Combinatorial Proofs of Sane Bounds on Some Polynomial van der Waerden Numbers
by William Gasarch[1], Clyde P. Kruskal[2], Justin D. Kruskal[3], Zach Price[4]

1 Introduction

We use the following standard notation and definitions.

Def 1.1 Let $\mathbb{Z}$ be the set of integers, $\mathbb{N}$ be the set of non-negative integers, and $\mathbb{N}^+$ be the set of positive integers. Let $[W]$ be the set $\{1,\ldots,W\}$ (where $W \in \mathbb{N} \in \mathbb{N}$).

Recall van Der Waerden's Theorem [10] (see also [4], [5]).

Theorem 1.2 *For any $k \in \mathbb{N}$, for any $c \in \mathbb{N}$, there exists $W = W(k,c)$, such that for any c-coloring of $[W]$, there exists $a, d \in \mathbb{N}$, $d \neq 0$, such that $a, a+d, \ldots, a+(k-1)d$ are all the same color.*

The original proof by van der Waerden was purely combinatorial and yielded bounds on W that were INSANE (called EEEEEEEEEENORMOUS by [4]). In particular, the proof used an ω^2 induction and $W(k,c)$ was bounded by a function that is not primitive recursive. Shelah [9] gave a purely combinatorial proof that yielded bounds that were HUGE, though not INSANE. In particular the bounds were primitive recursive. Gowers [3] gave a proof using non-combinatorial (and difficult) techniques that yielded bounds that were relatively SANE:

$$W(k,c) \le 2^{2^{c^{2^{2^{k+9}}}}}$$

We discuss a known generalization of van der Waerden's theorem. Note that the conclusion of van der Waerden's theorem is that

$$a, a+d, a+2d, \ldots, a+(k-1)d \text{ are the same color.}$$

Can we replace $d, 2d, \ldots, (k-1)d$ by other functions of d? Yes. We can replace them with polynomials in $\mathbb{Z}[x]$ that have no constant term. Here is the Polynomial van Der Waerden Theorem:

[1]U. of MD at College Park, Dept of CS and UMIACS, `gasarch@cs.umd.edu`
[2]U. of MD at College Park, Dept of CS, `kruskal@cs.umd.edu`
[3]EPIC Computing, `tinsuj@gmail.com`
[4]George Mason University, Dept of Mathematics, `zprice11@gmail.com`

Theorem 1.3 *Let $p_1(x), \ldots, p_k(x) \in \mathbb{Z}[x]$ such that, for $1 \le i \le k$, $p(0) = 0$. Let $c \in \mathbb{N}$, Then there exists $W = W(p_1(x), \ldots, p_k(x); c)$ such that, for any c-coloring of $[W]$, there exists $a, d \in \mathbb{N}$, $d \neq 0$, such that $a, a + p_1(d), \ldots, a + p_k(d)$ are all the same color.*

For $k = 1$, this theorem was proven independently by Furstenberg [2] and Sárközy [7]. Bergelson and Leibman [1] proved the general result using ergodic methods. These proofs yielded no upper bounds on $W(p_1(x), \ldots, p_k(x); c)$. Walters [11] proved Theorem 1.3 using combinatorial techniques which yielded bounds on W that were INSANE. In particular, the proof used an ω^ω induction and $W(p_1(x), \ldots, p_k(x); c)$ was bounded by a function that is not primitive recursive. One again Shelah [8] gave a purely combinatorial proof that yielded bounds that were HUGE, though not INSANE. In particular the bounds are primitive recursive. Peluse [6] has obtained even better bounds using non-combinatorial techniques (and difficult) techniques.

In this paper we show that, for some $p(x) \in \mathbb{Z}[x]$ and $c = 2, 3, 4$, one can obtain sane bounds on $W(p(x); c)$. Our proofs will be purely combinatorial and much easier than those of Walters, Shelah, and Peluse. We hasten to point out that they proved the full poly van der Warden theorem whereas we only prove it in special cases.

We will show the following.

- $W(x^2; 2) = 5$ (this is very easy) and, for all $a \in \mathbb{Z}$, $W(ax^2; 2) = 4a + 1$.
- For all $a \in \mathbb{Z}$, $W(ax; c) = |ac| + 1$.
- Let $p(x) \in \mathbb{Z}[x]$ such that $p(0) = 0$. Then $W(p(x); 2)$ is bounded above by the min of $|p(i)| + |p(j)| - g + 1$ such that (a) $i, j \in \mathbb{N}$, (b) $p(i), p(j) \neq 0$, (c) $g = \gcd p(i), p(j)$, (d) either $p(i)/g$ or $p(j)/g$) is even. In the appendix is a table of some other exact values of $W(ax^2 + bx; 2)$.
- $W(x^2; 3) = 29$ and, for all $a \in \mathbb{Z}$, $W(ax^2; 3) = 28a + 1$. In the appendix is a table of some other exact values of $W(ax^2 + bx; 3)$.
- For $a, b \in \mathbb{Z}$, $W(ax^2 + bx; 3) \le O(a^2 b^5)$.
- $W(x^2; 4) \le\le$ 84,149,474,894,213,522 In the appendix is a table of some upper bounds on $W(ax^2 + bx; 4)$.

2 Preliminaries

Def 2.1 Let $c \in \mathbb{N}^+$ and $W \in \mathbb{N}^+$.

1. A *c-coloring of* $[W]$ is a mapping $[W] \rightarrow [c]$.

2. Let $p(x) \in \mathbb{Z}[x]$. A *$(p(x); c)$-proper coloring of* $[W]$ is a c-coloring of $[W]$ such that, for all $x, y \in [W]$, if $y - x = p(d)$ for some $d \in N^+$, then x and y have different colors. When the context is clear, we will often write *proper c-coloring* or simply *proper coloring*.

Note that the polynomial van der Waerden number, $W = W(p(x); c)$, is the least number such that there is no $(p(x); c)$-proper coloring of $[W]$.

Although we care about proper $(p(x); c)$-colorings, we need a more general notion:

Def 2.2 Let $F \subseteq \mathbb{Z}$, $c \in \mathbb{N}^+$, and $W \in \mathbb{N}^+$.

- An *$(F; c)$-proper coloring of* $[W]$ is a c-coloring of $[W]$ such that, for all $x, y \in [W]$ with $y - x \in F$, x and y have different colors.

- $W = W(F; c)$ is the least number such that there is no $(F; c)$-proper coloring of $[W]$. If no such number exists, we set $W(F; c) = \infty$.

We leave the following easy lemma to the reader.

