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ON TRIPLES IN ARITHMETIC PROGRESSION

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0 Summary

A well-known theorem of K. Roth [R] assures us that for any fixed $\delta > 0$, $N \in \mathbb{Z}_+$ sufficiently large and $A \subset \{1, 2, \dots, N\}$,

$$|A| > \delta N\,, \tag{0.1}$$

there are always 3 distinct elements $n_1, n_2, n_3 \in A$ in arithmetic progression

$$n_1 + n_2 = 2n_3\,. \tag{0.2}$$

His argument yields the density condition

$$\delta > c\frac{1}{\log\log N}\,. \tag{0.3}$$

More recently, it was shown by E. Szemerédi and D. Heath-Brown (see [H] for details) that (0.3) may be replaced by the condition

$$\delta > \frac{1}{(\log N)^c} \tag{0.4}$$

for some (small) constant $c > 0$; Szemerédi produced an explicit value $c = 1/20$. Previous arguments are based on the circle method and a comparison of the integrals

$$\delta(A)^3 \int_{\mathbb{T}} S(x)^2 S(-2x)dx \quad \text{and} \quad \int_{\mathbb{T}} S_A(x)^2 S_A(-2x)dx\,, \tag{0.5}$$

where

$$S(x) = \sum_{n=1}^{N} e^{2\pi i n x} \tag{0.6}$$

$$S_A(x) = \sum_{N\in A} e^{2\pi i n x} \tag{0.7}$$

$$\delta(A) = \frac{|A|}{N}\,. \tag{0.8}$$

The main point is the fact that if $\|S_A - \delta(A)S\|_\infty$ is large, i.e.

$$\big\|S_A - \delta(A)S\big\|_\infty > \gamma N \tag{0.9}$$

then there is a density increment of A in some arithmetic progression $P \subset \{1, \dots, N\}$

$$\frac{|A \cap P|}{|P|} > \delta(A) + 0(\gamma)\,. \tag{0.10}$$

The key additional idea in the work of Szemerédi and Heath-Brown was to consider the contribution in (0.10) of sets of points $\{\theta_1, \dots, \theta_J\} \subset \mathbb{T}$ rather than a single point.

In this paper, we prove the existence of nontrivial triples in progression under the density assumption in (0.1)

$$\delta > c \left(\frac{\log \log N}{\log N} \right)^{1/2} . \tag{0.11}$$

Again we rely on the circle method but instead of considering arithmetic progressions, we aim to increase the density of A in consecutive "Bohr sets" of the form

$$\Lambda = \Lambda_{\theta,\varepsilon,M} = \big\{ n \in \mathbb{Z} \bigm| |n| \le M \quad \text{and} \quad \|n\theta_j\| < \varepsilon \text{ for } j = 1, \dots, d \big\} \tag{0.12}$$

where $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{T}^d$. This procedure turns out to be more economical than dealing with progressions. Given Λ, we introduce a probability measure λ on $\mathbb{Z}$ defined by

$$\lambda = \tfrac{1}{|\Lambda|} \mathbb{1}_\Lambda \, . \tag{0.13}$$

Our starting point is then to compare

$$\lambda'(A)^2 \lambda''(A) \int_{\mathbb{T}} S'(x)^2 S''(-2x) dx \tag{0.14}$$

and

$$\int_{\mathbb{T}} S'_A(x)^2 S''_A(-2x) dx \tag{0.15}$$

where

$$S'(x) = \sum \lambda'_n \, e^{2\pi i n x} \tag{0.16}$$

$$S''(x) = \sum \lambda''_n \, e^{2\pi i n x} \tag{0.17}$$

$$S'_A(x) = \sum_{n \in A} \lambda'_n \, e^{2\pi i n x} \tag{0.18}$$

$$S''_A(x) = \sum_{n \in A} \lambda''_n \, e^{2\pi i n x} \tag{0.19}$$

$$\lambda'(A) = \sum_{n \in A} \lambda'_n, \lambda''(A) = \sum_{n \in A} \lambda''_n \, . \tag{0.20}$$

Here λ', λ'' are associated by (0.13) to respective Bohr sets Λ', Λ'' and assumed constructed such that

$$\lambda' * \lambda''' \approx \lambda' \, , \tag{0.21}$$

when λ''' is defined by

$$\begin{cases} \lambda'''_n = \lambda''_{\frac{n}{2}} & \text{if } \ n \in 2\mathbb{Z} \\ \qquad = 0 & \text{otherwise} \, . \end{cases}$$

