When Ramsey Fails, Use More Colors

1 $\binom{\omega}{k}$

Theorem 1.1 $T(\binom{\omega}{k}) = 1$

2 $\binom{\zeta}{2}$ and $\omega \times \omega$

Theorem 2.1 $T(\omega \times \omega) = 2$.

Theorem 2.2 $T(\binom{\zeta}{2}) = 4$. *The proof uses Theorem 2.1.*

3 $\binom{\zeta}{3}$ and $\binom{\omega}{2} \times \omega$

Theorem 3.1 $T(\binom{\omega}{2} \times \omega) = 3$.

Theorem 3.2 $T(\binom{\zeta}{3}) = 8$. *The proof uses Theorem 3.1.*

4 $\binom{\zeta}{a}$ and $\binom{\omega}{a_1} \times \binom{\omega}{a_2}$

Theorem 4.1 $T(\binom{\omega}{a_1} \times \binom{\omega}{a_2}) = \binom{a_1+a_2}{a_1} = \frac{(a_1+a_2)!}{a_1!a_2!}$.

Theorem 4.2 $T(\binom{\zeta}{a} = 2^a$. *The proof uses Theorem 4.1.*

5 $\binom{\omega}{a_1} \times \cdots \times \binom{\omega}{a_k}$

Theorem 5.1 $T(\binom{\omega}{a_1} \times \binom{\omega}{a_2} \times \ldots \times \binom{\omega}{a_k}) = \binom{a_1+a_2+\ldots+a_k}{a_1,a_2,\ldots,a_k} = \frac{(a_1+a_2+\ldots+a_k!)}{a_1!a_2!\ldots a_k!}$

6 $\binom{n\cdot\omega}{a}$

Theorem 6.1 $T(\binom{n\cdot\omega}{a}) = FILLITIN$

BILL TO GANG: PLEASE FILL IN. DOES IT USE THEOREM 5.1? I SUSPECT YES. IF SO THEN ADD THAT FACT, AS I HAVE DONE FOR OTHER CASES.

7 $\binom{\omega^2}{1}$, $\binom{\omega^2}{2}$, $\binom{\omega^2}{3}$

Theorem 7.1

1. $T(\binom{\omega^2}{1}) = 1$
2. $T(\binom{\omega^2}{2}) = 4$
3. $T(\binom{\omega^2}{3}) = 26$

8 $\binom{\omega^2}{a}$

We have a recurrence for $T(\binom{\omega^2}{a})$. Actually we have two recurrences and they both work.

Theorem 8.1

1. $T(\binom{\omega^2}{1}) = 1$.
2. *For all* $a \geq 2$

$$T\left(\binom{\omega^2}{a}\right) = (a+2)T\left(\binom{\omega^2}{a-1}\right) + 2\sum_{k=2}^{a-1} T\left(\binom{\omega^2}{k-1}\right)T\left(\binom{\omega^2}{a-k}\right)$$

3.

$$T\left(\binom{\omega^2}{a}\right) = -(a)T\left(\binom{\omega^2}{a-1}\right) + \sum_{k=1}^{a} \binom{a+1}{k} T\left(\binom{\omega^2}{k-1}\right)T\left(\binom{\omega^2}{a-k}\right)$$

Theorem 8.1 gives a way to compute $T(\binom{\omega^2}{a})$). Here are the first 10.

a	$T(\binom{\omega^2}{a})$)
1	1
2	4
3	26
4	236
5	2752
6	39208
7	660032
8	12818912
9	282137824
10	6939897856

These numbers match (off by 1) sequence A00311 in OEIS, though we note that we have not proven this (though it is surely true). We now describe the sequence.

BILL TO GANG: MY CONFUSION WAS THAT I THOUGHT $1, 2, 3, \ldots, b$ WAS A SEQUENCE AND HENCE YOU COULD NOT SPLIT IT (13)(2). I NOW SEE THAT YOU CAN. WITH THIS IN MIND I REWRITE IT BELOW. ALSO THIS REALLY HAS NOTHING TO DO WITH THE USE OF THE WORD PARTITION AS IS USED IN NUMBER THEORY. I WILL ALSO CLARIFY THAT.

Def 8.2 Let $b \in \mathbb{N}$.

1. A *total partition of* b [1] is formed as follows: Take the set $\{1, \ldots, b\}$. Partition it into two sets with ≥ 2 elements. Then partition each set with ≥ 2 elements into two or more sets. Repeat until every set has only 1 element. For example, the following are all of the total partitions of $\{1, 2, 3\}$: $\{1, 2, 3\}$ are $\{1|2, 3\}$, $\{1, 2|3\}$, $\{1, 3|2\}$, and $\{1|2|3\}$.

