operatorname{tp}(L_{\vec{x}}) = \overline{(T_{\vec{x}}, l)}$, $\operatorname{tp}(L_{\vec{y}}) = \overline{(T_{\vec{y}}, l)}$.

Suppose now that $L' \subseteq_c L_{\vec{x}}$. Then $\operatorname{Com}(L', L)$, the reduced complement of L' in L, is to be a certain complete subset of $L_{\vec{y}}$, defined inductively as follows:

(i) if $\vec{x} = \varnothing$, $\operatorname{Com}(L', L) = L$, and if $\vec{y} = \varnothing$, $\operatorname{Com}(L', L) = \varnothing$,

(ii) if $(T, l) = \sum_{\alpha<\kappa}^{I} (T, l_\alpha)$ or $\sum_{\alpha<\kappa^*} (T, l_\alpha)$ (whence $L = \sum_{\alpha<\kappa}^{I} L_\alpha$ or $L = \sum_{\alpha<\kappa^*}^{I} L_\alpha$) then

$$\operatorname{Com}(L', L) = \bigcup\nolimits_{\alpha<\kappa: L'_\alpha \subseteq_c L_\alpha} \operatorname{Com}(L'_\alpha, L_\alpha)\ ,$$

where $L'_\alpha =_{\mathrm{df}} L' \cap L_\alpha$.

(iii) if $(T, l) = [\omega; (T_1, l_1), \cdots, (T_n, l_n)]$ or $[\omega^*; (T_1, l_1), \cdots, (T_n, l_n)]$ (whence $L = [\omega; L_1, \cdots, L_n]$ or $L = [\omega^*; L_1, \cdots, L_n]$) then $L'_i =_{\mathrm{df}} L' \cap L_i \subseteq_c L_{\vec{x}(i)}$ where $\vec{x}(i) =_{\mathrm{df}} \vec{x} \cap \{\text{treetops of } T_i\}$; take $\mathrm{Com}\,(L', L) = \bigcup_{i \leq n} \mathrm{Com}\,(L'_i, L_i)$.

LEMMA 4.3. *Let* $\mathrm{tp}(L) = \overline{(T, l)}$, $L = L_{\vec{x}} \cup L_{\vec{y}}$ *as above. Then*
(a) *If* $A \subseteq_a L_{\vec{x}}$, $B \subseteq_a L_{\vec{y}}$ *then* $A \cup B \subseteq_a L$
(b) *If* $A \subseteq_c L_{\vec{x}}$, $B \subseteq_c \mathrm{Com}\,(A, L)$ *then* $A \cup B \subseteq_c L$.

The proof of this lemma by induction is left to the reader.

The partition relation

$$\varphi \longrightarrow (\varphi, \text{infinite path})^2$$

means: if $\mathrm{tp}(L) = \varphi$ and $[L]^2 = R \cup S$, then either $\exists\, L' \subseteq L$, $\mathrm{tp}(L') = \varphi$ and $[L']^2 \subseteq R$, or there is a nonrepeating sequence $\cdots, x_{-n}, \cdots, x_{-1}, x_0, x_1, \cdots, x_n, \cdots$ from L with each $\{x_i, x_{i+1}\} \in S$. Erdös, Milner, and Hajnal [2] showed that if α is a countable limit ordinal then $\alpha \rightarrow (\alpha, \text{infinite path})^2$; we show here how to get it for countable limit types.

THEOREM 4.4. *If φ is a countable limit type, then $\varphi \rightarrow (\varphi, \text{infinite path})^2$.*

Proof. Since $\eta \rightarrow (\eta, \aleph_0)^2$, we may assume $\varphi < \eta$. Let $\mathrm{tp}(L) = \varphi$, $[L]^2 = R \cup S$, and suppose there is no infinite S path. Let $x_1, x_2, \cdots, x_n, \cdots$ $(n < \omega)$ be a standard enumeration of L.

Claim. For every infinite $A \subseteq L$, infinite SI $B \subseteq L$, B canonically represented, there is a $B' \subseteq_c B$ such that for all but finitely many $w \in A$, $\{z \in B' \colon \{w, z\} \in R\} \subseteq_a B'$.

Proof of Claim. If otherwise, then from Lemma 4.2 it follows that $\exists\, A, B, \forall\, B' \subseteq_c B$ (there are $\aleph_0$ $w \in A \colon \{z \in B' \colon \{w, z\} \in S\} \subseteq_c B'$). We leave it to the reader to show that this would imply the existence of an infinite S-path alternating between A and B.