Lemma 2.3 *Let* $c \in \mathbb{N}^+$*.*

1. *If* $0 \in F$ *then* $W(F; c) = 1$*.*

2. *Assume* $f \in F$*. Let* $F' = F \cup \{-f\}$*. Then* $W(F; c) = W(F'; c)$*.*

We now prove an easy theorem which will lead to a nice lemma.

Theorem 2.4

1. $W(x^2; 2) = 5 = 4 + 1$.

2. $W(ax^2; 2) = (W(x^2; 2) - 1)a + 1 = 4a + 1$.

Proof:
1) $W(x^2; 2) \leq 5$: Assume, by way of contradiction, that COL is an $(x^2; 2)$-proper coloring of $[5]$. Assume that COL(1) $= R$. Since 1 is a square we have COL(2) $= B$, COL(3) $= R$, COL(4) $= B$, COL(5) $= R$. Then COL(1) $=$ COL(5) with $5 - 1 = 2^2$, which is a contradiction.

$W(x^2; 5) \geq 5$ via the following $(x^2; 2)$-proper coloring of $[4]$:

1	2	3	4
R	B	R	B

2) $W(ax^2;2) \le (W(x^2;2)-1)a+1 = 4a+1$: Assume, by way of contradiction, that COL is an $(ax^2;2)$-proper coloring of $[4a+1]$. We use COL to define COL′, an $(x^2;2)$-proper coloring of $[5]$.

COL′(1) = COL(1)
COL′(2) = COL($a+1$)
COL′(3) = COL($2a+1$)
COL′(4) = COL($3a+1$)
COL′(5) = COL($4a+1$)

By using that a and $4a$ are forbidden distances for COL, one can show that COL′ is an $(x^2;2)$-proper coloring of $[5]$, which is a contradiction.

$W(ax^2;2) \ge (W(x^2;2)-1)a+1 = 4a+1$: Let COL be an $(x^2;2)$-proper coloring of $[4]$. We use COL to define COL′, an $(ax^2;2)$-proper coloring of $[4a]$.

Let $1 \le x \le 4a$. Let $0 \le i \le 3$, and $1 \le j \le a$ be such that $x = ia+j$. Let

$$\mathrm{COL}'(x) = \mathrm{COL}'(ia+j) = \mathrm{COL}(i+1).$$

By using that 1 and 4 are forbidden distances for COL, one can show that COL′ is an $(ax^2;2)$-proper coloring of $[4a]$. ∎

Using the ideas behind Theorem 2.4.2 one can show the following:

Lemma 2.5 *Let $p(x) \in \mathbb{Z}[x]$, $a \in \mathbb{Z}$, and $c \in \mathbb{N}$. Then $W(ap(x);c) = a(W(p(x);c)-1)+1$.*

3 Linear polynomials

For completeness we cover linear polynomials, for which we obtain a complete solution.

Theorem 3.1 *Let $a \in \mathbb{Z}$ and $c \in \mathbb{N}^+$. Then*

$$W(ax;c) = |ac|+1 \ .$$

Proof: By Lemma 2.3 (1) we have the case of $a=0$, and (2) we can assume that $|a|$ is a forbidden distance.

$W(ax;c) \le |ac|+1$: By setting $x = 1,2,\ldots,c$ we get forbidden distances $|a|,|2a|,\ldots,|ca|$. So $1, |a|+1, |2a|+1, \ldots, |ca|+1$ must all be different colors, but there are only c colors.

$W(ax;c) \geq |ac|$: We can properly c-color $[ca]$: color $1,\ldots,|a|$ by 1, color $|a|+1,\ldots,|2a|$ by 2, …, color $|(c-1)a|+1,\ldots,|ca|$ by $c-1$. ∎

4 Upper Bounds on $W(p(x);2)$ for any $p(x) \in \mathbb{Z}[x]$

The following is our main lemma.

Lemma 4.1 *Let $s,t \in \mathbb{N}^+$. Let $g = \gcd(s,t)$. Then*

$$W(\{s,t\};2) = \begin{cases} s+t-g+1 & \textit{if either } s/g \textit{ or } t/g \textit{ is even} \\ \infty & \textit{otherwise.} \end{cases}$$

Proof:
Temporarily assume s and t are relatively prime, so $g = 1$.

Let $z = s+t$. Let COL be a $(\{(s,t\};c)$-proper coloring of $[z-1]$. We are *not* aiming for a contradiction; we are aiming to see that the entire coloring is forced.

Consider the list

$$s \bmod z,\ 2s \bmod z,\ 3s \bmod z,\ \ldots,\ (z-1)s \bmod z.$$

Every pair of adjacent values has absolute value of the difference either s or t. Hence $2s \bmod z$ is B, $3s \bmod z$ is R, $4s \bmod z$ is B, etc.

Since s is relatively prime to t, it is also relatively prime to z. Hence

$$\{s \bmod z,\ 2s \bmod z,\ \ldots,\ (z-1)s \bmod z\} = [z-1],$$

Therefore we have forced a 2-coloring of (all of) $[z-1]$. We discuss if the coloring can be extended beyond $z-1$.

Extend Beyond $z-1$?: Whether this proper coloring can be extended beyond $[z-1]$ depends on the parity of z:

CASE (1): Assume that either s or t is even. (The other must be odd because we have assumed that $g = 1$.)

Then $z-1 = s+t-1$ must be even, so that the first number in the above alternating list of colors, $s \bmod z$, and the last number, $(z-1)s \bmod z$, must have different colors. But

$$(z-1)s \equiv zs - s \equiv -s \equiv t \bmod z.$$

So $z = s+t$ cannot be R or B, implying that the coloring cannot be extended to z.

CASE (2): Assume s and t are both odd.
The above alternating list of colors, makes the odd numbers all have the same color, say R, and the even numbers B (because each addition changes the parity of the number being colored). Any number at or above $s+t$ can be colored, but its color is forced by subtracting s (or equivalently t). So the coloring can be uniquely extended to ∞.

We have proven the theorem in the case of $g = 1$. If $g \geq 2$ then there is no interaction of numbers x, y where $x \neq y \bmod g$. We leave it to the reader to use this to prove the $g \geq 2$ case. ∎

Theorem 4.2 *Let $p(x) \in \mathbb{Z}[x]$ be a polynomial such that $p(0) = 0$. Then $W(p(x); 2)$ is bounded above by the min of $\{|p(i)| + |p(j)| - g + 1\}$*
such that

- $i, j \in \mathbb{N}$
- $p(i), p(j) \neq 0$
- $g = \gcd(p(i), p(j))$, *and*
- *either $p(i)/g$ or $p(j)/g$ is even.*

Proof:
Follows from Lemma 4.1. ∎

5 $W(ax^2; 3) = 28a + 1$

In this section we will show that $W(x^2; 3) = 29$ and then $W(ax^2; 3) \leq 28a+1$. We first show a weaker theorem which will be a good warm-up to our work on 4-colorings in Section 7.