Thus (0.21) ensures that

$$(0.14) \approx \lambda'(A)^2\lambda''(A)\Big[\sum(\lambda'_n)^2\Big] = \lambda'(A)^2\lambda''(A)\|\lambda'\|_2^2\,. \tag{0.22}$$

On the other hand, assuming A does not contain a nontrivial triple in progression,

$$(0.15) = \sum_{n\in A}(\lambda'_n)^2\lambda''_n \le \tfrac{1}{|\Lambda''|}\|\lambda'\|_2^2\,. \tag{0.23}$$

One then proceeds again in analyzing the difference $|(0.14)-(0.15)|$ and the differences $S'_A - \lambda'(A)S'$ and $S''_A - \lambda''(A)S''$ in order to increase the density $\tilde\lambda(A)$, $\tilde\lambda = \frac{1}{|\tilde\Lambda|}\mathbb{1}_{\tilde\Lambda}$ for some smaller Bohr set $\tilde\Lambda$.

Recall, in the other direction, Behrend's result [B], according to which there are sets $A = A_N \subset \{1,\dots,N\}$ for arbitrary N, without triples in progression and satisfying

$$\tfrac{|A_N|}{N} > \exp\big(-C\sqrt{\log N}\big)\,. \tag{0.24}$$

1 Definitions

Let $\theta\in\mathbb{R}^d$, $d\ge 1$, $\varepsilon>0$, N a positive integer. Denote

$$\Lambda_{\theta,\varepsilon,N} = \big\{n\in\mathbb{Z}\;\big|\;|n|\le N,\ \|n\theta_j\|<\varepsilon \text{ for } j=1,\dots,d\big\} \tag{1.1}$$

and $\lambda_{\theta,\varepsilon,N}=\lambda$ where

$$\lambda(n) = \begin{cases} |\Lambda_{\theta,\varepsilon,N}|^{-1} & \text{if } n\in\Lambda_{\theta,\varepsilon,N}\\ 0 & \text{otherwise}\,. \end{cases} \tag{1.2}$$

Thus λ is probability measure on $\mathbb{Z}$.

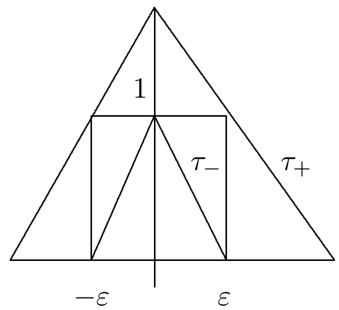
2 Estimates on Bohr Sets

Lemma 2.0.

$$\text{(i)}\qquad |\Lambda_{\theta,\varepsilon,N}| > \tfrac12\varepsilon^d N \tag{2.1}$$

$$\text{(ii)}\qquad |\Lambda_{\theta,\varepsilon,N}| < 8^{d+1}|\Lambda_{\theta,\frac{\varepsilon}{2},\frac{N}{2}}|\,. \tag{2.2}$$

Proof. Consider functions



Thus

$$\sum_{|n|<N}\left(1-\frac{|n|}{N}\right)\prod_{j=1}^{d}\tau_-(n\theta_j) < |\Lambda_{\theta,\varepsilon,N}| < 2\sum_{|n|<2N}\left(1-\frac{|n|}{2N}\right)\prod_{j=1}^{d}\tau_+(n\theta_j) \tag{2.3}$$

and

$$\sum_{|n|<N}\left(1-\frac{|n|}{N}\right)\prod_{j=1}^{d}\tau_-(n\theta_j) = \sum_{k\in\mathbb{Z}^d}\prod_{j=1}^{d}\widehat{\tau}_-(k_j)F_N(k.\theta)$$

$$= \sum_{k\in\mathbb{Z}^d}\prod_{j=1}^{d}\frac{\sin^2\pi\varepsilon k_j}{\varepsilon\pi^2 k_j^2}F_N(k.\theta) \tag{2.4}$$

$$2\sum_{|n|<2N}\left(1-\frac{|n|}{2N}\right)\prod_{j=1}^{d}\tau_+(n\theta_j) = 2\sum_{k\in\mathbb{Z}^d}\prod_{j=1}^{d}\frac{\sin^2 2\pi\varepsilon k_j}{\varepsilon\pi^2 k_j^2}F_{2N}(k.\theta)\,. \tag{2.5}$$

Clearly, from $k=0$ contribution and positivity

$$(2.4) > \varepsilon^d F_N(0) = \tfrac{1}{2}\varepsilon^d N \tag{2.6}$$

implying (2.1).

