

Countable Decompositions of R^2 and R^3

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Abstract. If the continuum hypothesis holds, R^2 is the union of countably many sets, none spanning a right triangle. Some partial results are obtained concerning the following conjecture of the first author: R^2 is the union of countably many sets, none spanning an isosceles triangle. Finally, it is shown that R^3 can be colored with countably many colors with no monochromatic rational distance.

Introduction

In the presence of the axiom of choice it is possible to decompose the plane into a “few” “small” pieces. “Few” usually means countable, while “small” can have various meanings. Davies proved, for example, that the plane can be the union of countably many graphs (here graph means a congruent copy of $\{(x, f(x)): x \in R\}$ for some real function f , see [2]). In another paper [3] Davies confirmed the conjecture of the first author, showing that, assuming the continuum hypothesis (CH), the plane can be partitioned into countably many pieces, with every distance occurring in every piece at most once. This was later extended to R^n by Kunen [10]. In the first part of this paper we show that under the CH, the plane can be decomposed into countably many pieces, none spanning a right-angled triangle. This was first proved, but not published, by the first author, answering a question of Fajtlowicz (Houston). It was announced in [5] and [8]. Later, but independently, the second author rediscovered Theorem 1 when answering a question of Heteyi, Jr. (Budapest). If the CH does not hold, no such decomposition exists; this fact is a trivial corollary of a well-known polarized partition theorem (see [7]). The positive result can be extended to the exclusion of triangles containing an angle in a predetermined countable set. With only the help of the axiom of

choice, Ceder showed that the equilateral triangle can be excluded (or, any triangle similar to any one in a predetermined countable set, see [1]). We conjecture that—again without using the CH—even isosceles triangles can be excluded, but we have only been able to prove a weaker result, namely, that the plane can be decomposed into countably many pieces, none containing x, y_0, y_1, \dots , with all the distances $d(x, y_i)$ the same. This makes it possible to show that 2^{\aleph_0} can be arbitrarily large while in the plane of the union of \aleph_1 sets, none spanning an isosceles triangle.

Finally, we show that R^3 can be decomposed into countably many pieces, none spanning a rational distance. The corresponding result for R^2 follows easily from an old result of the first author and Hajnal [4] (see also [8]), namely, if a graph does not contain a complete bipartite graph on two, resp. \aleph_1 , vertices, then it is countably chromatic. This idea, however, no longer works for the three-dimensional space, therefore we use another approach. We conjecture that the result can be extended to every R^n .

We use standard set-theoretical notation. If S is a set, $|S|$ is its cardinality, $[S]^{<\omega}$, $[S]^\omega$ denote the collections of finite, resp. countably infinite, subsets of S . $d(x, y)$ is the distance between x and y .

1

Theorem 1. *If CH holds, then the plane can be colored with countably many colors with no monochromatic right-angled triangle.*

We notice that it is not known if this result holds for every R^n .

Proof. By transfinite recursion on $\alpha < \omega_1$, we construct H_α , a countable set of points on the plane, and \mathcal{E}_α , a countable collection of lines and circles. We require that our sets extend each other and be sufficiently closed:

- (1) For $\alpha < \beta < \omega_1$, $H_\alpha \subseteq H_\beta$, $\mathcal{E}_\alpha \subseteq \mathcal{E}_\beta$.
- (2) If $\lambda < \omega_1$ is the limit, then $H_\lambda = \bigcup \{H_\alpha : \alpha < \lambda\}$ and $\mathcal{E}_\lambda = \bigcup \{\mathcal{E}_\alpha : \alpha < \lambda\}$.
- (3) $\bigcup \{H_\alpha : \alpha < \omega_1\} = R^2$.
- (4) If $x, y \in H_\alpha$, $x \neq y$, then their connecting line as well as their Thales circle is in \mathcal{E}_α .
- (5) If $x, y, z \in H_\alpha$ are not collinear, then the circle containing them is in \mathcal{E}_α .
- (6) The elements of the intersection of any two different members of \mathcal{E}_α are in H_α .
- (7) The center of every circle in \mathcal{E}_α is in H_α .
- (8) If $x \in C \in \mathcal{E}_\alpha$, $x \in H_\alpha$, C is a circle, then the antipodal of x on C is also in H_α .
- (9) If $L \in \mathcal{E}_\alpha$ is a line, $x \in H_\alpha \cap L$, then the line perpendicular to L at x is also in \mathcal{E}_α .