Theorem 5.1 $W(x^2; 3) \leq 1 + 41^2 = 1682$.

Proof:
Assume, by way of contradiction, that COL is an $(x^2; 3)$-proper coloring of $[1 + 41^2]$. we can assume $\mathrm{COL}(1) = R$ and $\mathrm{COL}(17) = B$. By Figure 1 we know that $\mathrm{COL}(26) \notin \{R, B\}$, hence $\mathrm{COL}(26) = G$. Again by Figure 1 we have that $\mathrm{COL}(42) \notin \{B, G\}$, hence $\mathrm{COL}(42) = R$.

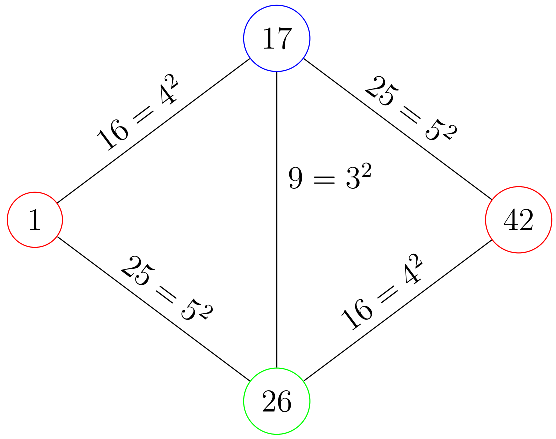

Figure 1: In any $(x^2, 3)$-proper coloring, $\mathrm{COL}(x) = \mathrm{COL}(x + 41)$

Note that we have shown that $\mathrm{COL}(1) = \mathrm{COL}(42)$. More generally we have shown that, for all x, $\mathrm{COL}(x) = \mathrm{COL}(x + 41)$. Hence

$$\mathrm{COL}(1) = \mathrm{COL}(1 + 41) = \mathrm{COL}(1 + 2 \times 41) = \cdots = \mathrm{COL}(1 + 41 \times 41) = \mathrm{COL}(1 + 41^2).$$

This contradicts COL being an $(x^2; 3)$-proper coloring. ▮

The following theorem was proven by Matt Jordan and Bill Gasarch.

Theorem 5.2

1. *$W(x^2; 3) = 29$.*
2. *For all $a \in \mathbb{Z}$, $W(ax^2; 3) = 28a + 1$. This follows from Part 1 and Lemma 2.5.*

Proof:
$W(x^2; 3) \leq 29$: Assume, by way of contradiction, that there exists COL, a proper 3-coloring of [29].

By Figure 2, $\mathrm{COL}(10) = \mathrm{COL}(17)$. By similar reasoning one can show that

$$(\forall x)[10 \leq x \leq 13 \implies \mathrm{COL}(x) = \mathrm{COL}(x + 7)].$$

We refer to this fact as FORCE.

We can assume, without loss of generality, that $\mathrm{COL}(10) = R$. Since $11 - 10 = 1^2$ we know that $\mathrm{COL}(11) \neq R$. We can, without loss of generality, assume that $\mathrm{COL}(11) = B$.

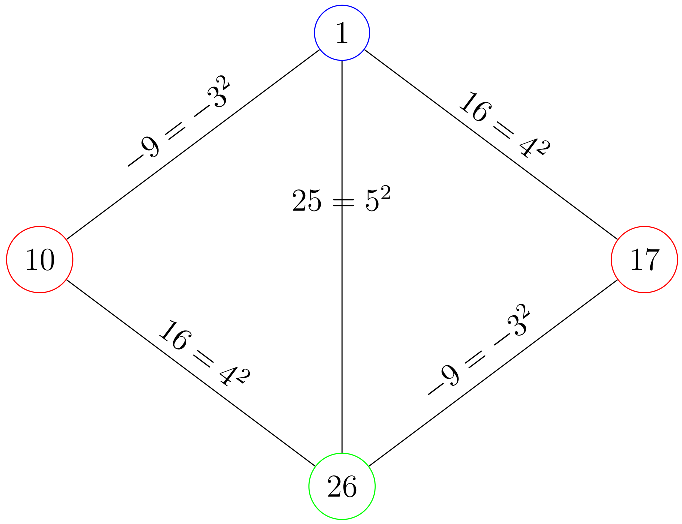

Figure 2: In any proper $(x^2, 3)$-coloring, COL(10) = COL(17)

17: By FORCE COL(17) = COL(10) = R

18: By FORCE COL(18) = COL(11) = B.

10	11	12	13	14	15	16	17	18	19	20
R	B						R	B		

19: Since COL(10) = R and COL(18) = B, COL(19) = G.

12: By FORCE COL(12) = COL(19) = G.

10	11	12	13	14	15	16	17	18	19	20
R	B	G					R	B	G	

20: Since COL(11) = B and COL(19) = G, COL(20) = R.

13: By FORCE COL(13) = COL(20) = R.

10	11	12	13	14	15	16	17	18	19	20
R	B	G	R				R	B	G	R

Now we have that COL(17) = COL(13) = R. But $17 - 14 = 2^2$. This is a contradiction.

$W(x^2, 3) \geq 29$:

We present a proper 3-coloring:

1	2	3	4	5	6	7	8	9	10	11	12	13	14
B	G	R	G	R	B	B	B	G	R	B	G	B	G

15	16	17	18	19	20	21	22	23	24	25	26	27	28
R	B	R	B	G	R	B	R	B	G	R	G	R	B

■

Note 5.3 By Figure 2 we easily show $W(x^2;3) \leq 68$: For $10 \leq x \leq 68$ then $\mathrm{COL}(x) = \mathrm{COL}(x+7)$, so

$$\mathrm{COL}(10) = \mathrm{COL}(17) = \cdots = \mathrm{COL}(59),$$

and note that $59 - 10 = 49 = 7^2$. This result is not as strong as $W(x^2;3) \leq 29$; however, it has a less detailed proof.

6 Upper Bounds on $W(ax^2 + bx; 3)$

Def 6.1

(a) A coloring of $[n]$ has *repeat distance* r if x and $x+r$ have the same color, for all $1 \leq x \leq n-r$.

(b) A coloring of $[n]$ has *repeat distance* r *under one-sided boundary condition* b if x and $x+r$ have the same color, for all $1 \leq x \leq n-r-b$.

