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COUNTING CANONICAL PARTITIONS IN THE RANDOM GRAPH

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Joyce trees have concrete realizations as J-trees of sequences of 0's and 1's. Algorithms are given for computing the number of minimal height J-trees of d-ary sequences with n leaves and the number of them with minimal parent passing numbers to obtain polynomials $\rho_n(d)$ for the full collection and $\alpha_n(d)$ for the subcollection.

The number of traditional Joyce trees is the tangent number $\alpha_n(1)$; $\alpha_n(2)$ is the number of cells in the canonical partition by Laflamme, Sauer and Vuksanovic of n-element subsets of the infinite random (Rado) graph; and $\rho_n(2)$ is the number of weak embedding types of rooted n-leaf J-trees of sequences of 0's and 1's.

1. Introduction

By definition, a Joyce tree is a finite rooted tree for which no two nodes have the same level, all levels up to that of the top leaf have a node (either a parent or a leaf), and every parent has exactly two children, which are ordered: left and right. They were named *Joyce trees* by Ross Street [13] after the physicist William P. Joyce who was using them in his calculations.

One way to represent Joyce trees concretely is to use a collection of sequences of 0's and 1's as the nodes and to use the length of the sequence to indicate the level on which a node is placed. The children of a node s are extensions of $s^\frown\langle 0\rangle$ and $s^\frown\langle 1\rangle$ ordered lexicographically. We generalize this approach below. As a preliminary, by an abuse of notation, we write

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$d = \{0, 1, \ldots, d-1\}$. Note that a d-ary sequence is one whose entries are from the set d.

Definition 1.1. A *J-tree* is a finite rooted tree T of sequences of non-negative integers with the properties that different nodes have different lengths (as sequences); t is a child of s in T if it is a minimal proper extension of s in T; each parent node has exactly two children; and each parent node is the longest initial segment common to its two children. A J-tree with n-leaves has *minimal height* if the lengths of the nodes are $0, 1, 2, \ldots, 2n-2$. For $d > 1$, let $\mathcal{J}(n, d)$ denote the set of n-leaf J-trees of minimal height whose nodes are d-ary sequences.

In Theorem 6.5, we give an expression for counting the number $\rho_n(d) = |\mathcal{J}(n, d)|$ of minimal height J-trees with n leaves all of whose nodes are d-ary sequences. As a corollary, we derive the fact that the number of weak embedding types of n-leaf J-trees of d-ary sequences is also $\rho_n(d)$. Weakly embedded trees were introduced by Milliken [6], who proved a Ramsey Theorem for them which has been applied widely. The expression for $\rho_n(d)$ is translated into Maple code for easy computation.

Given two nodes x and y of a J-tree, if x is shorter than y, in symbols, $k = |x| < |y| = \ell$, then the *passing number of* $y = \langle y_0, y_1, \ldots, y_{\ell-1}\rangle$ *over* x is $y(|x|) = y_k$. A J-tree T has *minimal parent passing numbers* if for all parents $s \in T$, the two children of s are extensions of $s^\frown\langle 0\rangle$ and $s^\frown\langle 1\rangle$ and if whenever $t = \langle t_0, t_1, \ldots, t_{n-1}\rangle \in T$ is longer than $s = \langle s_0, t_1, \ldots, s_{i-1}\rangle \in T$ and t does not extend s, then also $t(|s|) = t_i = 0$.

In Theorem 6.1, we give an expression for the number $\alpha_n(d)$ of J-trees $T \in \mathcal{J}(n, d)$ such that T has minimal parent passing numbers. This expression is translated into Maple code for ease of computation.

The algorithm for counting $\alpha_n(d)$ can be used to count the number of cells in a canonical partition of the n-element subsets of universal purely binary relational structures. This application is discussed in the final section of the paper, where the notion of canonical partition is defined and universal purely binary relational structures are discussed.

Here is a brief description of the ingredients of the proofs of the main theorems. We associate with each $(n+1)$-leaf J-tree T a *parent indicator sequence* σ_T and show (see Section 5) they coincide with Raney sequences. We identify the critical subcollection $\mathcal{J}(n+1, 1) \subseteq \mathcal{J}(n+1, 2)$ of $(n+1)$-leaf J-trees with *minimal passing numbers* (they are the concrete realizations of the traditional Joyce trees), and introduce some useful equivalence relations in Section 2, including *shape similarity*. We compute the cardinality of $\mathcal{J}(n+1, d)$ by summing over all Raney sequences σ the product of the number of $U \in \mathcal{J}(n+1,1)$ with $\sigma_U = \sigma$ times the number of $T \in \mathcal{J}(n+1,d)$ shape similar to U. The computation of the number of J-trees with minimal parent passing numbers in $\mathcal{J}(n+1,d)$ is a modification of the larger computation. See Sections 3, 4, 6 for details.

This remainder of this section collects basic notation and definitions in one place for the convenience of the reader. Some of the definitions are repeated for consistency. Each ordinal is the set of its predecessors, so, in particular, $0 = \emptyset$, $1 = \{0\}$, $n = \{0,1,\dots,n-1\}$, and $\omega = \{0,1,\dots\}$ is the set of non-negative integers. Denote by ${}^{\omega>}d$ the set of all finite sequences $s = \langle s_0, s_1, \dots, s_{n-1}\rangle$ with entries from $d = \{0,1,\dots,d-1\}$. For any J-tree T, the transitive closure of the parent-child relation is the relation of end-extension, $\subset$, and $(T,\subset)$ is the set theoretic version of the tree. Note that the parent-child relation can be recovered from the relation of end-extension, since parents are nodes with proper extensions in T and their children are their minimal proper extensions in T. When partially ordered by end-extension, $({}^{\omega>}d,\subset)$ is the regular d-ary set theoretic tree with root the empty sequence, $\emptyset$. We write ${}^{(2n)\geq}d$ for the subtree of ${}^{\omega>}d$ of sequences of length at most $2n$. The *meet*, $s \wedge t$, of $s,t \in {}^{\omega>}d$ is the longest sequence which is an initial segment of both s and t. We write $|s| = m$ to indicate that the length of the sequence s is m. We freely move between functional notation for sequences and writing them explicitly as an ordered list, such as $\langle +1,+1,-1\rangle$. We write $s^\frown t$ for the concatenation of s and t.

2. Similarity

The equivalence relations of *similarity*, *shape similarity* and *having the same week embedding type* are introduced for J-trees.

Definition 2.1. Two J-trees $S,T \subseteq {}^{\omega>}d$ are *shape similar* if there is a *shape similarity* $f : S \to T$, that is a bijection which preserves the parent-child relationship, the lexicographic order of the leaves and the length order of the nodes. The function f is a *similarity* and S and T are *similar* if f satisfies

- (leaf passing number preservation) for all leaves $x,z \in S$, if $|z| > |x|$, then $z(|x|) = f(z)(|f(x)|)$.

The function f is a *weak embedding* and S and T *have the same weak embedding type* if f satisfies

- (passing number preservation) for all nodes $x,z \in S$, if $|z| > |x|$, then $z(|x|) = f(z)(|f(x)|)$.

Note that a weak embedding is a similarity in addition to being a shape similarity. The definition of weak embedding is based on one introduced by Milliken [6] and made more specific by Vuksanovic [14] by use of the lexicographic order. The definition of similarity for J-trees parallels that of Laflamme, Sauer and Vuksanovic (see pages 188–189 of [5]) of similarity for diagonal subsets of ${}^{\omega>}\omega$.