As, by assumption, the CH holds, there is a well-ordering of R^2 into type ω_1 . Using that, we can build $H_{\alpha_1} \mathcal{E}_\alpha$ by Skolem-type closure arguments.

Let ω denote the set of natural numbers. $[\omega]^\omega$ will therefore be the collection of infinite sets of natural numbers. By transfinite recursion on $\alpha < \omega_1$ we build functions $\varphi: \bigcup \mathcal{E}_\alpha \rightarrow [\omega]^\omega$ and $f: \bigcup H_\alpha \rightarrow \omega$ such that:

- (10) If $C \in \mathcal{E}_\alpha$ is a circle, then $|\omega - \varphi(C)| \leq 2$.
- (11) If $C \in \mathcal{E}_\alpha$ is a circle, $i \in \varphi(C)$, $x, y \in C$, $f(x) = f(y) = i$, then x, y are not antipodal.
- (12) If $C \in \mathcal{E}_\alpha$ is a circle, $i \notin \varphi(C)$, there can be at most two points x on C with $f(x) = i$.
- (13) If $L \in \mathcal{E}_\alpha$ is a line, $i \notin \varphi(L)$, then there is at most one $x \in L$ with $f(x) = i$.
- (14) If $L_1, L_2 \in \mathcal{E}_\alpha$ are lines, $\{x\} = L_1 \cap L_2$, then $f(x) \notin \varphi(L_1) \cap \varphi(L_2)$.
- (15) If $x, y \in H_{\alpha+1} - H_\alpha$, $x \neq y$, then $f(x) \neq f(y)$.
- (16) If $e \in \mathcal{E}_\alpha$, $x \in e$, $x \notin H_\alpha$, then $f(x) \in \varphi(e)$.

First we prove that our conditions ensure that there will be no monochromatic (by f) right-angled triangles. In order to show this, assume that x, y, z form a right angle at y and $f(x) = f(y) = f(z) = i$. Let C be the circle containing x, y, z ; note that x and z are antipodal. Then $i \in \varphi(C)$ contradicts (11) and $i \notin \varphi(C)$ contradicts (12).

Assume that $\alpha < \omega_1$, and φ, f have already been defined on $H_\alpha, \mathcal{E}_\alpha$. We extend them to $H_{\alpha+1}, \mathcal{E}_{\alpha+1}$. Our first remark is that we do not have to care about (11) when extending f, φ . Assume that $C \in \mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha$ is a circle, $x_1, y_1, x_2, y_2, x_3, y_3$ are six points on C , and, for $t = 1, 2, 3$, $f(x_t) = f(y_t)$. Then, by (15), three of the points are in H_α , so $C \in \mathcal{E}_\alpha$ by (5). This means that when f on $H_{\alpha+1}$ is defined we can define $\varphi(C)$ to satisfy (10) and (11). To save (12), observe that a circle $C \in \mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha$ can contain only two points from H_α (by (5)), so if they have the same color, we can ensure that, when coloring $(H_{\alpha+1} - H_\alpha) \cap C$, no new point gets that color.

By (6), for every $x \in H_{\alpha+1} - H_\alpha$, only one $e \in \mathcal{E}_\alpha$ can contain x , so we must choose $f(x)$ as an element of this (possible) $\varphi(e)$. This gives an infinite set of candidates for $f(x)$. If we color the elements of $H_{\alpha+1} - H_\alpha$ in an ω -sequence, at every step only finitely many possibilities are ruled out by (15).

If $L \in \mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha$ is a line, let g_0, g_1, \dots , enumerate those lines $g_i \in \mathcal{E}_\alpha$ which are perpendicular to L , and define x_i by $\{x_i\} = g_i \cap L$. By (6) and (9), $x_i \in H_{\alpha+1} - H_\alpha$. To ensure (14) and (16) we have to ensure $f(x_i) \in \varphi(g_i), f(x_i) \in \varphi(L)$. Also, $\varphi(L) \subseteq \omega - \{f(x_0), f(x_1), \dots\}$ must be infinite. These requirements can be met by an inductive selection of colors. Condition (13) can also be met if $x \in H_\alpha, x \in L$, and $L \in \mathcal{E}_{\alpha+1} - \mathcal{E}_\alpha$ is a line, then we can ensure $f(x) \neq f(y)$ for $y \in L \cap (H_{\alpha+1} - H_\alpha)$. □

Next we give the easy converse to Theorem 1.