Suppose now that we have chosen $\{y_1, y_2, \cdots, y_n\} \subset Y_n \subseteq L$, with

$$(Y_n, <_L, y_1, \cdots, y_n) \cong (L, <_L, x_1, \cdots, x_n)\ ,$$

and so that

$$i \leqq n\ ,\ \ y \in Y_n \longrightarrow \{y_i, y\} \in R\ ;$$

we want to continue and find Y_{n+1}, y_{n+1} satisfying the same conditions. Let $\{x_1, \cdots, x_n\}$ be $x_{i_1} <_L x_{i_2} <_L \cdots <_L x_{i_n}$, and suppose, say, that $x_{i_j} <_L x_{n+1} <_L x_{i_{j+1}}$. As in Theorem 4.1(a), there is an ω or ω^* sequence $A \subseteq (Y_n \cap (y_{i_j}, y_{i_{j+1}}))$ of possible candidates for y_{n+1}. Using Ramsey's theorem we may assume A

has been cut down so that $[A]^2 \subseteq R$. Y may be partitioned: $-\infty <_L y_{i_1} <_L B_1 <_L \cdots <_L y_{i_j} <_L B_j <_L A <_L B_{j+1} <_L y_{j+1} <_L \cdots <_L y_{i_n} <_L B_{n+1} <_L +\infty$, where each B_i has limit type φ_i.

We are going to find an $A' \subseteq_a A$ and a $B_0' \subseteq_c B_0$ such that

$$w \in A' \longrightarrow \{z \in B_0' \colon \{w, z\} \in R\} \subseteq_a B_0' \ .$$

Then, by cutting down and repeating this process for $B_1, B_2, \cdots, B_{n+1}$, we will be in position to choose y_{n+1}, Y_{n+1} correctly, completing the inductive step. Let $B_0 = B$. We may reduce to the case where B is AI. Let $\mathrm{tp}(B) \equiv (\overline{T, l})$, where T has r treetops corresponding to SI sets $B^1, \cdots, B^r \subseteq B$. Suppose $k < r$ and we have found $B(k) \subseteq_c B^1 \cup B^2 \cup \cdots \cup B^k$, and $A(k) \subseteq_a A$ such that

$$(*) \qquad w \in A(k) \longrightarrow \{z \in B(k) \colon \{w, z\} \in R\} \subseteq_a B(k) \ .$$

Let $C \subseteq_c B^{k+1}$ be $\mathrm{Com}\big(B(k), B^1 \cup \cdots \cup B^{k+1}\big)$. We may now apply the claim and Lemma 4.3 to get $A(k+1) \subseteq_a A(k)$ and $C' \subseteq_c C$ such that, letting $B(k+1) = \big(B(k) \cup C'\big)$, the condition $(*)$ holds for $k+1$. Then it is seen that $A' = A(r)$, $B' = B(r)$ have the desired properties. This completes the proof.

The partition relation

$$\varphi \longrightarrow \big(\varphi, \tbinom{n}{\aleph_0}\big)^2$$

means: if $\mathrm{tp}(L) = \varphi$, $[L]^2 = R \cup S$, then either $\exists\, L' \subseteq L$, $\mathrm{tp}(L') = \varphi$ and $[L']^2 \subseteq R$ or $\exists\, a$, $A \subseteq L$, Card $a = n$, Card $A = \aleph_0$ and $(a \otimes A) \subseteq S$. Erdös, Milner and Hajnal [2] proved that if α is a limit ordinal then for all $n < \omega$, $\alpha \to \big(\alpha, \binom{n}{\aleph_0}\big)^2$ (this may be contrasted with Theorem 4.4, which is not known to hold for all limit ordinals), and Galvin proved it for all countable limit types, using Theorem 2.12(b). We show here the extension to scattered limit types. A number of new features arise; we think it reasonable just to indicate the proof, leaving the rather complicated details to the interested reader.

THEOREM 4.5. *If $\varphi \in \mathfrak{S}$ is a limit type, $n < \omega$, then $\varphi \to \big(\varphi, \binom{n}{\aleph_0}\big)^2$.*

Proof. Let $\mathrm{tp}(L) = \varphi$, $[L]^2 = R \cup S$, Card $L = \kappa$. Assume no $(n \otimes \aleph_0) \subseteq S$.

Case 1. $\varphi \in$ SI. We will assume here that κ is singular, the regular case is similar. By a component of L we will mean any subset of L of power κ which appears at some point as a summand in the inductive construction of L. A component X is a κ-block if X is a $cf\,\kappa$ or $(cf\,\kappa)^*$ sum of orderings of power $< \kappa$. We will construct $L' \subseteq L$, $\mathrm{tp}\, L' = \varphi$, $[L']^2 \subseteq R$, in $cf\,\kappa$ stages.