Next consider a notion of collapse for J-trees.

Definition 2.2. For any sequence $s : q \to 2$ any strictly increasing function $h : p \to q$, define the *collapse of s with respect to h* by $\mathrm{clp}_h(s) = s \circ h$, i.e., $\mathrm{clp}_h(s)(k) = s(h(k))$ for all $k < p$.

If $S \subseteq {}^{\omega>}d$ is an n-leaf J-tree whose elements are listed in increasing order as $b_0, b_1, \ldots, b_{2n-2}$ and $I : 2n-1 \to \omega$ is the function $i \mapsto |b_i|$, then $\mathrm{clp}(S) = \{\mathrm{clp}_I(b) : b \in S\}$ is the *collapse of S*.

Lemma 2.3. *If $S \subseteq {}^{\omega>}d$ is an n-leaf J-tree, then S is both similar and shape similar to* $\mathrm{clp}(S)$ *and S and* $\mathrm{clp}(S)$ *have the same weak embedding type.*

Proof. Let $d_0, d_1, \ldots, d_{2n-2}$ enumerate the elements of S, an n-leaf J-tree in increasing order of length, and let $I : (2n-1) \to \omega$ be defined by $I(j) = |d_j|$. Then $\mathrm{clp}(d_0) = \emptyset$. By recursion, $|\mathrm{clp}(d_j)| = j$. Thus for $i < j$, $\mathrm{clp}(d_j)(|\mathrm{clp}(d_i)|) = \mathrm{clp}(d_j)(i) = d_j(|d_i|)$. Therefore, the function $\mathrm{clp}_I : S \to \mathrm{clp}(S)$ is a similarity, a shape similarity and a weak embedding. ∎

Lemma 2.4. *For any n, d with $2 \leq n, d < \omega$, the relations of shape similarity, similarity and having the same weak embedding type are equivalence relations on n-leaf J-trees of ${}^{\omega>}d$, and each has finitely many equivalence classes.*

Proof. To see that these are equivalence relations, first check that the identity is simultaneously a shape similarity, a similarity and a weak embedding. Next note that inverses of shape similarities are shape similarities, and the same is true for similarities and weak embedding, and all three types of mappings are closed under composition. To see that there are only finitely many shape similarity classes, only finitely many similarity classes, and only finitely many weak embedding types, observe that every J-tree with $n \geq 2$ leaves is shape similar and similar to one which is a subtree of the d-ary tree of sequences of length at most $2n-2$, by Lemma 2.3. ∎

In the introduction we defined J-trees with minimal parent passing numbers. Here is another interesting collection.

Definition 2.5. A J-tree $S \subseteq {}^{\omega>}d$ has *minimal passing numbers* if for all $s, t \in S$,

1. if $s \subsetneq t$, then $t(|s|) < 2$; and
2. if $s \not\subseteq t$ and $|s| < |t|$, then $t(|s|) = 0$.

Let $\mathcal{J}(n,1)$ denote the set of $T \in \mathcal{J}(n,2)$ with minimal passing numbers.

Note that J-trees with minimal passing numbers have collapses which are subsets of ${}^{\omega>}2$.

Lemma 2.6. *Suppose S and T are shape similar J-trees. Then $\mathrm{clp}(S) = \mathrm{clp}(T)$ if one of the following conditions holds:*

- *S and T have the same weak embedding type;*
- *S and T are similar and have minimal parent passing numbers; or*
- *S and T are shape similar and have minimal passing numbers.*

Proof. Let $s_0, s_1, \dots, s_{2n-2}$ enumerate S, and $t_0, t_1, \dots, t_{2n-2}$ enumerate T, both in increasing order of length. Let $g : S \to T$ be the map defined by $g(s_i) = t_i$, the unique length order preserving map. Since S and T are shape similar, the function g is a shape similarity.

In the first item, the hypothesis says that g is a weak embedding, so it preserves all passing numbers. In the second item, g is a similarity, so it preserves all leaf passing numbers, and S and T are assumed to have minimal parent passing numbers, so g preserves parent passing numbers as well. In the third item, S and T both have minimal passing numbers, so g again preserves all passing numbers.

Thus g is a weak embedding between S and T.

Let $I, J : (2n-1) \to \omega$ be defined by $I(j) = |s_j|$ and $J(k) = |t_k|$. Note that $\mathrm{clp}_I(s_0) = \emptyset = \mathrm{clp}_J(t_0)$ has length 0 and for $\ell > 0$, $\mathrm{clp}_I(s_\ell)$ and $\mathrm{clp}_J(t_\ell)$ have length ℓ. Moreover, for $\ell > 0$, every value of the form $\mathrm{clp}_I(s_\ell)(k)$ or $\mathrm{clp}_j(t_\ell)(k)$ is a passing number. Since g preserves passing numbers, it follows that for all $\ell > 0$, if $k < \ell$, then $\mathrm{clp}_I(s_\ell)(k) = \mathrm{clp}_j(t_\ell)(k)$, so $\mathrm{clp}_I(s_\ell) = \mathrm{clp}_J(t_\ell)$. Hence $\mathrm{clp}_I(S) = \mathrm{clp}_J(T)$. ∎

Lemma 2.7. *Let R be an n-leaf J-tree.*

1. *R has the same weak embedding type as a unique tree $S = \mathrm{clp}(R)$ in $\mathcal{J}(n,d)$;*
2. *R is similar to a unique tree $T \in \mathcal{J}(n,d)$ with minimal parent passing numbers; and*
3. *R is shape similar to a unique tree $U \in \mathcal{J}(n,1)$.*

Proof. Since any tree in $\mathcal{J}(n,d)$ is its own collapse, the uniqueness in all three cases follows from Lemma 2.6.

Existence for first item follows from Lemma 2.3.

To prove existence in the other two items, assume, without loss of generality, that $\mathrm{clp}(R) = R$. Enumerate R in increasing order of length as $r_0, r_1, \ldots, r_{2n-2}$. Let $s_0 = \emptyset = t_0$. For positive $\ell < n$ Let t_ℓ and u_ℓ for be the sequences of length $\ell = |r_\ell|$ such that for all $k < \ell$,

$$t_\ell(k) = \begin{cases} r_\ell(k), & \text{if } r_k \text{ is a leaf,} \\ 1, & \text{if } r_\ell \text{ is the right child of } r_k, \\ 0, & \text{otherwise;} \end{cases}$$

and $u_\ell(k) = 1$ if r_ℓ is the right child of r_k and otherwise $u_\ell(k) = 0$. Let $T = \{t_\ell : r_\ell \in R\}$ and $U = \{u_\ell : r_\ell \in R\}$. Note that T is in $\mathcal{J}(n,d)$ and U is in $\mathcal{J}(n,2)$, since $R = \mathrm{clp}(R)$ is in $\mathcal{J}(n,d)$. By construction, S has minimal parent passing numbers and U and has minimal passing numbers. The reader may check $r_\ell \mapsto t_\ell$ is a similarity and $r_\ell \mapsto u_\ell$ is a shape similarity. ∎

3. Alternating permutations

The goal of this section is to count the number of J-trees in $\mathcal{J}(n+1,1)$ which have a specified *parent indicator sequence*. A key tool is a bijection between $\mathcal{J}(n+1,1)$ and the set of *alternating permutations* on the set $\{0,1,\ldots,2n\}$.