Since

$$F_{2N}(x) \le 4F_{N/2}(x)$$
$$\sin^2 2x = 4\sin^2 x\cos^2 x \le 4\sin^2 x \le 16\sin^2\tfrac{x}{2}\,,$$

it follows that

$$(2.5) \le 8^{d+1}\sum_{k\in\mathbb{Z}^d}\prod_{j=1}^{d}\frac{\sin^2\pi\frac{\varepsilon}{2}k_j}{\frac{\varepsilon}{2}\pi^2 k_j^2}F_{\frac{N}{2}}(k.\theta) \tag{2.7}$$

$$\le 8^{d+1}|\Lambda_{\theta,\frac{\varepsilon}{2},\frac{N}{2}}|\,, \tag{2.8}$$

proving (2.2).

3 Regular Values of (ε, N)

LEMMA 3.0. *For given* (ε, N)*, there are*

$$\tfrac{\varepsilon}{2} < \varepsilon_1 < \varepsilon \tag{3.1}$$

$$\tfrac{N}{2} < N_1 < N \tag{3.2}$$

such that for $0 < \kappa < 1$

$$1 - \kappa < \frac{|\Lambda_{\theta,\varepsilon_2,N_2}|}{|\Lambda_{\theta,\varepsilon_1,N_1}|} < 1 + \kappa \tag{3.3}$$

if

$$|\varepsilon_1 - \varepsilon_2| < \tfrac{1}{100}\tfrac{\kappa}{d}\varepsilon_1 \tag{3.4}$$

and

$$|N_1 - N_2| < \tfrac{1}{100}\tfrac{\kappa}{d}N_1\,. \tag{3.5}$$

Proof. Assume for each $t \in [1/2, 1]$ there is $\kappa = \kappa(t) \lesssim 1$ such that

$$\begin{aligned}&\big|\Lambda_{\theta,(1-\frac{1}{100}\frac{\kappa}{d})t\varepsilon,\,(1-\frac{1}{100}\frac{\kappa}{d})tN}\big| \\ &\qquad < (1+\kappa)^{-1}\big|\Lambda_{\theta,(1+\frac{1}{100}\frac{\kappa}{d})t\varepsilon,\,(1+\frac{1}{100}\frac{\kappa}{d})tN}\big|\,.\end{aligned} \tag{3.6}$$

From standard covering argument of $[1/2, 1]$ by collection of intervals we deduce that

$$\begin{aligned}&\frac{|\Lambda_{\theta,\frac{\varepsilon}{4},\frac{N}{4}}|}{|\Lambda_{\theta,2\varepsilon,2N}|} \\ &\qquad \le \prod_\alpha \frac{|\Lambda_{\theta,(1-\frac{1}{100}\frac{\kappa_\alpha}{d})t_\alpha\varepsilon,\,(1-\frac{1}{100}\frac{\kappa_\alpha}{d})t_\alpha N}|}{|\Lambda_{\theta,\,(1+\frac{1}{100}\frac{\kappa_\alpha}{d})t_\alpha\varepsilon,\,(1+\frac{1}{100}\frac{\kappa_\alpha}{d})t_\alpha N}|} \\ &\qquad \le \prod_\alpha (1+\kappa_\alpha)^{-1}\end{aligned} \tag{3.7}$$

where the intervals $\big[\big(1-\frac{1}{100}\frac{\kappa_\alpha}{d}\big)t_\alpha, \big(1+\frac{1}{100}\frac{\kappa_\alpha}{d}\big)t_\alpha\big]$ are disjoint of total measure

$$\tfrac{1}{50d}\sum \kappa_\alpha t_\alpha > \tfrac{1}{4}\,. \tag{3.8}$$

Hence

$$\sum \kappa_\alpha > 12d$$

and

$$\prod (1+\kappa_\alpha) > e^{\frac{2}{3}\sum\kappa_\alpha} > e^{8d}\,. \tag{3.9}$$

On the other hand, (2.2) implies that

$$\frac{|\Lambda_{\theta,\frac{\varepsilon}{4},\frac{N}{4}}|}{|\Lambda_{\theta,2\varepsilon,2N}|} > 8^{-3(d+1)}\,. \tag{3.10}$$

Thus from (3.7), (3.9), (3.10)

$$8^{-3(d+1)} < e^{-8d}\,, \tag{3.11}$$

a contradiction.