(c) A coloring of $[n]$ has *repeat distance* r *under two-sided boundary condition* b if x and $x+r$ have the same color, for all $b < x \leq n-r-b$.

Lemma 6.2 *In any 3-coloring of $[n]$ with forbidden distances $s, t, s+t$, where $0 < s < t$:*

(a) $2s+t$ is a repeat distance.

(b) $t-s$ is a repeat distance under two-sided boundary condition s.

(c) $3s$ is a repeat distance under one-sided boundary condition t.

Proof: Let $u = s+t$.

(a) Consider a 3-coloring satisfying the conditions of the lemma. Let $1 \le x \le n-(2s+t)$. Without loss of generality, we can assume that x is R. Then $x+s$ is not R, say B, and $x+u = (x+s)+t$ cannot be R or B so it must be G. Then $(x+s)+u = (x+u)+s$ cannot be B or G so it must be R. Since x and $x+u+s$ are both R, $(x+u+s)-x = u+s = 2s+t$ is a repeat distance,

(b) Consider a 3-coloring satisfying the conditions of the lemma. Let $s < x \le n-(t-s)-s$. Without loss of generality, we can assume that x is R. Then $x-s$ is not R, say B, and $(x-s)+u = x+t$ cannot be R or B so it must be G. Then $(x-s)+t = (x+t)-s$ cannot be B or G, so it must be R. This process requires that $x-s > 0$ and $x+t \le n$. So $(x+t-s)-x = t-s$ is a repeat distance under two-sided boundary condition s.

(c) Take $2s+t$ from part (a) and subtract $t-s$ from part (b). The repeat distance is $(2s+t)-(t-s) = 3s$. There is a one-sided boundary of size $(t-s)+s = t$ from one side of part (b).

∎

Lemma 6.3 *Assume $[w]$ has a proper 3-coloring where s is a forbidden distance and r is repeat distance under two-sided boundary condition b. If $r|s$ then*

$$w \le s + 2b + 1 \ .$$

Proof: Assume $w > s+2b+1$. Assume, without loss of generality, that $b+1$ is R. Then, by Lemma 6.2b, $r+b+1$, $2r+b+1$, ..., $s+b+1$ are also R, since $b+1 > b$ and $(s+b+1)+b = s+2b+1 \le n$. But s is a forbidden distance so $b+1$ and $s+b+1$ cannot both be R. Contradiction. ∎

We give an example of the using Lemma 6.3 to get an upper bound on a set of poly Van der Warden number. For one of them we have an exact value.

Theorem 6.4

1. *For $a, b > 0$ and $a|b$, $W(ax^2+bx;3) \le 72b^2/a+1$.*

2. *$W(x^2+x;3) = 73$.*

Proof: 1) Let $p(x) = ax^2 + bx$. Let

$$x = 5b/a, \qquad y = 6b/a, \qquad z = 8b/a \ .$$

Then

$$p(x) = 30b^2/a, \qquad p(y) = 42b^2/a, \qquad p(z) = 72b^2/a \ .$$

Since $p(x) + p(y) = p(z)$, by Lemma 6.2b, $p(y) - p(x) = 12b^2/a$ is a repeat distance under two-sided boundary condition $30b^2/a$. But $p(3b/a) = 12b^2/a$ is a forbidden distance. Thus, by Lemma 6.3, $W(ax^2 + bx; 3) \le 12b^2/a + 2 \cdot 30b^2/a + 1 = 72b^2/a + 1$.
2) By Part 1 $W(x^2 + x; 3) \le 73$. We show $W(x^2 + x; 3) \ge 73$ by giving a $(x^2 + x; 3)$-proper coloring of [72].

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
R	R	G	G	R	R	B	B	R	R	B	B	G	G	B	B	G	G

19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
R	R	G	G	R	R	B	B	R	R	B	B	G	G	B	B	G	G

37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54
R	R	G	G	R	R	B	B	R	R	B	B	G	G	B	B	G	G

55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
R	R	G	G	R	R	B	B	R	R	B	B	G	G	B	B	G	G

∎

Theorem 6.5 *Let $p(x) = ax^2 + bx$. Then $W(p(x); 3) \le O(|a^5b^2|)$.*

Proof: We prove the theorem for $a, b \ge 0$. The other cases are similar.
Let

$$x_0 = (2a+1)b, \qquad y_0 = (2a^2 + 2a + 1)b, \qquad z_0 = (2a^2 + 2a + 2)b \ .$$

Then

$$\begin{aligned}
p(x_0) &= (4a^3 + 4a^2 + 3a + 1)b^2 \\
p(y_0) &= (4a^5 + 8a^4 + 8a^3 + 6a^2 + 3a + 1)b^2 \\
p(z_0) &= (4a^5 + 8a^4 + 12a^3 + 10a^2 + 6a + 2)b^2
\end{aligned}$$

Thus $p(x_0) + p(y_0) = p(z_0)$. By Lemma 6.2b, $2p(x_0) + p(y)$ is a repeat distance, and by Lemma 6.2c, $3p(x_0)$ is a repeat distance under one-sided boundary condition $p(y)$. By a Euclid's algorithm-type argument (see Lemma A.1 in Appendix A) it is easy to check that $\gcd(2p(x_0)+p(y_0), 3p(x_0)) = db^2$ for some constant d (independent of a, b). Thus there exist integers j, k such that $j((2p(x_0)+p(y_0))+k(3p(x_0)) = db^2$. By starting at 1 and adding repeat distance $2p(x_0) + p(y_0)$ j times and subtracting repeat distance $3p(x_0)$ k times, we see that db^2 is also a repeat distance. Furthermore, by interspersing the adds and subtracts so that we subtract whenever the sum is greater than $2p(x_0)+p(y_0)$, the one-sided boundary condition is $(2p(x_0)+p(y_0))+p(y_0) = 2(p(x_0)+p(y_0))$. Thus for any integer α, αdb^2 is a repeat distance with the one-sided boundary condition $2(p(x_0)+p(y_0)) = O(a^5b^2)$. But $p(db) = ad^2b^2+b^2d = (ad+1)db^2$ is a forbidden distance. So, $W(p(x); 3) \leq p(db) + 2(p(x_0) + p(y_0)) = O(a^5b^2)$. ∎

Appendix B gives a table of some exact polynomial van der Waerden numbers for quadratic polynomials and three colors.

7 Upper Bounds on $W(x^2; 4)$

Recall that Figure 1 was the key to showing $W(x^2; 3) \leq 1682$. We now derive parameters for a new figure that will be the key to an upper bound on $W(x^2; 4)$.