Definition 3.1. An *alternating permutation on* $\{0,1,\ldots,2n\}$ is a permutation $\pi = \langle p_0, p_1, \ldots, p_{2n}\rangle$ with the properties that $\{p_0, p_1, \ldots, 2n\}$ and $p_0 > p_1 < p_2 > \cdots < p_{2n}$.

Lemma 3.2. *For each n, define A_n $\mathcal{J}(n+1,1)$ by*

$$A_n(U) := \langle |\ell_0|, |\ell_0 \wedge \ell_1|, |\ell_1|, \ldots, |\ell_{2n-1} \wedge \ell_{2n}|, |\ell_{2n}|\rangle,$$

where $\ell_0, \ell_1, \ldots, \ell_{2n}$ lists the leaves of U in increasing lexicographic order. For each n, A_n is a bijection onto the set of alternating permutations on the set $\{0,1,\ldots,2n\}$.

Proof. Fix n. Note that A_n maps into the alternating permutations on $\{0,1,\ldots,2n\}$ since a meet is always smaller than the leaves of which it is a meet, and the values listed in $A_n(U)$ are the distinct lengths of the elements of U, i.e., the values in $\{0,1,\ldots,2n\}$.

To show that A_n is an injection, suppose that $A_n(U) = A_n(V)$. Let $g : U \to V$ be the unique length order preserving map. Since $A_n(U) = A_n(V)$, it follows that g preserves the lexicographic order of leaves and the parent-child relationship. Thus g is a shape similarity. By Lemma 2.6, $U = V$. Thus A_n is an injection.

To show that A_n is a surjection, fix an alternating permutation $\pi = \langle p_0, p_1, \ldots, p_{2n}\rangle$. Use recursion to define $t_0, t_1, \ldots, t_{2n}$ as follows. Let t_0 be the sequence of 0's of length p_0. If t_i has been defined for some even $i < 2n$, then t_{i+1} is the initial segment of t_i of length p_i. If t_j has been defined for some odd j, then t_{j+1} is the sequence of length p_{j+1} obtained by the extension by zeros of $t_j{}^\frown\langle 1\rangle$. Let $U = \{t_0, t_1, \ldots, t_{2n}\}$. Then U is a tree under extension. By construction, each t_i has length p_i; for even indices i, the sequence t_i is a leaf of U, while for odd indices j, the sequence t_j is a parent of U with exactly two children, t_{j-1} and t_{j+1}. Thus U is in $\mathcal{J}(n+1, 2)$. Furthermore, U has minimal passing numbers, since it is a subset of ${}^{2n\geq}2$ and if $t(p_j) = 1$ for some j, then j is odd, and $t_j \subseteq t$. Thus for every alternating permutation π, there is some $U \in \mathcal{J}(n+1, 1)$ for which $A_n(U) = \pi$. It follows that A_n is a surjection, hence a bijection. ∎

For the purposes of counting, it will be useful to look at collections of *convex segments* of a sequence.

Definition 3.3. A *convex segment of a sequence* π is a sequence ρ for which there are ρ_0 and ρ_1 (possibly empty) for which $\pi = \rho_0{}^\frown\rho^\frown\rho_1$. Given an alternating permutation π on $\{0, 1, \ldots, 2n\}$, let $P(\pi) = \langle P_1, P_2, \ldots, P_{2n+1}\rangle$ be the sequence in which P_i is the set of all maximal convex segments of π all of whose members are from $\{2n, 2n-1, \ldots, 2n-i+1\}$.

In the definition of $P(\pi)$, we have $P_1 = \{\langle 2n\rangle\}$, $P_2 = \{\langle 2n\rangle, \langle 2n-1\rangle\}$, and $P_{2n+1} = \{\pi\}$, so $P(\pi)$ and π are definable one from the other.

Definition 3.4. Suppose that $S \subseteq {}^{\omega>}d$ is an $(n{+}1)$-leaf J-tree enumerated in decreasing length order as $x_0, \ldots, x_{2n}$. The *parent indicator sequence of* S, σ_S, is the sequence of $+1$'s and -1's of length $2n+1$ defined by $\sigma_S(j) := -1$ if and only if x_j is a parent.

Remark 3.4.1. By an abuse of notation, we say an alternating permutation π has parent indicator sequence σ if the corresponding $U \in \mathcal{J}(n+1, 1)$ has $\sigma_U = \sigma$. Without constructing U, it is easy to identify σ, since for all $k \leq 2n$, $\sigma(k) = -1$ if and only if $2n{-}k$ appears between larger neighbors in π.

Definition 3.5. Given any $\sigma : 2n-1 \to \{-1, +1\}$, define the *tally sequence of* σ $\tau_\sigma : (2n-1) \to \omega$ by $\tau_\sigma(0) = 0$ and for positive j, $\tau_\sigma(j) := \sum_{i<j} \sigma(j)$.

Lemma 3.6. *If π is an alternating permutation on $\{0, 1, \ldots, 2n\}$ with parent indicator sequence σ and $P(\pi) = \langle P_1, P_2, \ldots, P_{2n+1}\rangle$, then $|P_j| = \tau_\sigma(j)$.*

Proof. Use induction on j. For $j = 1$, $|P_1| = 1 = \sigma(0) = \tau_\sigma(1)$. If $\sigma(j) = +1$ and $|P_j| = \tau_\sigma(j)$, then $P_{j+1} = P_j \cup \{\langle 2n{-}j\rangle\}$ and $|P_{j+1}| = \tau_\sigma(j){+}1 = \sum_{i<j+1} \sigma(i) =$

$\tau_\sigma(j+1)$. If $\sigma(j)=-1$ and $|P_j|=\tau_\sigma(j)$, then $P_{j+1}=(P_j\setminus\{L_{j+1},R_{j+1}\})\cup\{L_{j+1}{}^\frown\langle 2n-j\rangle{}^\frown R_{j+1}\}$ for some $L_{j+1},R_{j+1}\in P_j$, so $|P_{j+1}|=\tau_\sigma(j)-2+1=\sum_{i<j}\sigma(i)=1=\tau_\sigma(j+1)$. ∎

Corollary 3.7. *Suppose σ is the parent indicator sequence of some element of $\mathcal{J}(n+1,1)$. The number of $U\in\mathcal{J}(n+1,1)$ for which $\sigma_U=\sigma$ is*

$$\prod_{\substack{j<2n\\ \sigma(j)<0}}\left(\sum_{i<j}\sigma(i)\right)\left(-1+\sum_{i<j}\sigma(i)\right)=\prod_{\substack{j<2n\\ \sigma(j)<0}}\tau_\sigma(j)\left(\tau_\sigma(j)-1\right).$$

Proof. It suffices to compute the cardinality of the set $Q(\sigma)$ of sequences $P(\pi)$ for π an alternating permutation with parent indicator sequence σ.

