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Author(s): M. Dunton
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BOUNDS FOR PAIRS OF CUBIC RESIDUES

M. DUNTON

The purpose of this paper is to prove Theorems 1–4. To this end one first establishes the following simple lemma.

LEMMA. *Let k be an odd prime, and m, n any two nonzero integers. Then for a prime modulus p, at least one member of the set, $S=\{m, n, mn, mn^2, \cdots, mn^{k-1}\}$, is a kth power residue.*

PROOF. If p is not of the form $tk+1$, every residue is a kth power residue, and the lemma is trivial. So suppose $p=tk+1$.

Let χ be a kth power character function defined on residues modulo p. Let $\chi(m)=\theta^\alpha$, $\chi(n)=\theta^\beta$, where θ is a primitive kth root of unity. If $\beta\equiv 0 \bmod k$, $\chi(n)=1$. If not, $\chi(mn^j)=\theta^\alpha$, $\theta^{j\beta}=\theta^{\alpha+j\beta}$, and as j takes values in $\{0, 1, \cdots, k-1\}$, $\alpha+j\beta$ runs over a complete set of residues modulo k, i.e., there is some j_0 such that $\alpha+j_0\beta\equiv 0 \bmod k$. Then $\chi(mn^{j_0})=1$ and mn^{j_0} is a kth power residue modulo p.

In the following theorems, it is assumed that residues are the least positive representatives of their classes and the ordering implied by a "bound" is that of the real integers.

THEOREM 1. *If $p\nmid 7\cdot 13$, then there exists a consecutive pair of nontrivial cubic residues $\leqq$ (77, 78) modulo p.*

PROOF. In the lemma, let $k=3$, $n=2$, $m=7$. Then at least one of the following is a cubic residue: 2, 7, 14, 28. If $\chi(2)=1$, (1, 2) is a consecutive pair; if $\chi(7)=1$, (7, 8) is a consecutive pair; and if $\chi(28)=1$, (27, 28) is a consecutive pair. In the remaining case $\chi(14)=1$, $\chi(2)=\omega$ and $\chi(7)=\omega^2$. To get the shortest proof, one looks at 13, 3 and 11 in that order. The results are summarized in the diagram on the next page.

THEOREM 2. *There are infinitely many primes which have* (77, 78) *as their first pair of consecutive nontrivial cubic residues; this implies that* (77, 78) *is the best possible bound.* 13,817,029 *is the smallest such prime.*

Note. The calculation of this prime was made possible by a grant of free time from the University of California at Berkeley Computer Center. The University of California, Davis Center provided card punching.

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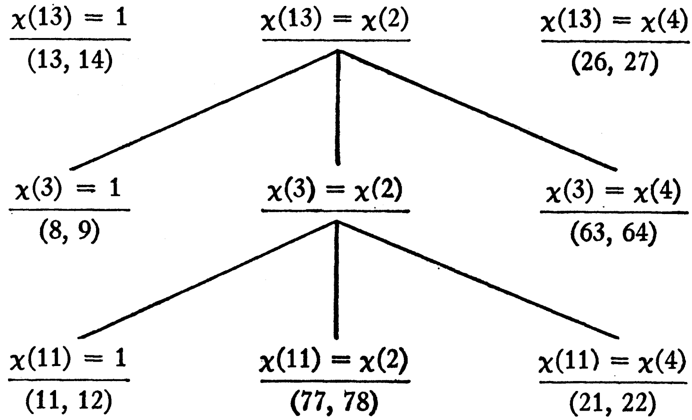

PROOF. The first part of Theorem 2 follows from Kummer's Theorem that there exist infinitely many primes having prescribed prime power character assigned to a finite set of smaller primes. The computer was programmed to find the smallest prime with the following cubic character: $\chi(2)=\chi(3)=\chi(11)=\chi(13)=\chi(17)=\omega$; $\chi(7)=\chi(19)=\omega^2$; $\chi(5)=1$; $\chi(23)\neq\omega^2$, $\chi(29)\neq\omega^2$; $\chi(37)\neq 1$ and $\chi(73)\neq 1$ if $\chi(37)=\omega^2$; $\chi(41)\neq 1$, $\chi(43)\neq 1$, $\chi(53)\neq 1$, $\chi(59)\neq 1$, $\chi(61)\neq 1$, $\chi(67)\neq 1$, $\chi(71)\neq 1$.