We need to find $a, b, c, d, e, f, x, y, z \in \mathbb{N}^+$ such that the following figure can be drawn:

Hence we need to find solutions in $\mathbb{N}^+$ to the following system of equations:

$$\begin{gathered}
x^2 + a^2 = y^2 \\
x^2 + b^2 = z^2 \\
y^2 + c^2 = z^2 \\
x^2 + d^2 = w \\
y^2 + e^2 = w \\
z^2 + f^2 = w
\end{gathered}$$

Each equation is a Pythagorean triple, for which we have a known formula with parameters k, m, n where $m > n$, and m, n are coprime but not both odd; we can use the Farey sequence as an efficient algorithm to generate coprime pairs m, n. We used a computer

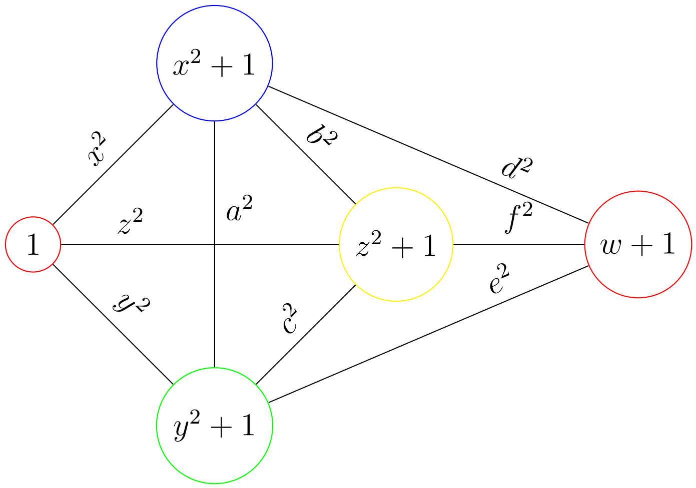

Figure 3: In any $(x^2;4)$-proper coloring, $\mathrm{COL}(1) = \mathrm{COL}(1+w)$

program and obtained the following:

Theorem 7.1 $PW(4, \{x^2\}) \le 1 + (290{,}085{,}289)^2 = 84{,}149{,}474{,}894{,}213{,}522$

Proof:

Assume, by way of contradiction, that COL is an $(x^2;4)$-proper coloring of $[1+(290{,}085{,}289)^2]$ By Figure 4 we know that

$$\mathrm{COL}(1) = \mathrm{COL}(1 + 290{,}085{,}289)^2).$$

More generally we have shown that, for all x,

$$\mathrm{COL}(x) = \mathrm{COL}(x + 290{,}085{,}289)^2).$$

Hence

$$\mathrm{COL}(1) = \mathrm{COL}(1 + 290{,}085{,}289)^2) = \cdots = \mathrm{COL}(1 + (290{,}085{,}289)^2).$$

This contradicts COL being an $(x^2;4)$-proper coloring. ▮

To find upper bounds on $W(Ax^2 + Bx; 4)$ we have several overlapping equations of the form $(Ax^2 + Bx) + (Ay^2 + By) = (Az^2 + Bz)$. We need a way to generate such triples (x, y, z) much like the generation of Pythagorean triples. First, we use the quadratic formula to express z in terms of x and y.

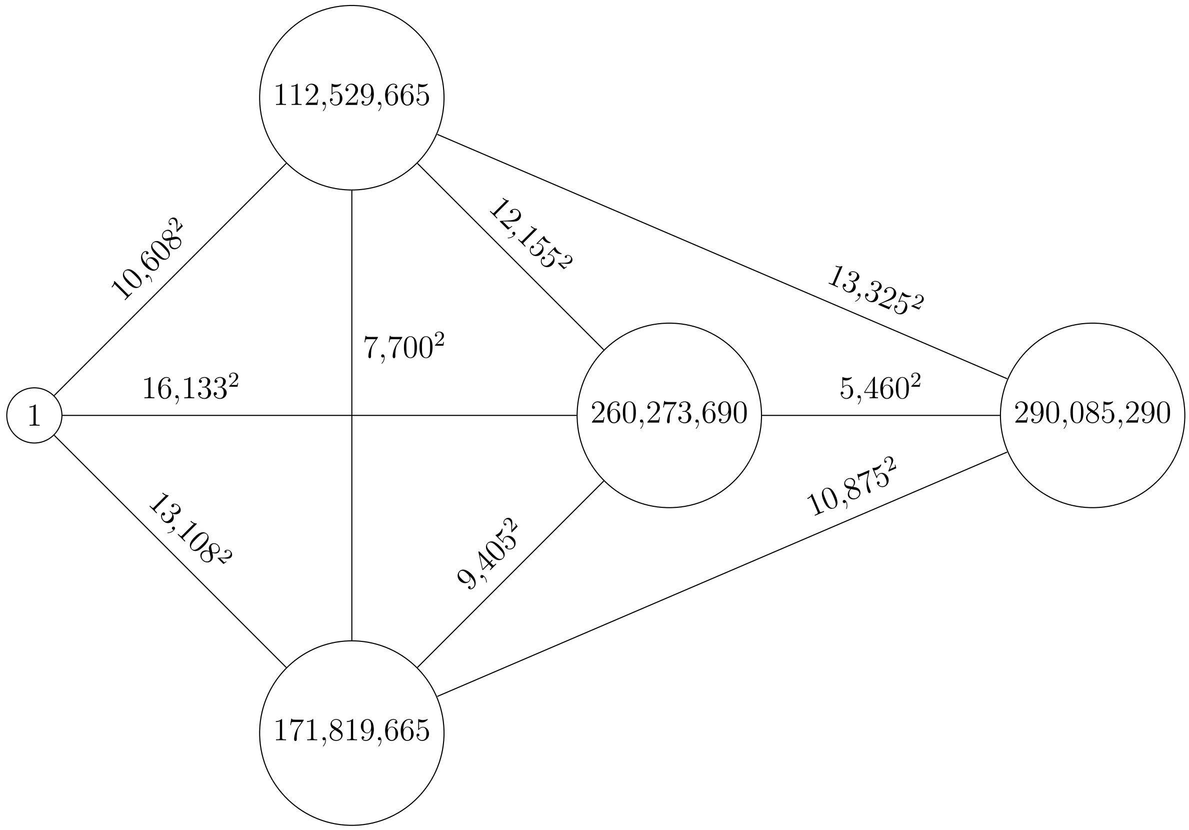

Figure 4: In any $(x^2; 4)$-proper coloring, $\mathrm{COL}(1) = \mathrm{COL}(1 + 290,085,290)$

$$z = f(x, y) = \frac{-B + \sqrt{4A^2(x^2 + y^2) + 4AB(x + y) + B^2}}{2A}$$

This equality holds in general, but we only want values with $z \in \mathbb{N}$.

z is an integer *iff* $4A^2(x^2 + y^2) + 4AB(x + y) + B^2 = (2Az + B)^2$ is an odd square. This can be factored into $(2Ax + B)^2 + (2Ay + B)^2 = (2Az + B)^2 + B^2$, or $m^2 + n^2 = k^2 + B^2$ with some constraints on m, n, k.