Claim 3.7.1. *For all $j\leq 2n$, the size of the set $Q_j(\sigma)$ of initial segments of length $j+1$ of elements of $Q(\sigma)$ is $\prod_{i<j}q(i)$, where $q(i)=1$ if $\sigma(i)=+1$ and $q(i)=\tau_\sigma(i)\left(\tau_\sigma(i)-1\right)$ if $\sigma(i)=-1$.*

Proof. Use induction on j. Since all sequences in $Q(\sigma)$ start with $\langle\{\langle 2n\rangle\}\rangle$, $|Q_0(\sigma)|=1=q(0)$, as required.

Next suppose $\sigma(j)=+1$ and $|Q_j(\sigma)|=\prod_{i<j}q(i)$. Then $q(j)=1$. Since for any sequence $P(\pi)$ of $Q(\sigma)$, $P_{j+1}=P_j\cup\{\langle 2n-j\rangle\}$, it follows that $|Q_{j+1}(\sigma)|=|Q_j(\sigma)|=\prod_{i<j}q(i)=\prod_{i<j+1}q(i)$, and the claim holds in this case.

Finally suppose $\sigma(j)=-1$ and $|Q_j(\sigma)|=\prod_{i<j}q(i)$. Note that, under these circumstances, $q(j)=\tau_\sigma(j)\left(\tau_\sigma(j)-1\right)$. For each initial segment $\langle P_1,P_2,\dots,P_j\rangle$ in $Q_j(\sigma)$, the set P_j has cardinality $\tau_\sigma(j)$ so there are $q(j)$ many ways to select $L_{j+1},R_{j+1}\in P_j$ and extend the sequence by adding the set $(P_j\setminus\{L_{j+1},R_{j+1}\})\cup\{L_{j+1}{}^\frown\langle 2n-j\rangle{}^\frown R_{j+1}\}$. It follows that $|Q_{j+1}(\sigma)|=|Q_j(\sigma)|\cdot q(j)=\prod_{i<j+1}q(i)$, and the claim holds in this case as well. ∎

The corollary follows from the claim since

$$|Q(\sigma)|=|Q_{2n}(\sigma)|=\prod_{i<2n}q(i)=\prod_{\substack{j<2n\\ \sigma(j)<0}}\tau_\sigma(j)\left(\tau_\sigma(j)-1\right).\qquad ∎$$

4. Local computations

For this section, fix attention on some $U\in\mathcal{J}(n+1,1)$. The goal is to compute, for any $d\geq 2$, both the number of trees $T\in\mathcal{J}(n+1,d)$ which are shape similar to U and the smaller number of them which also have minimal parent passing numbers.

The next definition is useful in these computations.

Definition 4.1. For $T \in \mathcal{J}(n+1,d)$, let $\overline{T} = \{s \in {}^{2n\geq}2 : (\exists t \in T)(s \subseteq t)\}$ be the closure of T under initial segments. For each $\ell \leq 2n$, let $\overline{T}_n = \{s \in \overline{T} : |s| = \ell\}$ be the set of elements on *level* ℓ.

The next lemma follows straightforwardly from the definitions and the fact that T is the set of leaves of $\overline{T}$ together with the parents in $\overline{T}$ of exactly two children.

Lemma 4.2. *Suppose* $T \in \mathcal{J}(n+1,d)$.

- T *may be defined from* $\overline{T}$.
- *Suppose* U *is shape similar to* T *via* $g : U \to T$. *Let* $\overline{g} : \overline{U} \to \overline{T}$ *be defined by* $\overline{g}(s) = g(u) \restriction |s|$ *where* $u \in U$ *is the shortest element with* $s \subseteq u$. *Then* $\overline{g}$ *is a bijection which preserves length (level), extension, the parent-child relationship, and the lexicographic order of leaves.*

Lemma 4.3. *Suppose* $T \in \mathcal{J}(n+1,d)$ *has* $\sigma_T = \sigma$. *For all* $\ell \leq 2n$, $|\overline{T}_{2n+1-\ell}| = \tau_\sigma(\ell)$.

Proof. Use induction on $j \leq 2n+1$ to show that $|\overline{T}_{2n+1-j}| = \tau_\sigma(j)$. To start the induction with $j=0$, note that $\overline{T}_{2n+1-0}$ is empty, so $|\overline{T}_{2n+1-0}| = 0$.

For the induction step, let $j < 2n+1$ be given and assume $w = \tau_\sigma(j)$. Let $y_0, y_1, \ldots, y_{2n}$ enumerate the nodes of T in decreasing length order. The node y_j has length $2n-j = (2n+1)-(j+1)$. To see that $|\overline{T}_{(2n+1)-(j+1)}| = |\overline{T}_{2n+1-j}| + \sigma(j)$, observe that if $\sigma(j) = +1$ and y_j is a leaf, then y_j is the unique element of $\overline{T}_{(2n+1)-(j+1)}$ which is not the restriction of a member of $\overline{T}_{2n+1-j}$, while if $\sigma(j) = -1$ and y_j is a parent, then y_j is the restriction of both of its children in $\overline{T}_{2n+1-j}$. Thus $|\overline{T}_{(2n+1)-(j+1)}| = |\overline{T}_{2n+1-j}| + \sigma(j) = \tau_\sigma(j) + \sigma(j) = \tau_\sigma(j+1)$. ∎

Corollary 4.4. *Suppose that* $T \in \mathcal{J}(n+1,d)$ *and* $\sigma_T = \sigma$. *For positive* $k \leq 2n+1$, *the partial sum* $\sum_{j<k} \sigma_T(j)$ *is positive, and* $\sum_{j<2n+1} \sigma_T(j) = 1$.

Proof. The value of $\sum_{j<\ell} \sigma_T(j)$, for positive $\ell \leq 2n$, is $\tau_\sigma(\ell)$, by definition of τ_S. By the previous lemma, the value of $\tau_\sigma(\ell)$ is also $|\overline{T}_{2n_1-\ell}|$. Since $\overline{T}_{2n_1-\ell}$ is non-empty, for positive $\ell \leq 2n$, it follows that the proper partial sums are all positive. Since σ_S is a sequence of $n+1$ $(+1)$'s and n (-1)'s, the sum of the entire sequence is $+1$, as desired. ∎

Notation. For each $T \in \mathcal{J}(n+1,d)$ which is shape similar U, let $g_T : U \to T$ denote a shape similarity and let $h_T : \overline{U} \setminus \{\emptyset\} \to d$ be defined by $h_T(s) = \overline{g}_T(|s|-1)$.

Lemma 4.5. *The mapping $T \mapsto h_T$ defined on the set of all $T \in \mathcal{J}(n+1,d)$ shape similar to U is a bijection onto the set of functions $h : \overline{U} \setminus \{\emptyset\} \to d$ which* preserve the lexicographic order of children, *i.e., the set of those h such that for all $s \in \overline{U}$ with two children, $h(s^\frown\langle 0\rangle) < h(s^\frown\langle 1\rangle)$.*

Proof. Since shape similarity preserves the lexicographic order of leaves, for any T which is shape similar to U, the mapping h_T preserves the lexicographic order of children. Thus the mapping is well-defined.