Theorem 2. *If the CH does not hold, then the plane cannot be colored with countably many colors with no monochromatic right-angled triangle.*

Proof. Assume that $2^{\aleph_0} > \aleph_1$, select \aleph_2 parallel lines, $\{e_\alpha: \alpha < \omega_2\}$, and \aleph_1 lines perpendicular to them, $\{f_\beta: \beta < \omega_1\}$. Color the edges of the complete bipartite

graph $\{x_\alpha, y_\beta: \alpha < \omega_2, \beta < \omega_1\}$ as follows. If the color of the point in the intersection of e_α and f_β is i , then we color the edge between x_α and y_β by i . By an old result of the first author and Hajnal, there are $\alpha(0) < \alpha(1) < \omega_2, \beta(0) < \beta(1) < \omega_1$ with all the edges $\{x_{\alpha(s)}, y_{\beta(t)}\}$ having the same color. But this obviously gives a monochromatic rectangle. \square

2

Theorem 3. *There is a well-ordering, $<$, of R^2 , and functions F, G such that $F(C) \in [C]^{<\omega}$ for every circle C , $G(x) \in [R^2]^{<\omega}$ for every point $x \in R^2$, and, moreover:*

- (1) *If the center of the circle C is x , and $y \in C - F(C)$, then $x < y$ and $x \in G(y)$.*

Proof. We show by transfinite induction on $\kappa = |X|$ that for every infinite $X \subseteq R^2$ there exists a well-ordering, $<$, of X , and functions F, G with $F(C) \in [C]^{<\omega}$ for every circle C , $G(x) \in [X]^{<\omega}$ for every $x \in X$, and:

- (1') *If the center of circle C is x , and $y \in (C \cap X) - F(C)$, then $x \in G(y)$ and, if $x \in X$, then $x < y$.*

We call $X \subseteq R^2$ *closed*, if:

- (2) For every $x, y, z \in X$ not collinear, the center of the circle through them is in X .
- (3) If $x_1, x_2 \in X, y \in R^2$ is a point such that $d(x_1, y), d(x_2, y)$ both occur as distances in X , then $y \in X$.

These conditions mean that X is a subalgebra of a certain algebra. Standard arguments show that every infinite X can be embedded into a closed set of the same cardinality. It suffices, therefore, to prove (1') for closed X .

If X is countable, we can enumerate as $\{x_0, x_1, \dots\}$, we define the well-order as $x_0 < x_1 < \dots$ and take $F(C) = C \cap \{x_0, \dots, x_{i-1}\}$ for circle C , where x_i is the center of C , and $G(x_i) = \{x_0, \dots, x_{i-1}\}$.

Assume now that X is closed, and $|X| = \kappa > \aleph_0$. Decompose X as the continuous, increasing union of $\{X_\alpha: \alpha < \kappa\}$ with $|X_\alpha| < \kappa$ for $\alpha < \kappa$, all X_α closed.

By the inductive hypothesis, there are appropriate $<_\alpha, F_\alpha, G_\alpha$ on $X_{\alpha+1}$. We order X as follows: $x < y$ whenever $x \in X_{\alpha+1} - X_\alpha, y \in X_{\beta+1} - X_\beta$, and either $\alpha < \beta$, or else $\alpha = \beta$ and $x <_\alpha y$. This is clearly a well-order. If $x \in X_{\alpha+1} - X_\alpha$, then, as X_α is closed, by (3), there can only be one $y \in X_\alpha$ such that $d(x, y)$ occurs as a distance in X_α . Put $G(x) = G_\alpha(x) \cup \{y\}$.

Assume that C is a circle. If $|C \cap X| \leq 2$, we can take $F(C) = C \cap X$. Otherwise, C 's center, x , is in X . Let $\gamma < \kappa$ be the smallest ordinal with $|C \cap X_{\gamma+1}| \geq 3$, and $\alpha < \kappa$ such that $x \in X_{\alpha+1} - X_\alpha$. By (2), $\alpha \leq \gamma$. Put

- (4) $F(C) = (C \cap X_\gamma) \cup F_\gamma(C)$.