A parameterizations of $m^2 + n^2 = k^2 + B^2$ would imply one for (x, y, z), and luckily this equation is easier. Using the *Bramagupta-Fibonacci identity* with $bc - ad = B$, we get:

$$(ac - bd)^2 + (ad + bc)^2 = (ac + bd)^2 + B^2$$

So, with parameters a, b, c, d and constraints $bc - ad = B$, $ac - bd > B$, $2A | ac - bd - B, ad + bc - B$, we have:

$$x = \frac{ac - bd - B}{2A},\ y = \frac{ad + bc - B}{2A},\ z = \frac{ac + bd - B}{2A}$$

Rather than searching all (a, b, c, d), we can eliminate parts of the parameter space that do not contain solutions. With fixed a and d, the first constraint implies that bc is some factorization of $ad + B$. We can pre-compute a table of factorizations and use that to cut the search space down to almost $O(n^2)$. You can see the code for this on GitHub at `https://github.com/zaprice/polyvdw`

We can get bounds for $PW(4, \{x^2 + Bx\})$ with this method with rather large values of B, but only a few bounds for the more general $Ax^2 + Bx$ case; if such configurations exist, it seems the numbers involved are much larger. See Appendix D for some of the upper bounds we have. We note two things about these upper bounds:

1. The largest upper bound on $W(x^2 + Bx; 4)$ that we found was when $B = 0$. Note that these are just the upper bounds we found. It is not clear how the real values compares.

2. For $W(2x^2 + Bx; 4)$ and $W(3x^2 + Bx; 4)$ the B for which we could find an upper bound seem scattered and arbitrary. For example we were not able to find an upper bound for any of $W(2x^2 + Bx; 4)$ for $0 \le B \le 56$, but were able to for 57. And then not again until $B = 95$. Again, this may be a limit to our methods and not a statement about the actual values of $W(2x^2 + Bx; 4)$.

A GCD calculations

Lemma A.1 *Let*

$$p(x) = (4a^3 + 4a^2 + 3a + 1)b^2 \qquad \textit{and} \qquad p(y) = (4a^5 + 8a^4 + 8a^3 + 6a^2 + 3a + 1)b^2$$

Then $\gcd(2p(x) + p(y), 3p(x)) = db^2$ *for some* $d \le 34560$.

Proof:

$$2p(x) + p(y) = (4a^5 + 8a^4 + 16a^3 + 14a^2 + 9a + 3)b^2 \qquad \text{and} \qquad 3p(x) = 3(4a^3 + 4a^2 + 3a + 1)b^2$$

They both have a common factor of b^2, so we can drop that term and put it back in at the end. We will use a Euclid's algorithm type argument along with the facts that

$$\gcd(ma, b) \le m \gcd(a, b) \qquad \text{and} \qquad \gcd(a, b) \le \gcd(ma, b)$$

Divide out 3 from $3p(x)$ this gives a starting point of

$$4a^5 + 8a^4 + 16a^3 + 14a^2 + 9a + 3 \qquad \text{and} \qquad 4a^3 + 4a^2 + 3a + 1$$

with $3b^2$ to be multiplied back in later.

Subtract a^2 times the second from the first:

$$4a^4 + 13a^3 + 3a^2 + 9a + 3 \qquad \text{and} \qquad 4a^3 + 4a^2 + 3a + 1$$

Subtract a times the second from the first:

$$9a^3 + 10a^2 + 8a + 3 \qquad \text{and} \qquad 4a^3 + 4a^2 + 3a + 1$$

Subtract 2 times the second from the first:

$$a^3 + 2a^2 + 2a + 1 \qquad \text{and} \qquad 4a^3 + 4a^2 + 3a + 1$$

Subtract 4 times the first from the second:

$$a^3 + 2a^2 + 2a + 1 \qquad \text{and} \qquad -4a^2 - 5a - 3$$

Multiply the first by 4 and the second by -1:

$$4a^3 + 8a^2 + 8a + 4 \qquad \text{and} \qquad 4a^2 + 5a + 3$$

Subtract a times the second from the first:

$$3a^2 + 5a + 4 \qquad \text{and} \qquad 4a^2 + 5a + 3$$

Subtract the first from the second:

$$3a^2 + 5a + 4 \qquad \text{and} \qquad a^2 - 1$$

Subtract 3 times the second from the first:

$$5a + 7 \qquad \text{and} \qquad a^2 - 1$$

Multiply the second by 5:

$$5a + 7 \qquad \text{and} \qquad 5a^2 - 5$$

Subtract a times the first from the second:

$$5a + 7 \qquad \text{and} \qquad -7a - 5$$

Add the first to the second:

$$5a + 7 \qquad \text{and} \qquad -2a + 2$$

Add two times the second to the first:

$$a + 11 \qquad \text{and} \qquad -2a + 2$$

Add two times the first to the second:

$$a + 11 \qquad \text{and} \qquad 24$$

Multiply the first by 24:

$$24a + 11 \cdot 24 \qquad \text{and} \qquad 24$$

Subtract a times the second from the first:

$$11 \cdot 24 \qquad \text{and} \qquad 24$$

These two numbers have gcd $= 24$. So the total is $24 \cdot 3b^2 = 72b^2$.
Thus the gcd $\le db^2$ for some natural number $d \le 72$.