To see that the mapping is one-to-one, suppose S and T are both shape similar to U, and $h_S = h_T = h$. Define f^* on $\overline{U}$ by recursion on the levels. Let $f^*(\emptyset) = \emptyset$. If $f^*(s)$ has been defined and $s^\frown\langle\delta\rangle \in \overline{U}$, then $f^*(s^\frown\langle\delta\rangle) = f^*(s)^\frown\langle h(s^\frown\langle\delta\rangle)\rangle$. Use induction on the levels to show that $f^* = \overline{g}_T = \overline{g}_S$. To start the induction, $f^*(\emptyset) = \emptyset = \overline{g}_T(\emptyset) = \overline{g}_S(\emptyset)$. Suppose $s^\frown\langle\delta\rangle \in \overline{U}$. Further suppose that $f^*(s) = \overline{g}_T(s) = \overline{g}_S(s)$. Then

$$\begin{aligned} f^*(s^\frown\langle\delta\rangle) &= f^*(s)^\frown\langle h(s^\frown\langle\delta\rangle)\rangle \\ &= \overline{g}_T(s)^\frown\langle h_T(s^\frown\langle\delta\rangle)\rangle \\ &= \overline{g}_T(s^\frown\langle\delta\rangle) \\ &= \overline{g}_S(s)^\frown\langle h_S(s^\frown\langle\delta\rangle)\rangle \\ &= \overline{g}_S(s^\frown\langle\delta\rangle). \end{aligned}$$

Thus by induction, $f^* = \overline{g}_T = \overline{g}_S$ and $\overline{S} = \overline{T}$, so $S = T$.

To see that the mapping is onto, suppose $h : \overline{U} \to d$ preserves the lexicographic order of children. Define f^* on $\overline{U}$ as in the previous paragraph. By definition, f^* preserves extension and hence the parent-child relationship. Moreover, for all $s \in \overline{U}$, the length of $f^*(s)$ is the same as the length of s. Since h preserves the lexicographic order of children, it follows that f^* preserves the lexicographic order of leaves. Hence the restriction of f^* to domain U and codomain $f^*[U]$ is a shape similarity. It follows that $T = f^*[U]$ is in $\mathcal{J}(n+1,d)$ and $h = h_T$. ∎

Lemma 4.6. *Suppose $U \in \mathcal{J}(n+1,1)$ and $\sigma_U = \sigma$.*

1. *The number of $T \in \mathcal{J}(n+1,d)$ shape similar to U is $d^{R(\sigma)}\left[\frac{d(d-1)}{2}\right]^n$ where $R(\sigma) := -2n + \sum_{j<2n} \tau_\sigma(j)$.*
2. *The number of $T \in \mathcal{J}(n+1,d)$ with minimal parent passing numbers which are shape similar to U is $d^{Q(\sigma)}$ for $Q(\sigma) = \sum_{0<j<2n \wedge \sigma(j)=1} \tau_\sigma(j)$.*

Proof. For the first item, by Lemma 4.5, it suffices to show the number of functions $h:\overline{U}\setminus\{\emptyset\}\to d$ which preserve the lexicographical order of children has the specified value.

The set $\overline{U}\setminus\{\emptyset\}$, by Lemma 4.3, has size $\sum_{j\leq 2n}\tau_\sigma(j)$. There are n nodes of $\overline{U}$ which are parents with exactly two children. There are $[d(d-1)/2]^n$ ways to chose the values of a function defined on the set of those children which preserves the lexicographic order of children. The other values may be freely chosen from d for a total of $d^{R(\sigma)}\,[d(d-1)/2]^n$ functions $h:\overline{U}\setminus\{\emptyset\}\to d$ which preserve the lexicographic order of children.

Now consider the second item. Enumerate the elements of U in decreasing order of length as $y_0, y_1, \ldots, y_{2n}$. If T is shape similar to U and has minimal parent passing numbers, then for all $j\leq 2n$, if $\sigma(j)=-1$, then y_j is the parent of exactly two children, and the restriction of $\overline{h}_T$ to $\overline{U}_{2n+1-j}$ is determined, namely $\overline{h}_T(z)=0=\overline{h}_U(z)$ if z is not the left child of y_j and $\overline{h}_T(z)=1=\overline{h}_U(z)$ if z is the left child of y_j. By Lemma 4.5, it suffices to show that there are $d^{Q(\sigma)}$ many functions $h:\overline{U}\setminus\{\emptyset\}\to d$ which preserve the lexicographical order of children and agree with $\overline{h}_U$ on $\overline{U}_{2n+1-j}$ for all $j\leq 2n$ with $\sigma(j)=-1$. By Lemma 4.3, there are $Q(\sigma)$ remaining elements of $\overline{U}\setminus\{\emptyset\}$. Since the other values may be freely chose, the second item follows as well. ∎

5. Raney sequences

In Section 4, for a fixed $U\in\mathcal{J}(n+1,2)$ with minimal passing numbers, the size of the set of T shape similar to U was computed as an expression in the tally sequence τ_σ for the parent indicator sequence $\sigma=\sigma_U$. An expression in τ_σ was also found for the size of the set of those T shape similar to U which have minimal parent passing numbers. The sequences σ that occur in this fashion have been studied (see *Concrete Mathematics* [4] pages 345–347). They are closely related to the *ballot sequences* discussed in *Enumerative Combinatorics, volume 2* [12] (see page 173 for a definition).

Definition 5.1. A sequence $\sigma:(2n+1)\to\{-1,+1\}$ is a 2-Raney sequence of length $2n+1$ if all of its partial sums are positive and its total sum is $+1$. Let $\mathcal{R}(n)$ denote the set of 2-Raney sequences of length $2n+1$.

Lemma 5.2. *For all $n<\omega$, the number of sequences in $\mathcal{R}(n)$ is a Catalan number:*

$$|\mathcal{R}(n)| = C(n) = \binom{2n}{n}\frac{1}{n+1}.$$

Proof. George Raney [7] showed in 1959 that if $\langle x_0, x_1, \ldots, x_m\rangle$ is any sequence of integers whose sum is $+1$, then exactly one of the cyclic shifts

$$\langle x_0, x_1, \ldots, x_m\rangle,\ \langle x_1, x_3, \ldots, x_m, x_0\rangle,\ \ldots,\ \langle x_m, x_0, \ldots, x_{m-1}\rangle$$

has all of its partial sums positive. In *Concrete Mathematics*, Graham, Knuth and Patashnik [4] (see pages 345–6) show that the number of these sequences is the Catalan number above. ∎

Proposition 5.3. *For any positive $n<\omega$ and any $\sigma:(2n+1)\to\{-1,1\}$, there is $U\in\mathcal{J}(n+1,1)$ such that $\sigma_U=\sigma$ if and only if $\sigma\in\mathcal{R}(n)$.*

Proof. For the first direction, note that by Definitions 1.1, 3.4 and Corollary 4.4, every parent indicator sequence for a J-tree $T\in\mathcal{J}(n+1,d)$ is a 2-Raney sequence in $\mathcal{R}(n)$.

For the other direction, suppose $\sigma\in\mathcal{R}(n)$. Since the sum of the entire sequence is $+1$, there are $n+1$ indices for which σ takes the value $+1$ and n indices for which σ takes the value -1. Let $i_0, i_1, \ldots, i_{n-1}$ list the indices with $\sigma(i)=-1$ in increasing order, and let $k_0, k_1, \ldots, k_n$ list the indices with $\sigma(k)=+1$.