Let $t_1 \in [1/2, 1]$ be such that for all $0 \le \kappa \le 1$

$$(1+\kappa)|\Lambda_{\theta,(1-\frac{1}{100}\frac{\kappa}{d})t_1\varepsilon,(1-\frac{1}{100}\frac{\kappa}{d})t_1N}| \ge |\Lambda_{\theta,(1+\frac{1}{100}\frac{\kappa}{d})t_1\varepsilon,(1+\frac{1}{100}\frac{\kappa}{d})t_1N}| \tag{3.12}$$

and take

$$\varepsilon_1 = t_1\varepsilon\,, \quad N_1 = t_1 N\,. \tag{3.13}$$

If (3.4), (3.5) hold, then

$$\Lambda_{\theta,(1-\frac{\kappa}{100d})\varepsilon_1,(1-\frac{\kappa}{100d})N_1} \subset \Lambda_{\theta,\varepsilon_2,N_2} \subset \Lambda_{\theta,(1+\frac{\kappa}{100d})\varepsilon_1,(1+\frac{\kappa}{100d})N_1} \tag{3.14}$$

and by (3.12)

$$\frac{1}{1+\kappa} < \frac{|\Lambda_{\theta,\varepsilon_2,N_2}|}{|\Lambda_{\theta,\varepsilon_1,N_1}|} < 1+\kappa\,. \tag{3.15}$$

This proves the lemma.

Definition. *We call (ε_1, N_1) satisfying Lemma 3.0 regular.*

Lemma 3.16. *Let $\lambda = \lambda_{\theta,\varepsilon,N}$ with (ε, N) regular and $\lambda' = \lambda_{\theta,\frac{\kappa}{100d}\varepsilon,\frac{\kappa}{100d}N}$. Then*

$$\|\lambda * \lambda' - \lambda\|_1 \equiv \|\lambda * \lambda' - \lambda\|_{\ell^1(\mathbb{Z})} < 2\kappa\,. \tag{3.17}$$

Proof. Write

$$(\lambda * \lambda')(n) = \sum_m \lambda'(m)\lambda(n-m)\,.$$

If $(\lambda * \lambda')(n) \neq 0$, then there is m

$$|m| < \tfrac{\kappa}{100d}N\,, \quad |n-m| < N \tag{3.18}$$

such that

$$\|m\theta_j\| < \tfrac{\kappa}{100d}\varepsilon \tag{3.19}$$

$$\big\|(n-m)\theta_j\big\| < \varepsilon\,. \tag{3.20}$$

Hence, from (3.18)-(3.20)

$$|n| < \left(1+\tfrac{\kappa}{100d}\right)N \tag{3.21}$$

$$\|n\theta_j\| < \left(1+\tfrac{\kappa}{100d}\right)\varepsilon \tag{3.22}$$

and

$$n \in \Lambda_{\theta,(1+\frac{\kappa}{100d})\varepsilon,(1+\frac{\kappa}{100d})N}\,. \tag{3.23}$$

Similarly, one sees that if

$$n \in \Lambda_{\theta,(1-\frac{\kappa}{100d})\varepsilon,(1-\frac{\kappa}{100d})N}\,, \tag{3.24}$$

then

$$(\lambda * \lambda')(n) = \tfrac{1}{|\Lambda|} = \lambda(n)\,. \tag{3.25}$$

From the preceding

$$\|\lambda * \lambda' - \lambda\|_1 = \big\|(\lambda * \lambda') - \lambda\big\|_{\ell^1(\Lambda_{\theta,(1+\frac{\kappa}{100d})\varepsilon,(1+\frac{\kappa}{100d})N} \setminus \Lambda_{\theta,(1-\frac{\kappa}{100d})\varepsilon,(1-\frac{\kappa}{100d})N})} \tag{3.26}$$

$$\leq \tfrac{1}{|\Lambda|}\big[|\Lambda_{\theta,(1+\frac{\kappa}{100d})\varepsilon,(1+\frac{\kappa}{100d})N}| - |\Lambda_{\theta,(1-\frac{\kappa}{100d})\varepsilon,(1-\frac{\kappa}{100d})N}|\big] \tag{3.27}$$

$$< 2\kappa\,, \tag{3.28}$$

using Lemma (3.0).

This proves (3.17).