$\blacksquare$

B Some Exact Values of $W(ax^2 + bx; 2)$

Chart of $W(p(x); 2)$ for $p(x) = ax^2 + bx$ for $0 \le a \le 10$ and $-10 \le b \le 10$.
The values for $a, b \ge 0$ were obtained from our formulas.

		a										
		0	1	2	3	4	5	6	7	8	9	10
	-10	21	1	1	9	9	1	25	11	13	17	1
	-9	19	1	9	1	7	5	7	37	15	1	23
	-8	17	1	1	7	1	7	9	13	1	21	25
	-7	15	1	7	5	5	25	11	1	19	61	29
	-6	13	1	1	1	5	9	1	17	21	25	73
	-5	11	1	5	13	7	1	15	49	25	29	31
	-4	9	1	1	5	1	13	17	23	25	33	37
	-3	7	1	3	1	11	37	19	25	31	73	41
	-2	5	1	1	9	13	19	49	29	33	39	41
	-1	3	1	7	25	17	21	27	61	37	41	47
b	0	1	5	9	13	17	21	25	29	33	37	41
	1	3	13	13	17	23	49	33	37	43	85	53
	2	5	11	25	21	25	31	33	41	45	51	97
	3	7	13	19	37	29	33	37	73	49	49	59
	4	9	17	21	27	49	37	41	47	49	57	61
	5	11	25	25	29	35	61	45	49	55	97	61
	6	13	23	25	31	37	43	73	53	57	61	65
	7	15	25	31	49	41	45	51	85	61	65	71
	8	17	29	33	39	41	49	53	59	97	69	73
	9	19	37	37	37	47	73	55	61	67	109	77
	10	21	35	49	45	49	51	57	65	69	75	121

The numbers tend to increase with increasing a and $|b|$. When $a = b$ the values tend to be large; this is because neither $p(1)/g$ nor $p(2)/g$ is even so $W(p(x); 2) = p(3) + 1$, which is somewhat larger than $p(1) + p(2) - g + 1$ (the other possibility).

C Some Exact Values of $W(ax^2 + bx; 3)$

Chart of $W(p(x); 3)$ for $p(x) = ax^2 + bx$ for $0 \leq a \leq 5$ and $-5 \leq b \leq 5$.
The values were obtained by computer.

		a					
		0	1	2	3	4	5
b	-5	16	1	64	61	217	1
	-4	13	1	1	91	1	289
	-3	10	1	10	1	135	171
	-2	7	1	1	68	97	171
	-1	4	1	49	105	190	183
	0	1	29	57	85	113	141
	1	4	73	76	65	156	253
	2	7	64	145	123	151	?
	3	10	37	95	217	?	?
	4	13	65	127	?	289	?
	5	16	55	?	109	?	361

D Some Upper Bounds on $W(ax^2 + bx; 4)$

We give bounds for $W(g(x); 4)$ where g is of the form $Ax^2 + Bx$. Only bounds for coprime coefficients (A, B) are presented. Each row of he table gives g, x, y, z, w (as in We give three such tables.

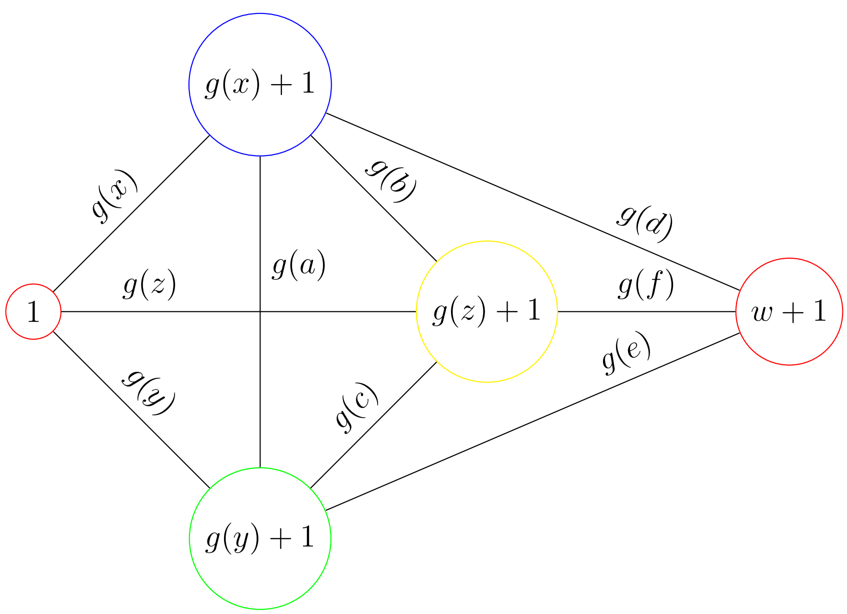

Figure 5: In any $(g(x); ; 4)$-proper coloring, $\mathrm{COL}(1) = \mathrm{COL}(1 + w)$

Table for $x^2 + Bx$ where $0 \leq B \leq 20$.

g	x	y	z	w	$W(g(x);4) \leq$
x^2	10,608	13,108	16,133	290,085,289	84,149,474,894,213,522
$x^2 + x$	299	302	327	113,262	12,828,393,907
$x^2 + 2x$	91	127	211	257,463	66,287,711,296
$x^2 + 3x$	35	43	53	3,308	10,952,789
$x^2 + 4x$	80	84	92	10,197	104,019,598
$x^2 + 5x$	70	81	100	11,250	126,618,751
$x^2 + 6x$	70	86	106	13,232	175,165,217
$x^2 + 7x$	638	785	923	988,338	976,818,920,611
$x^2 + 8x$	160	168	184	40,788	1,663,987,249
$x^2 + 9x$	35	37	44	3,242	10,539,743
$x^2 + 10x$	144	150	165	36,075	1,301,766,376
$x^2 + 11x$	364	472	727	1,263,252	1,595,819,511,277
$x^2 + 12x$	140	172	212	52,928	2,802,008,321
$x^2 + 13x$	119	129	143	38,016	1,445,710,465
$x^2 + 14x$	66	96	135	25,395	645,261,556
$x^2 + 15x$	120	138	215	54,364	2,956,259,957
$x^2 + 16x$	75	99	141	45,177	2,041,684,162
$x^2 + 17x$	123	165	255	232,908	54,250,095,901
$x^2 + 18x$	70	74	88	12,968	168,402,449
$x^2 + 19x$	65	66	69	6,852	47,080,093
$x^2 + 20x$	84	96	115	24,261	589,081,342

Table for $x^2 + Bx$ where $1980 \leq B \leq 2000$.