Claim 5.3.1. *The following sequence π_σ is an alternating permutation on $\{0,1,\ldots,2n\}$ with parent indicator sequence σ:*

$$\pi_\sigma := \langle 2n-k_0, 2n-i_0, 2n-k_1, 2n-i_1, \ldots, 2n-k_{n-1}, 2n-i_{n-1}, 2n-k_n\rangle.$$

Proof. Note that the sequences $2n-i_0, 2n-i_1, \ldots, 2n-i_{n-1}$ and $2n-k_0, 2n-k_1, \ldots, 2n-k_n$ are decreasing and every value from 0 to $2n$ occurs in one or the other list, but not both. Thus π_σ is a permutation on $\{0,1,\ldots,n\}$.

Since all partial sums $\tau_\sigma(\ell)=\sum_{j<\ell}\sigma(j)$ are positive for positive $\ell\le 2n$, $k_0=0$ and $k_1=1$, so $2n-k_0>2n-i_0<2n-k_1$. Moreover, $i_{n-1}=2n$, so $2n-k_{n-1}>0=2n-i_0<2n-k_n$. Also $k_{j+1}<i_j$ for all $j<n-1$, else for the least ℓ with $k_{\ell+1}>i_\ell$ one has $k_\ell<i_{\ell-1}<i_\ell<k_{\ell+1}$ which leads to the contradiction that $\sum_{i\le i_\ell}\sigma(i)=0$. Since $k_j<k_{j+1}<i_j$ for $j<n-1$, it follows that $2n-k_j>2n-i_j<2n-k_{j+1}$ for all $j<n-1$. Thus π_σ is an alternating permutation whose parent indicator is σ. ∎

By Lemma 3.2, there is $U\in\mathcal{J}(n+1,1)$ such that $A_n(U)=\pi_\sigma$. By Remark 3.4.1, the parent indicator sequence of U is σ. ∎

Lemma 5.4. *Suppose $\sigma\in\mathcal{R}(n)$ for some positive n.*

1. *Let $R(\sigma):=-2n+\sum_{j<2n}\tau_\sigma(j)$. The number of $T\in\mathcal{J}(n+1,d)$ with $\sigma_T=\sigma$ is $d^{R(\sigma)}\left[\frac{d(d-1)}{2}\right]^n\prod_{j<2n\wedge\sigma(j)<0}\tau_\sigma(j)\,(\tau_\sigma(j)-1)$.*

2. *Let* $Q(\sigma) = \sum_{0<j<2n \wedge \sigma(j)=1} \tau_\sigma(j)$. *The number of* $T \in \mathcal{J}(n+1,d)$ *with minimal parent passing numbers is* $d^{Q(\sigma)} \prod_{j<2n \wedge \sigma(j)<0} \tau_\sigma(j)\,(\tau_\sigma(j)-1)$.

Proof. By Proposition 5.3, σ is the parent indicator sequence of some element of $\mathcal{J}(n+1,1)$. By Corollary 3.7, the number of $U \in \mathcal{J}(n+1,1)$ which have $\sigma_U = \sigma$ is $\prod_{j<2n \wedge \sigma(j)<0} \tau_\sigma(j)\,(\tau_\sigma(j)-1)$. By Lemma 2.7, every $T \in \mathcal{J}(n+1,d)$ is shape similar to a unique tree $U \in \mathcal{J}(n+1,1)$. By Lemma 4.6, for each $U \in \mathcal{J}(n+1,1)$ with $\sigma_U = \sigma$, the number of $T \in \mathcal{J}(n+1,d)$ which are shape similar to U is $d^{R(\sigma)}\,[d(d-1)/2]^n$ and the number of $T \in \mathcal{J}(n+1,d)$ with minimal parent passing numbers which are shape similar to U is $d^{Q(\sigma)}$. The sums of these expressions over all $U \in \mathcal{J}(n+1,1)$ and $\sigma_U = \sigma$ give the cardinalities of the sets specified in the lemma. ∎

For $n=0$, there is a single Raney sequence of length $2{\cdot}0+1$, namely, $\langle +1 \rangle$. It is the parent indicator sequence for the trivial J-tree with one leaf and no parents, which corresponds to the trivial permutation $\langle 0 \rangle$, so the correspondences extend to these trivial cases.

6. The theorems and the algorithms

Theorem 6.1. *For* $1 \le n, d < \omega$, $\sigma \in \mathcal{R}(n)$, *define* $\alpha_{n+1}(d)$ *to be the cardinality of the set of all* $T \in \mathcal{J}(n+1,d)$ *such that* T *has minimal parent passing numbers. For positive* $n, d < \omega$ *and* $Q(\sigma) := \sum_{j<2n \wedge \sigma(j)>0} \tau_\sigma(j)$,

$$\alpha_{n+1}(d) = \sum_{\sigma \in \mathcal{R}(n)} d^{Q(\sigma)} \prod_{\substack{j<2n \\ \sigma(j)<0}} \tau_\sigma(j)(\tau_\sigma(j)-1) = \sum_{\sigma \in \mathcal{R}(n)} \prod_{j<2n} \theta_\sigma(j),$$

where $\theta_\sigma(j) = d^{\tau_\sigma(j)}$ *if* $\sigma(j)>0$ *and* $\theta_\sigma(j) = \tau_\sigma(j)(\tau_\sigma(j)-1)$ *if* $\sigma(j)<0$.

Proof. For $d=1$, $d^{Q(\sigma)} = 1$ and the result follows from Corollary 3.7. For $d>1$, sum the results of the second part of Lemma 5.4. ∎

In order to take the above theorem and turn it into a Maple procedure, it is useful to introduce some small values that fit in. As noted above, a single node may be considered a J-tree with one leaf, and any two single node trees are shape similar and similar, so $\alpha_1(d) = 1$ for all d. For convenience, set $\alpha_n(d) = 0$ for all $n < 1$.

Note that $\tau_\sigma(j)$ is the difference between the number of indices smaller than j with $\sigma(j)>0$ and the number of indices smaller than j with $\sigma(j)>0$. Hence $\tau_\sigma(j) = j-2k$ where k is the number of indices smaller than j with $\sigma(j)>0$. We introduce a two parameter family of polynomials, $p(k,m,x)$, where $p(k,m,x)$ computes the sum over all initial segments μ of length k of 2-Raney sequences with m entries of (-1) and $k-m$ entries of $(+1)$ of the product $\prod_{j\le k}\theta_\mu(j)$. Then $\alpha_n(x)=p(2n-1,n-1,x)$. The test that a sequence μ of length k with m entries of (-1) and $k-m$ entries of $(+1)$ is an initial segment of a 2-Raney sequence is easy: μ must satisfy $2m<k$. Thus we set $p(k,m,x)=0$ if $2m\ge k$ or if $m<0$. For $m=0$, we can directly compute the value of $p(k,0,x)=x^{[k(k-1)]/2}$. Since the recurrence relation in terms of initial segments of length one less uses both $p(k-1,m,x)$ and $p(k-1,m-1,x)$, this initial data is needed for the Maple procedure.

```
p := proc(k,m,x) option remember;
  if (k < 1 or m < 0 or 2*m >= k) then
    0
  elif (m = 0) then
    x^((k)*(k-1)/2)
  else
    p(k-1,m-1,x)*(k-1-2*(m-1))*(k-2-2*(m-1))  # ends in -1
    +p(k-1,m,x)*(x^(k-1-2*(m)))               # ends in +1
  end if end proc:
P := n -> p(2*n-1,n-1,x);

for n from 1 to 10 do
  sort(expand(P(n)));
end do;
```

Examples 6.2. Here are the values of $\alpha_n(2)$ for $n=1,\dots,10$:

$$
\begin{array}{rc}
1: & 1\\
2: & 4\\
3: & 112\\
4: & 12352\\
5: & 4437760\\
6: & 4686103552\\
7: & 13624250626048\\
8: & 104218697796173824\\
9: & 2028257407393613676544\\
10: & 97849915247810309454561280
\end{array}
$$

Examples 6.3. The following are a few examples of the polynomials $\alpha_n(d)$:

- $\alpha_2(d)=2d$;
- $\alpha_3(d)=12d^3+4d^2$;
- $\alpha_4(d)=144d^6+72d^5+48d^4+8d^3$;

- $\alpha_5(d) = 2880d^{10} + 1728d^9 + 1728d^8 + 1008d^7 + 432d^6 + 144d^5 + 16d^4$.