If $\kappa_\alpha < \kappa$, to κ_α-fix a component X will mean to find a $Z \subseteq L$, Card $Z \leqq \kappa_\alpha$ and to find an $X' \subseteq_c X$, in such a way that if $Y \subseteq L - Z$, Card $Y < \kappa$, then $(Y \otimes X'') \subseteq R$ for some $X'' \subseteq_a X$.

It is seen that for each component X there is a $\kappa_\alpha < \kappa$ such that X can be κ_α-fixed; otherwise, using Lemma 4.2, n times, an $(n \otimes \aleph_0) \subseteq S$ could be found. Let $cf\,\kappa < \kappa_0 < \kappa_1 < \cdots < \kappa_\alpha < \cdots$ $(\alpha < cf\,\kappa) \uparrow \kappa$, each κ_α regular. Whenever a component X is κ_α-fixed in the course of the construction below, X is replaced by X' in all future steps, and the set Z is discarded entirely $(Z \cap L' = \varnothing)$.

Stage 0. $\leqq \kappa_0$-fix L if possible, otherwise stage 0 terminates. Suppose stage 0 has progressed and some component X has been $\leqq \kappa_0$-fixed. If X is a $> \kappa_0$ or $> \kappa_0^*$ sum, no components of X will be fixed at stage 0, if X is a $\leq \kappa_0$ or κ_0^* sum fix each component of X which it is possible to $\leqq \kappa_0$-fix. Stage 0 terminates when this process cannot be continued. It is shown by induction that at most κ_0 components of L become fixed at stage 0, in particular the κ-blocks which are fixed can be enumerated

$$X_1, X_2, \cdots, X_\gamma, \cdots, \gamma < \delta, \delta \leqq \kappa_0 .$$

Pick $X_1(0) \subseteq X_1$, Card $X_1(0) = \kappa_0$ so that, if possible, $\mathrm{tp}\big(X_1(0)\big) \geq \mathrm{tp}$(the 0^{th} subcomponent of X). For each $\gamma < \delta$ $\exists\, X_\gamma' \subseteq_a X_\gamma$ $\big(X_1(0) \otimes X_\gamma'\big) \subseteq R$, so we may continue this process and choose $X_2(0) \subseteq X_2'$ similarly; letting

$$B_0 = \bigcup_{\gamma<\delta} X_\gamma(0) ,$$

we have $[B_0]^2 \subseteq R$, Card $B_0 \leqq \kappa_0$, and $(B_0 \otimes L_0) \subseteq R$ for some $L_0 \subseteq L$, tp $L_0 = \varphi$.

Stage $\alpha + 1$, $\alpha < cf(k)$. Proceed in the same manner inside of L_α, beginning with the components which have been fixed but which contain unfixed subcomponents, obtaining sets $B_{\alpha+1}$, $L_{\alpha+1}$, $[B_{\alpha+1}]^2 \subseteq R$, Card $B_{\alpha+1} \leqq \kappa_{\alpha+1}$, $(B_{\alpha+1} \otimes L_{\alpha+1}) \subseteq R$, where $L_{\alpha+1} \subseteq L_\alpha$, $\mathrm{tp}(L_{\alpha+1}) = \varphi$. At limit stages $\lambda < cf\,\kappa$ it is shown that $B_\lambda = \bigcup_{\alpha<\lambda} B_\alpha$, $L_\lambda = \bigcap_{\alpha<\lambda} L_\alpha$ satisfy the hypotheses. Finally, we have that $L' = \bigcup_{\alpha<cf\,\kappa} B_\alpha$ has type φ and $[L']^2 \subseteq R$, as desired.

Case 2. *φ arbitrary*. We will only consider the case where $\varphi = \mathrm{tp}(L, l)$, the treetops of T are x, y, and accordingly, L is of the form $L_x \cup L_y$, where we assume Card $L_x = \kappa$, Card $L_y = \delta < \kappa$, and $[L_x]^2 \subseteq R$, $[L_y]^2 \subseteq R$. Lest there be an $(n \otimes \aleph_0) \subseteq S$, find $Z \subseteq L_x$, Card $Z = \delta$ $\big($whence $L_x' = (L_x - Z) \subseteq_a L_x\big)$ such that

$$\forall\, z \in L_x' \ \ \mathrm{Card}\,\{w \in L_y \colon \{z, w\} \in S\} \leqq n - 1 .$$

Write $L_y = \sum_{u \in M} M_u$, each M_u of type ω or ω^*, and let $L_y = \bigcup_{i \leqq n} L_y(i)$, where each $L_y(i) \cap M_u$ is infinite and $i \neq j \rightarrow L_y(i) \cap L_y(j) = \varnothing$. Obtain

$L_x'' \subseteq_c L_x$ such that for some i, $(L_x'' \otimes L_y(i)) \subseteq R$. $\mathrm{tp}(L_x'' \cup L_y(i)) \equiv \varphi$, as desired. This completes the sketch of the theorem; the remaining cases can be done by these methods, using Lemma 4.3 as in the proof of Theorem 4.4.