Tables by Cunningham and Gosset [1] provide values of $\mu\equiv L/M$ and $\lambda\equiv M/L$ mod q (where $4p=L^2+27M^2$) for primes $q<50$ such that q will have prescribed cubic character modulo p. For primes >50 a theorem of Emma Lehmer [2] made the necessary extension of the tables a simple matter. Hand calculation provided the machine with 32 possible values of μ mod $60060=2^2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13$. Since 2 is to be a cubic nonresidue of the prime p, L and M must be odd. The machine computed $L\equiv M\mu$ mod 60060 where M is odd and <5000 and μ is taken from the 32 values computed by hand. Next, the machine found $L/M\equiv\mu_q$ mod q for $13<q\leqq 73$ and checked the exclusion tables which had been obtained from Cunningham and Gosset and Lehmer. If μ_q passed all the tests for q up to 73, $(1/4)(L^2+27M^2)=p$ was tested for primality, and in the case of a prime, a card was punched with the values of L, M and p. 13,817,029 was the smallest prime of 28 solutions $<5\cdot 10^8$.

THEOREM 3. *If* $p\nmid 7\cdot 13$, *then there exist* 2 *pairs of nontrivial consecutive cubic residues* $\leqq(125,126)$.

PROOF. At least one of the 10 pairs in the proof of Theorem 1 must occur. Starting from each of these, and considering possible values of

$\chi(2)$, $\chi(3)$, $\chi(5)$, $\chi(7)$, $\chi(11)$, $\chi(13)$, one arrives at a second pair $\leqq(125, 126)$. For example, if (7, 8) occurs, the lemma can be applied to $k=3$, $m=2$, $n=3$. If $\chi(2)=1$, then (1, 2) is another pair; $\chi(3)=1$ implies (8, 9) is a second pair; $\chi(6)=1$ implies (6, 7) is a pair, and $\chi(18)=1$ implies (125, 126) is a second pair. The remaining nine cases can be settled in similar fashion.

THEOREM 4. *If d is a positive integer, $d \not\equiv \pm 3 \bmod 7$ is a necessary condition for the existence of a bound $B(d)$ such that for all except a finite set of primes, there exists a pair of cubic residues r_1, r_2, with $r_1-r_2=d$ and $r_2 \leqq B(d)$.*

PROOF. It must be shown that for all $B(d)>1$ there exist infinitely many primes p such that the first pair of cubic residues which differ by $d\equiv \pm 3 \bmod 7$ are greater than $B(d)$. Such primes can be constructed by taking $\chi(q)=1$ if q is a prime $\leqq B(d)$ and $\equiv \pm 1 \bmod 7$, $\chi(q)=\omega$ if q is a prime $\leqq B(d)$ and $\equiv \pm 2 \bmod 7$, $\chi(q)=\omega^2$ if q is a prime $\leqq B(d)$ and $\equiv \pm 3 \bmod 7$. Kummer's Theorem states that infinitely many such primes p with prescribed character exist and primes so determined cannot have a pair of cubic residues $\leqq B(d)$ which differ by $d\equiv \pm 3 \bmod 7$.

The following table gives values of the bound, $B(d)$, associated with difference d. In each case $B(d)$ was obtained by methods similar to the proof of Theorem 1.

$B(d)$	77	90	none	none	114	21	1	56	72	none
d	1	2	3	4	5	6	7	8	9	10

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2. Emma Lehmer, *Criteria for cubic and quartic residuacity*, Mathematika 5 (1958), 20–29.

SACRAMENTO, CALIFORNIA