g	x	y	z	w	$W(g(x); 4) \leq$
$x^2 + 1{,}980x$	1,683	2,145	2,915	25,524,829	651,567,434,640,662
$x^2 + 1{,}981x$	1,674	1,735	2,026	14,236,652	202,710,462,976,717
$x^2 + 1{,}982x$	1,248	1,495	1,731	6,882,723	47,385,517,451,716
$x^2 + 1{,}983x$	3,498	3,549	3,664	24,967,678	623,434,455,617,159
$x^2 + 1{,}984x$	860	975	2,585	12,424,497	154,392,775,905,058
$x^2 + 1{,}985x$	867	1,098	2,365	11,200,200	125,466,712,437,001
$x^2 + 1{,}986x$	1,900	2,432	2,908	19,712,552	388,623,855,480,977
$x^2 + 1{,}987x$	3,048	3,393	3,987	39,165,018	1,533,976,455,831,091
$x^2 + 1{,}988x$	508	738	1,194	6,489,996	42,132,950,192,065
$x^2 + 1{,}989x$	2,023	2,288	3,094	18,950,528	359,160,204,078,977
$x^2 + 1{,}990x$	1,364	1,610	2,100	13,163,856	173,313,300,862,177
$x^2 + 1{,}991x$	1,330	1,519	1,814	7,817,030	61,121,521,727,631
$x^2 + 1{,}992x$	975	1,065	1,871	10,120,498	102,444,639,800,021
$x^2 + 1{,}993x$	1,985	2,349	4,373	68,596,488	4,705,614,878,734,729
$x^2 + 1{,}994x$	1,246	1,350	1,716	8,551,440	73,144,177,644,961
$x^2 + 1{,}995x$	891	1,185	1,464	10,543,450	111,185,372,085,251
$x^2 + 1{,}996x$	705	995	1,793	7,390,317	54,631,536,433,222
$x^2 + 1{,}997x$	1,081	1,136	1,391	8,040,026	64,658,074,012,599
$x^2 + 1{,}998x$	1,292	1,732	3,704	39,649,768	1,572,183,322,690,289
$x^2 + 1{,}999x$	1,235	1,757	2,789	14,633,322	214,163,364,766,363
$x^2 + 2{,}000x$	184	280	984	5,592,000	31,281,648,000,001

Table for $2x^2 + Bx$ for assorted B.

g	x	y	z	w	$W(g(x);4) \leq$
$2x^2 + 57x$	3,969	4,035	4,295	38,199,155	2,918,353,062,779,886
$2x^2 + 95x$	707	758	1,008	14,365,638	412,744,475,029,699
$2x^2 + 171x$	11,907	12,105	12,885	343,792,395	236,386,480,508,171,596
$2x^2 + 285x$	2,121	2,274	3,024	129,290,742	33,432,228,781,682,599
$2x^2 + 399x$	27,783	28,245	30,065	1,871,758,595	7,006,961,222,744,427,456
$2x^2 + 455x$	3,320	3,663	4,170	39,229,128	3,077,866,816,534,009
$2x^2 + 511x$	2,772	3,367	6,282	131,899,720	34,795,139,672,913,721
$2x^2 + 627x$	43,659	44,385	47,245	4,622,097,755	5,834,090,064,188,269,204
$2x^2 + 805x$	1,210	1,303	2,920	87,446,025	15,293,684,970,651,376
$2x^2 + 855x$	5,548	7,087	13,262	530,042,423	561,890,393,545,693,524
$2x^2 + 1{,}011x$	5,164	6,568	9,889	318,517,859	202,907,575,025,443,212
$2x^2 + 1{,}153x$	12,705	12,726	12,970	352,488,525	248,496,726,932,620,576
$2x^2 + 1{,}199x$	8,245	8,710	9,748	221,108,291	97,778,017,806,722,272
$2x^2 + 1{,}295x$	14,030	14,355	22,244	1,162,712,925	2,703,804,197,637,349,126
$2x^2 + 1{,}301x$	25,622	26,105	28,172	1,638,880,116	5,371,858,201,423,377,829
$2x^2 + 1{,}365x$	9,960	10,989	12,510	353,062,152	249,306,248,279,579,689
$2x^2 + 1{,}459x$	954	1,174	1,379	58,465,486	6,836,511,407,576,467
$2x^2 + 1{,}545x$	11,298	11,815	12,860	425,440,418	361,999,755,841,475,259
$2x^2 + 1{,}685x$	10,695	10,968	11,570	289,144,125	167,209,137,251,881,876
$2x^2 + 1{,}753x$	3,586	5,236	8,232	181,967,394	66,224,583,947,144,155
$2x^2 + 1{,}851x$	50,031	51,441	55,164	6,379,649,159	7,612,882,297,751,201,408
$2x^2 + 1{,}913x$	2,261	3,366	5,324	81,424,299	13,259,988,699,966,790

Table for $3x^2 + Bx$ for assorted B.

g	x	y	z	w	$W(g(x);4) \leq$
$3x^2 + x$	42,273	42,660	43,375	5,738,872,934	6,570,267,294,984,419,923
$3x^2 + 143x$	13,244	13,332	13,442	554,651,696	922,915,590,942,221,777
$3x^2 + 172x$	4,452	4,712	5,189	88,862,311	23,689,546,233,099,656
$3x^2 + 200x$	1,896	2,204	5,004	115,177,723	39,797,746,661,938,788
$3x^2 + 235x$	11,155	11,270	11,610	583,594,418	1,021,747,471,306,964,403
$3x^2 + 274x$	9,322	11,610	16,903	1,125,018,929	3,797,003,080,080,107,670
$3x^2 + 344x$	8,904	9,424	10,378	355,449,244	379,032,617,455,054,545
$3x^2 + 361x$	3,540	4,658	7,703	397,333,094	473,620,906,200,085,443
$3x^2 + 400x$	3,792	4,408	10,008	460,710,892	636,763,762,306,663,793
$3x^2 + 407x$	2,806	3,401	6,131	122,898,626	45,312,266,837,804,411
$3x^2 + 412x$	2,077	2,829	5,839	392,773,686	462,813,667,064,838,421
$3x^2 + 520x$	7,616	9,244	12,716	515,261,395	796,483,183,467,963,476
$3x^2 + 556x$	9,400	9,408	9,451	273,674,799	224,693,838,986,259,448
$3x^2 + 592x$	15,744	16,472	17,944	994,061,387	2,964,474,711,857,432,412
$3x^2 + 643x$	50,932	51,357	52,351	8,273,167,696	2,421,731,687,255,606,001
$3x^2 + 688x$	17,808	18,848	20,756	1,421,796,976	6,064,520,901,084,553,217
$3x^2 + 725x$	3,172	3,185	3,278	34,869,750	3,647,723,675,756,251
$3x^2 + 728x$	16,744	17,360	18,928	1,174,742,491	4,140,060,615,695,188,692
$3x^2 + 797x$	2,847	3,082	3,524	148,907,272	66,520,245,642,541,737
$3x^2 + 814x$	5,612	6,802	12,262	491,594,504	724,995,869,246,944,305
$3x^2 + 932x$	1,820	2,229	2,799	37,745,311	4,274,160,686,090,016
$3x^2 + 1{,}085x$	1,190	1,344	1,540	10,401,450	324,581,771,880,751
$3x^2 + 1{,}087x$	9,800	9,909	11,434	604,108,526	1,094,841,990,223,645,791
$3x^2 + 1{,}112x$	18,800	18,816	18,902	1,094,699,196	3,595,100,206,474,645,201

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