5. A result on dimension

If P is a partial order, the dimension of P, written $\dim(P)$, is defined to be the least cardinal κ such that P is isomorphically embeddable in the direct product of κ linear orderings. This notion was introduced in [1], where it is verified that $\dim(P) \leqq \mathrm{Card}(P)$. In [11] it is shown that $\dim(P)$ is the least κ such that $<_P$ can be written as the intersection of κ linear orders on P. Recall that $\mathcal{S}_{(<\kappa)}(\mathfrak{M}_{(<\kappa)})$ is the set of members of $\mathcal{S}(\mathfrak{M})$ which have power $< \kappa$. A question raised in [13] was whether there are wqo sets of uncountable dimension; the next theorem shows that there are bqo sets of arbitrary dimension (for another example of such sets, see [16]).

THEOREM 5.1. *If* $\kappa > \omega$ *then* $\dim(\mathcal{S}_{(<\kappa)/\equiv}) = \dim(\mathfrak{M}_{(<\kappa)/\equiv}) = \kappa$.

Proof. Let A be the AI members of $\mathcal{S}(<\kappa)/\equiv$. By Theorem 3.1, $\mathrm{Card}\,\mathfrak{M}_{(<\kappa)} = \kappa$, so it will suffice to show $\dim A \geqq \kappa$. If κ is a limit cardinal we are done by induction, so assume $\kappa = \gamma^+$. Suppose, by way of contradiction, that $A \underset{f}{\cong} B \subseteq \bigotimes_{\beta<\gamma} L_\beta$, each L_β linearly ordered. We may clearly assume that $x \in L_\beta \to \exists\ \varphi \in A, f(\varphi)_\beta = x$, and thus, since A is wqo, each L_β is well ordered. We have $A = \bigcup_{\alpha<\kappa} A_\alpha$, where, for λ a limit ordinal,

$A_{\lambda+2n+1}$ is the closure of $A_{\lambda+2n}$ under regular unbounded sums of length $< \kappa^*$,
$A_{\lambda+2n+2}$ is the closure of $A_{\lambda+2n+1}$ under regular unbounded sums of length $< \kappa$,

$$\text{and} \quad A_\lambda = \bigcup_{\alpha<\lambda} A_\alpha .$$

It can be proved by induction that

$$\alpha \leqq \delta,\ \ \theta \in A_\alpha,\ \ \psi \in (A_{\delta+1} - A_\delta) \longrightarrow \psi \not\leqq \theta .$$

If $\alpha < \kappa$, $\beta < \gamma$, let $P(\alpha, \beta)$ mean

$$\forall\ x \in L_\beta\ \exists\ \varphi \in A_\alpha \quad f(\varphi)_\beta \geq_{L_\beta} x .$$

Let $G_\alpha = \{\beta < \gamma\colon P(\alpha, \beta)\}$. Clearly $\alpha < \delta \to G_\alpha \subseteq G_\delta$, so there must be an $\alpha_0 < \kappa$ so that α_0 is a successor ordinal and

$$\forall\ \delta \geqq \alpha_0 \quad G(\delta) = G(\alpha_0) .$$

For $R \subseteq \gamma$ define $\varphi <_R \psi$ to mean that

$$\forall\ \beta \in R \quad f(\varphi)_\beta <_{L_\beta} f(\psi)_\beta .$$

Let $\alpha_1 = \alpha_0 + 1$. By diagonalizing over the set $\gamma - G_{\alpha_0}$ and using the closure of A under regular unbounded sums of length $\leqq \gamma$, it is seen that an $\alpha_2 > \alpha_1$

and a $\varphi \in A_{\alpha_2}$ can be found so that for each $\psi \in A_{\alpha_1}$,

$$\psi <_{(\gamma - G_{\alpha_0})} \varphi .$$

However, by the closure properties of A_{α_1} there will be a $\psi \in (A_{\alpha_1} - A_{\alpha_0})$ with $\psi \not\leq \varphi$ (this is proved by an inductive argument). Since $G_{\alpha_0} = G_{\alpha_1}$, the closure of A_{α_0} allows a $\theta \in A_{\alpha_0}$ to be found satisfying

$$\psi <_{G_{\alpha_0}} \theta$$

(and yet $\psi \not\leq \theta$). We must now have $\psi <_{\gamma} (\varphi + \theta)$, and so $\psi \leq (\varphi + \theta)$, but, since ψ is AI, $\psi \leq \varphi$ or $\psi \leq \theta$, a contradiction, completing the proof.

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