To show (1'), assume that C is a circle, x is its center, $x \in X_{\alpha+1} - X_\alpha$, and $y \in (C \cap X) - F(C)$. Let γ be as in the previous paragraph. By (4), $y \notin X_\gamma$. If

$y \in X_{\gamma+1} - X_\gamma$, then $x \in G_\gamma(y) \subseteq G(y)$, If $y \notin X_{\gamma+1}$, then $d(x, y)$ occurs in $X_{\gamma+1}$, so $x \in G(y)$. □

Corollary 1. *There exists an $f: R^2 \rightarrow \omega$ such that if C is a circle with center x , then $\{y \in C: f(y) = f(x)\}$ is finite.*

Proof. Along the well-ordering of Theorem 3, define $f(x)$ such that if $y < x$, $y \in G(x)$, then $f(x) \neq f(y)$. If C is a circle with center x and $y \notin F(C)$ (a finite set), then $f(x) \neq f(y)$ by our construction. □

For the notation appearing in the following results, such as ccc (countable chain condition), MA (Martin’s axiom), etc., we refer the reader to [9].

Theorem 4. *There is a ccc poset which adds an ω -coloring of the (old) R^2 with no monochromatic isosceles triangle.*

Proof. Let $f: R^2 \rightarrow \omega$ be as in the previous corollary, i.e., if C is a circle with center x , and $y \notin F(C)$, then $f(x) \neq f(y)$. p will be an element of our poset P if p is a function with $\text{Dom}(p) \subseteq [R^2]^{<\omega}$, $\text{Ran}(p) \subseteq \omega$, p refines f , i.e., $p(x) = p(y)$ implies $f(x) = f(y)$, and, for every i , $p^{-1}(i)$ does not contain an isosceles triangle. We define $q \leq p$ if q extends p as a function.

In order to show that P is ccc, assume that p_α ($\alpha < \omega_1$) are \aleph_1 conditions. Without loss of generality, we can assume that $\text{Dom}(p_\alpha) = S \cup S_\alpha$, with $S \cap S_\alpha = S_\alpha \cap S_\beta = \emptyset$ for $\alpha \neq \beta$, $|S_\alpha| = n$, and $S_\alpha = \{x(\alpha, 0), \dots, x(\alpha, n-1)\}$. We can also assume that $p_\alpha|_S = p_\beta|_S$, $p_\alpha(x(\alpha, i)) = p_\beta(x(\beta, i))$, and $f(x(\alpha, i)) = f(x(\beta, i))$ for $i < n$, $\alpha < \beta < \omega_1$. For every $\alpha < \omega_1$, there is an $\varepsilon > 0$ such that all distances in $\text{Dom}(p_\alpha)$ are larger than ε , and the difference between any two *different* distances is also bigger than ε . Again, we may assume that this ε is the same for every p_α . By separability, we can further shrink to get

$$d(x(\alpha, i), x(\beta, i)) < \varepsilon/2 \quad \text{for } i < n, \alpha < \beta < \omega_1.$$

If $p_\alpha \cup p_\beta$ is not a condition, $S \cup S_\alpha \cup S_\beta$ contains a monochromatic isosceles triangle. This triangle must contain a point in S_α and another in S_β . If two points in the triangle are in S_α , one in S_β , say they are $x(\alpha, i), x(\alpha, j), x(\beta, k)$, the only possibility is that $d(x(\alpha, i), x(\beta, k)) = d(x(\alpha, j), x(\beta, k))$. Now

$$|d(x(\alpha, i), x(\alpha, k)) - d(x(\alpha, j), x(\alpha, k))| < \varepsilon,$$

so $x(\alpha, i), x(\alpha, j), x(\alpha, k)$ form an isosceles monochromatic triangle, which is impossible.

Assume now that $s \in S$, $x(\alpha, i), x(\beta, j)$ form a forbidden triangle. The only possibility is that $d(s, x(\alpha, i)) = d(s, x(\beta, j))$. By using ε , $i = j$. This means that if C is the circle around s , containing $x(\alpha, i)$, then $x(\beta, j)$ is in $F(C)$. For every α there can only be finitely many such β 's, so by Hajnal’s set mapping theorem

(this case is, in fact, trivial), there are $\alpha < \beta$ with no such configuration, i.e., p_α, p_β are compatible. (We even get \aleph_1 pairwise compatible elements.) \square

Corollary 2.

- (a) If MA_κ holds, then every $X \subseteq R^2$ with $|X| \leq \kappa$ can be colored with countably many colors, with no monochromatic isosceles triangle.
- (b) It is consistent that 2^{\aleph_0} is arbitrarily large, and R^2 can be colored with \aleph_1 colors without monochromatic isosceles triangles.

Proof. (a) Standard from Theorem 4.

(b) Add the poset of Theorem 4 ω_1 times iteratedly, with finite supports. \square

3

From now on, let Q denote the set of rational numbers (actually any countable subset of R works).