[1] In number theory the term *partition* means a way to write a number as a sum of smaller numbers. In that context the number of partitions of 5 is 7: $1+1+1+1+1$, $1+1+1+2$, $1+1+3$, $1+2+2$, $1+4$, $2+3$, 5. This notion of partition has *nothing to do* with what we are calling a *total partition*.

2. *Schroder's fourth problem* is to, given b, determine the number of total partitions of b. We denote this by $s(b)$. (There are other equivalent definitions which you can find at OEIS-A000311.)

$s(0) = 0$: There are no strong partitions of $\emptyset$.
$s(1) = 0$: The only strong partitions of $\{1\}$ is $\{1\}$.
$s(2) = 1$: The only strong partition of $\{1, 2\}$ is $\{1, 2\}$.
$s(3) = 4$: The only strong partitions of $\{1, 2, 3\}$ are $\{1|2, 3\}$, $\{1, 2|3\}$, $\{1, 3|2\}$, and $\{1|2|3\}$.
$s(4) = 26$. We leave this to the reader.

Empirically we have that, for all $1 \le a \le 10$, $T(\binom{\omega^2}{a}) = s(a + 1)$. Hence we have the following conjecture:

BILL TO GANG: I HAVE WRITTEN THAT WE HAVE A MATCH TO SCHRODER NUMBERS FROM a=1 to 10. IS THIS TRUE? DO WE KNOW MORE?

Conjecture 8.3 $T(\binom{\omega^2}{a}) = s(a + 1)$

9 $\binom{\omega^3}{2}$

BILL TO GANG: WOULD IT BE INSIGHTFUL FOR THE READER IF WE HAD THE PROOF FOR $T(\binom{\omega^3}{a})$ AND PROVED IT, OR SHOULD WE JUST GO STRAIGHT TO THE VERY GENERAL RECURRENCE AND THEOREM?

We define a sequence that is not in OEIS.

Def 9.1 We define $JNR(a)$ as follows.

1. $JNR(1) = FILLITIN$.
2. $JNR(a) = FILLITIN$ (AS A RECURRENCE.

The JNR numbers grow very fast. Here are the first 10:

a	$JNR(a)$
1	1
2	14
3	509
4	35839
5	4154652
6	718142257
7	173201493539
8	55580900954954
9	22900450653281951
10	11782966082685899537

Theorem 9.2 $T(\binom{\omega^3}{a}) = JNR(a)$.

10 $\binom{\omega^n}{a}$

YOU CURRENTLY DO THIS WITH THE VERY GEN THEOREM. DO YOU NEED IT? COULD YOU WRITE A LESS GEN VERSION WHERE WE HAVE A RECURRENCE FOR $T(\binom{\omega^n}{a})$?

11 Very General Theorem

Theorem 11.1

$$T\left(\binom{\omega^{d_1}}{a_1} \times \binom{\omega^{d_2}}{a_2} \times \ldots \times \binom{\omega^{d_m}}{a_m}\right) =$$

$$\sum_{\substack{k=1\\ d_k=1\\ a_k\geq 0}}^{m} T\left(\binom{\omega^{d_1}}{a_1} \times \binom{\omega^{d_2}}{a_2} \times \ldots \times \binom{\omega^{d_{k-1}}}{a_{k-1}} \times \binom{\omega^{d_k}}{a_k - 1} \times \binom{\omega^{d_{k+1}}}{a_{k+1}} \times \ldots \times \binom{\omega^{d_m}}{a_m}\right) +$$

$$\sum_{\substack{k=1\\ d_k>1}}^{m} \sum_{i=1}^{a_k} T\left(\binom{\omega^{d_1}}{a_1} \times \binom{\omega^{d_2}}{a_2} \times \ldots \times \binom{\omega^{d_{k-1}}}{a_{k-1}} \times \binom{\omega^{d_k - 1}}{i} \times \binom{\omega^{d_k}}{a_k - i} \times \binom{\omega^{d_{k+1}}}{a_{k+1}} \times \ldots \times \binom{\omega^{d_m}}{a_m}\right)$$

for all integers $d_k, a_k \geq 0$ where $\exists k\colon d_k, a_k > 0$. The base case is where $\forall k\colon d_k = 0 \vee a_k = 0$ where the value is 1.

BILL TO GANG: WE SHOULD GET AN ASY FORMULA FOR $T(\binom{\omega^n}{a})$. I THINK YOU SAID IT WAS EXP, SO PERHAPS its $Af(n))^a$ WHERE WE NEED TO FIND $f(n)$. COULD FIND IT BY LOOKING AT a VS $\log_2(T(\binom{\omega^n}{a}))$ WHICH SHOULD BE LINEAR

a vs $a\log_2(f(n) + \log_2(A)$.