Corollary 6.4. *The polynomial $\alpha_{n+1}(d)$ the following properties:*

1. *the degree of $\alpha_{n+1}(d)$ is $n(n+1)/2$;*
2. *the leading coefficient of $\alpha_{n+1}(d)$ is $n!(n+1)!$;*
3. *the lowest degree term of $\alpha_{n+1}(d)$ is $2^n d^n$; and*
4. *$n!(n+1)!\,d^{n(n+1)/2} \leq \alpha_{n+1}(d) < (2n)!\,d^{n(n+1)/2}$.*

Proof. Each 2-Raney sequence $\sigma \in \mathcal{R}(n)$ contributes to the $d^{Q(\sigma)}$ term of $\alpha_{n+1}(d)$ the quantity $\prod_{j\leq 2n \wedge \sigma(j)<0} \tau_\sigma(j)(\tau_\sigma(j)-1)$.

By the definitions of τ_σ and 2-Raney sequences, the value of $\tau_\sigma(j)$ is positive for all positive $j < 2n$ and $\tau_\sigma(0) = 0$. Thus $Q(\sigma) \geq n$, since there are $n+1$ indices j for which $\sigma(j) > 0$. This value n is achieved by the 2-Raney sequence $\sigma_* := \langle 1\rangle^\frown\langle 1,-1\rangle^n$ and by no other 2-Raney sequence. Item 3 follows since $\tau_* := \tau_{\mu_*} = \langle 0\rangle^\frown\langle 1,2\rangle^n$, and $\prod_{\substack{j<2n \\ \sigma_*(j)<0}} \tau_*(j)(\tau_*(j)-1) = 2^n$.

For a 2-Raney sequence $\sigma \in \mathcal{R}(n)$, the function $\tau_\sigma(j)$ counts the difference between the number of $k < j$ with $\sigma(k) = +1$ and the number of $k < j$ with $\sigma(k) = -1$. Thus the largest possible value for $Q(\sigma)$ is $0+1+\cdots+n = n(n+1)/2$. This value is achieved uniquely for $\sigma^* := \langle +1\rangle^{n+1\frown}\langle -1\rangle^n$. Note that $\tau^* := \tau_{\sigma^*} = \langle 0,1,2,\ldots,n,n+1,n,n-1,n-2,\ldots,2\rangle$. Thus the degree of $\alpha_{n+1}(d)$ is $Q(\sigma^*) = n(n+1)/2$, and the leading coefficient is

$$\prod_{\substack{j\leq 2n \\ \sigma(j)<0}} \tau^*(j)(\tau^*(j)-1) = \prod_{2\leq \ell\leq n+1} \ell(\ell-1) = n!(n+1)!\,.$$

To get the lower bound of item 4, simply truncate the polynomial to its leading term. Note that the computation of the leading coefficient gives the largest value that can be contributed toward the polynomial by any 2-Raney sequence. Since $|\mathcal{R}(n)| = C(n) = (2n)!/[(n!)((n+1)!)]$, the upper bound is obtained by using $n!(n+1)!d^{n(n+1)/2}$ as an estimate for every Raney sequence. ∎

Theorem 6.5. *For positive $n, d < \omega$ and $R(\sigma) := -2n + \sum_{j<2n} \tau_\sigma(j)$, the cardinality of $\mathcal{J}(n+1,d)$ is*

$$\rho_{n+1}(d) := \sum_{\sigma\in\mathcal{R}(n)} d^{R(\sigma)} \left[\frac{d(d-1)}{2}\right]^n \prod_{\substack{j<2n \\ \sigma(j)<0}} \tau_A(j)\,(\tau_A(j)-1)\,.$$

Proof. Sum the results of the first part of Lemma 5.4. ∎

We translate this theorem into an algorithm via a Maple procedure using the techniques applied above to Theorem 6.5. First note that $R(\sigma) = -2n + \sum_{j<2n} \tau_\sigma(j) = \sum_{j<2n} [\tau_\sigma(j) + \sigma(j) - 1]$. Thus $\rho_{n+1} = \sum_{\sigma \in \mathcal{R}(n)} \prod_{j<2n} \eta_\sigma(j)$ where $\eta_\sigma(j) = d^{\tau_\sigma(j)}$ if $\sigma(j) > 0$ and $\eta_\sigma(j) = d^{\tau_\sigma(j)-2}(\tau_\sigma(j)-1)$ if $\sigma(j) < 0$.

```
q := proc(k,m,x) option remember;
  if (k < 1 or m < 0 or 2*m >= k) then
    0
  elif (m = 0) then
    x^((k)*(k-1)/2)
  else
    q(k-1,m-1,x)*(k-1-2*(m-1))*(k-2-2*(m-1))*x^(k-1-2*(m-1)-2)
      # ends in -1
    +q(k-1,m,x)*(x^(k-1-2*(m-1)))
      # ends in +1
  end if end proc:
Q := n -> q(2*n-1,n-1,x)*((x^2-x)/2)^(n-1);

for n from 1 to 4 do
  sort(expand(Q(n)));
end do;
```

Examples 6.6. Here are the values of $\rho_n(2)$ for $n = 1, \ldots, 10$:

$$
\begin{array}{rc}
1: & 1 \\
2: & 4 \\
3: & 208 \\
4: & 84544 \\
5: & 225285376 \\
6: & 3562673554432 \\
7: & 313228604408713216 \\
8: & 146151093077541238226944 \\
9: & 349492125813998287750324092928 \\
10: & 416817372663146443348345786611033 7024
\end{array}
$$

Examples 6.7. The following are a few examples of the polynomials $\rho_n(d)$:

- $\rho_2(d) = d^3 - d^2$;
- $\rho_3(d) = 3d^8 - 6d^7 + 4d^6 - 2d^5 + d^4$;
- $\rho_4(d) = 18d^{15} - 54d^{14} + 63d^{13} - 45d^{12} + 33d^{10} + 19d^9 - 9d^8 + 3d^7 - d^6$.

7. Applications

The first application is a direct corollary of Theorem 6.1.