A subset $E \subseteq R^3$ is called a *combinatorial line*, if there is a line L , and a point $x \notin L$, such that $E = \{y \in L: d(x, y) \in Q\}$. Obviously, $|E| = \aleph_0$. Given $x_1, x_2 \in R^3$, $x_1 \neq x_2$, only countably many combinatorial lines contain both x_1 and x_2 (as y is determined by $d(y, x_1), d(y, x_2)$). If E , a combinatorial line, is given, the set of those points y , for which $d(x, y) \in Q$ holds for every $x \in E$, is a countable collection of disjoint circles.

We remind the reader that a set $A \subseteq \omega$ is of *density zero* (or of *density one*), if

$$\lim_{n \rightarrow \infty} \frac{|A \cap \{0, 1, \dots, n-1\}|}{n} = 0 \quad (\text{or } 1).$$

(As the reader will probably observe, this notion is too strong here, we only need the properties that every infinite set contains an infinite subset of density zero, and finitely many density zero sets cannot cover ω .)

Theorem 5. Assume that $X \subseteq R^3$ and a set $\varphi(x) \in [\omega]^\omega$ of density one is assigned to every $x \in X$. Then there is a coloring $f: X \rightarrow \omega$ such that $f(x) \in \varphi(x)$ ($x \in X$), if $d(x, y) \in Q$, then $f(x) \neq f(y)$, and, if E is a combinatorial line, then $\{f(x): x \in E\}$ has density zero.

Proof. We call $X \subseteq R^3$ *closed*, if:

- (1) For $x_1, x_2 \in X, x_1 \neq x_2, E$ is a combinatorial line with $x_1, x_2 \in E$, then $E \subseteq X$.
- (2) If $x_1, x_2, x_3 \in X$ are not collinear, y has $d(x_i, y) \in Q$ ($i = 1, 2, 3$), then $y \in X$.

Similarly to Theorems 1 and 3, from the facts mentioned after the definition of a combinatorial line, every X can be embedded into a closed set of size $\max(|X|, \aleph_0)$. It suffices, therefore, to prove the theorem for closed sets. Again, we do this by transfinite induction on $|X|$.

If $|X| \leq \aleph_0$, enumerate X as $\{x_0, x_1, \dots\}$ and choose by induction on i an $f(x_i) \in \varphi(x_i)$ with $f(x_j) \neq f(x_i)$ for $j < i$, and $f(x_i) > 2^i$. Assume now that $|X| = \kappa > \aleph_0$ and X is closed. We build an increasing, continuous chain of closed sets $\{X_\alpha : \alpha < \kappa\}$ with $|X_\alpha| < \kappa$ ($\alpha < \kappa$), and $\bigcup \{X_\alpha : \alpha < \kappa\} = X$. We color $X_{\alpha+1} - X_\alpha$ by transfinite recursion on α . If we are at the α th step, assume that $f_\alpha = f|_{X_\alpha}$ is already given. For $x \in X_{\alpha+1} - X_\alpha$, put

$$\varphi_\alpha(x) = \omega - \{f_\alpha(y) : y \in X_\alpha, d(x, y) \in Q\}.$$

As X_α is closed, and $x \in X_{\alpha+1} - X_\alpha$, the set $\{y \in X_\alpha : d(x, y) \in Q\}$ is either empty, a one-element set, or a combinatorial line. Therefore, $\varphi_\alpha(x)$ is of density one. We can now apply our theorem on $X_{\alpha+1} - X_\alpha$ with $\varphi(x) \cap \varphi_\alpha(x)$ to get $g_\alpha : X_{\alpha+1} - X_\alpha \rightarrow \omega$. Then $f_{\alpha+1} = f_\alpha \cup g_\alpha$ obviously has the desired properties. To end, we check that if $f = \bigcup \{f_\alpha : \alpha < \kappa\}$, then f maps every combinatorial line into a set of density zero. Assume E is a combinatorial line, put

$$\alpha = \min\{\beta < \kappa : |X_\beta \cap E| \geq 2\}.$$

Now α is of the form $\alpha' + 1$. $E - X_{\alpha'}$ contains all but at most one element of E , so our conditions will be met. \square

Corollary 3. *There is a coloring $f : R^3 \rightarrow \omega$ with $f(x) \neq f(y)$ whenever $d(x, y)$ is rational.*

Proof. Take $\varphi(x) = \omega$ for every $x \in R^3$ and apply Theorem 5.

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