Corollary 7.1. *The value $\alpha_n(1)$ is the cardinality of $\mathcal{J}(n,1)$, the set of $T \in \mathcal{J}(n,2)$ with minimal passing numbers. It is also the number of shape similarity classes of (n)-leaf subtrees of ${}^{\omega>}d$ which are J-trees. Moreover, $\langle \alpha_n(1) : 1 \leq n < \omega \rangle >$ is the sequence of tangent numbers, so $\alpha_n(1) = t_n$ may also be computed using the generating function*

$$\tan(x) = \sum_{1}^{\infty} t_n \frac{x^{2n-1}}{(2n-1)!}.$$

Proof. The value $\alpha_n(1)$ is by definition the size of $\mathcal{J}(n,1)$, which is the set of trees in $\mathcal{J}(n,2)$ with minimal passing numbers. It follows from Lemma 2.7, that $\alpha_n(1)$ is the number of shape similarity classes, since each class has a unique member from $\mathcal{J}(n,1)$. To see that this sequence is the tangent numbers, observe that Vuksanovic computes this quantity in [14] (for a version in the language of category theory, see [1]), or check Sloane's On-Line Encyclopedia of Integer Sequences [11] where the tangent numbers appear as the number of Joyce trees on $2n-1$ nodes. ∎

The next application is to weak embedding types of $(n+1)$-leaf J-trees.

Theorem 7.2. *The number of weak embedding types of $(n+1)$-leaf subtrees of ${}^{\omega>}d$ which are J-trees is $\rho_{n+1}(d) = |\mathcal{J}(n+1,d)|$.*

Proof. Use Theorem 6.5 and Lemma 2.7. ∎

Theorem 7.3. *The number of similarity equivalence classes of $(n+1)$-leaf subtrees of ${}^{\omega>}d$ which are J-trees is $\alpha_{n+1}(d)$.*

Proof. Use Theorem 6.1 and Lemma 2.7. ∎

Laflame, Sauer and Vuksanovic [5] use *diagonal sets* (see the definitions on pages 188–189) in the definition of their canonical partitions. One can show that a finite set $D \subseteq {}^{\omega>}d$ is diagonal if and only if D is the set of leaves of $T = D \cup \{s \wedge t : s,t \in D\}$ and T is a J-tree. Moreover, two diagonal sets are similar in the sense of Laflamme, Sauer and Vuksanovic if and only if their corresponding J-trees are similar as in Definition 2.1. Thus we have the following corollary to Theorem 7.3.

Corollary 7.4. *For $2 \leq n,d < \omega$, the number of similarity classes of n-element diagonal subsets of ${}^{\omega>}d$ is $\alpha_n(d)$.*

The application mentioned in the title of the paper is part of the the Ramsey theory of n-tuples of the countable universal binary homogeneous relational structures as discussed in [10] and [5]. In particular, these are structures in a language with only binary relations which are determined by their two element substructures and may be coded as a cofinal subset of ${}^{\omega>}d$ for some finite d. For concreteness, here is a short list of universal countable purely binary relational structures with the tree in which they may be coded as a cofinal set.

1. The Rado (random) graph may be coded in ${}^{\omega>}2$.
2. The random oriented graph may be coded in ${}^{\omega>}3$.
3. The random directed graph may be coded in ${}^{\omega>}4$.
4. The random tournament may be coded in ${}^{\omega>}2$.

For clarity, we point out that the countable homogeneous triangle-free graph is an example of a countable homogeneous purely binary relational structure which is *not* universal in the sense used here, since it forbids a three element substructure, so is not determined by its two element substructures. See [9] for more information about universal countable binary relational structures.

For background information on the Ramsey theory of countable universal homogeneous relational structures, see the introductory section of the paper *Coloring subgraphs of the Rado graph* by Sauer [10] and the final section of *Canonical partitions of universal structures* by Laflamme, Sauer and Vuksanovic [5].

Sauer in [10] proved the existence of canonical partitions for countable universal binary homogeneous relational structures and showed that the cells of the partitions he described were indivisible. Laflamme, Sauer and Vuksanovic [5] showed the cells were persistent and hence the partitions identified by Sauer were shown to be canonical. Below two theorems from [5] are quoted and then followed with some related definitions and notation.

Theorem 7.5 **([5, Theorem 7.9]).** *Let $\mathbb{U}=(U;\mathfrak{L})$ be a universal countable binary relational structure and $n\in\omega$. Then $\mathcal{C}_n(\mathbb{U})$ is a canonical partition of the n-element subsets of U.*

Here U is identified with a cofinal subset of ${}^{\omega>}d$, $\mathcal{C}_n(\mathbb{U})$ is a partition of the n-element subsets of U consisting of the similarity classes of n-element diagonal subsets of U with a specified similarity class enlarged by addition of all the non-diagonal subsets of U. Let $r_{\mathbb{U}}$ denote the number of similarity equivalence classes of the n-element diagonal subsets of U.

Corollary 7.6 **([5, Corollary 7.10]).** *Let $\mathbb{U}=(U,\mathfrak{L})$ be a universal countable binary relational structure and $n\in\omega$. Moreover, if $\mathbb{U}\nrightarrow(\mathbb{U})^n_{<\omega/s}$, then $s\geq r_{\mathbb{U}}(n)$.*

By definition, the partition relation $\mathbb{U} \to (\mathbb{U})^n_{<\omega/r}$ holds if for every coloring of the n-element subsets of U with finitely many colors, there is an induced substructure of $\mathbb{U}$ isomorphic to $\mathbb{U}$ on whose n-element subsets the coloring takes at most r values. The number of cells, $r_{\mathbb{U}}$, of the canonical partition is a critical number for this partition relation.

Below we apply Corollary 7.4 to find the value of $r_{\mathbb{U}}$.

Corollary 7.7. *For positive $n<\omega$ and d with $2 \le d < \omega$, if $\mathbb{U} = (U, \mathcal{L})$ is a universal countable binary relational structure coded as a cofinal subset of ${}^{\omega>}d$, then the number of cells in a canonical partition of [5, Theorem 7.9] and the critical value for the partition relation of [5, Corollary 7.10] is $r_{\mathbb{U}} = \alpha_n(d)$.*

This work of Sauer and Laflamme, Sauer and Vuksanovic generalizes that of Erdős and Rado [3], who determined the canonical partitions of n element sequences of natural numbers.

Devlin [1], in his thesis, proved a parallel Ramsey Theorem for $(\mathbb{Q}, <)$ in which the critical values are the tangent numbers (see also Vuksanovic [14]). Vuksanovic [15] has worked on canonical partitions for $(\mathbb{Q}, <)$.

The work of Laflamme, Sauer and Vuksanovic also generalizes work of Erdős, Hajnal and Pósa [2], who showed that any partition of the edges of the random graph must have at least two colors. The random (Rado) graph, $\mathbb{RG} = (\omega, E_{\mathbb{RG}})$, is a special case of a universal countable binary relational structure of degree 2. Vuksanovic, in Lemma 2.1 of [16], gives a characterization of the equivalence classes of a canonical partition for the random graph and includes a table with the first few values for the number of cells the canonical partition of n-tuples: there is one cell for singletons, four cells for pairs, and 112 cells for three element sets. $r_n = \alpha_n(2)$: $r_1 = 1$, $r_2 = 4$, $r_3 = 112$. More generally, by Corollary 7.7 above, there are $\alpha_n(2)$ cells in the canonical partition of n-element sets of the random graph.

Sauer [8] has proved a Ramsey theorem for colorings of the edges of the countable triangle free homogenous graph, but Ramsey questions for larger size subsets remain open.

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