Lemma 3.29. *Under the assumptions of Lemma 3.16, we also have*

$$\|(\lambda * \lambda') - \lambda\|_2 < 2\sqrt{\kappa}\|\lambda\|_2\,. \tag{3.30}$$

Proof. Write by (3.17) and definition of λ, i.e. (1.2)

$$\begin{aligned}\big\|(\lambda * \lambda') - \lambda\big\|_2 &\leq \big\|(\lambda * \lambda') - \lambda\big\|_1^{1/2}\,\big\|(\lambda * \lambda') - \lambda\big\|_\infty^{1/2}\\ &\leq \sqrt{2\kappa}\big(2\|\lambda\|_\infty\big)^{1/2}\\ &= 2\sqrt{\kappa}|\Lambda|^{-1/2}\\ &= 2\sqrt{\kappa}\|\lambda\|_2\,.\end{aligned}$$

4 Estimation of Exponential Sum

Let $\theta \in \mathbb{T}^d$, $\lambda = \lambda_{\theta,\varepsilon,N}$ with (ε, N) regular.

Lemma 4.0. *Assume $x \in \mathbb{T}$ and*

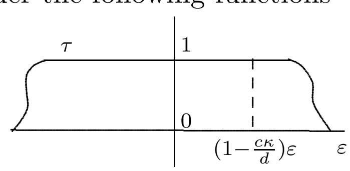
$$\Big|\sum \lambda_n e^{inx}\Big| > \kappa\,. \tag{4.1}$$

Then, there is $k \in \mathbb{Z}^d$ s.t.

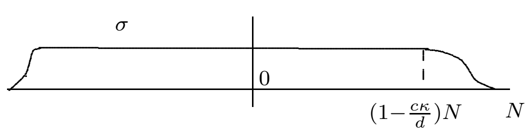
$$|k_j| < Cd^4\kappa^{-2}\left(\log\tfrac{1}{\varepsilon}\right)^2\tfrac{1}{\varepsilon} \tag{4.2}$$

$$\|x - k.\theta\| < Cd^4\kappa^{-2}\left(\log\tfrac{1}{\varepsilon}\right)^2\tfrac{1}{N}\,. \tag{4.3}$$

Proof. Consider the following functions



(τ 1-periodic)



(with c appropriately chosen constant) such that the Fourier transform $\widehat{\tau}, \widehat{\sigma}$ satisfy decay estimate

$$|\widehat{\tau}(k)| < 2\varepsilon \exp\Big(-\big(\tfrac{\kappa\varepsilon}{Cd}|k|\big)^{1/2}\Big) \tag{4.4}$$

$$|\widehat{\sigma}(\lambda)| < 2N \exp\left(-\left(\tfrac{\kappa N}{Cd}|\lambda|\right)^{1/2}\right) . \tag{4.5}$$

Thus

$$\Big|\sum \sigma_n e^{2\pi i n x}\Big| < CN \exp\left(-\left(\tfrac{\kappa N}{Cd}\|x\|\right)^{1/2}\right) . \tag{4.6}$$

Clearly, from definition of τ, σ, we get

$$\Big|\sum \lambda_n e^{2\pi i n x} - \frac{1}{|\Lambda|}\sum \sigma_n \prod_{j=1}^{d} \tau(n\theta_j) e^{2\pi i n x}\Big| < \frac{1}{|\Lambda|}\big(|\Lambda_{\theta,\varepsilon,N}| - |\Lambda_{\theta,(1-\frac{c\kappa}{d})\varepsilon,(1-\frac{c\kappa}{d})N|}\big) < \frac{\kappa}{10} \tag{4.7}$$

for appropriate choice of c (cf. §3).

Thus, if (4.1)

$$\Big|\sum \sigma_n \prod_{j=1}^{d} \tau(n\theta_j) e^{2\pi i n x}\Big| > \frac{\kappa}{2}|\Lambda| > \frac{\kappa}{2}\varepsilon^d N , \tag{4.8}$$

by (2.1). Hence

$$\sum_{k\in\mathbb{Z}^d} \prod |\widehat{\tau}(k_j)| \Big|\sum_n \sigma_n e^{2\pi i n(x+k.\theta)}\Big| > \tfrac{\kappa}{2}\varepsilon^d N , \tag{4.9}$$

and from (4.4), (4.6)

$$\sum_{k\in\mathbb{Z}^d} \exp -\left[\left(\frac{\kappa\varepsilon}{Cd}\right)^{1/2} \sum_{j=1}^{d} |k_j|^{1/2} + \left(\frac{\kappa N}{Cd}\right)^{1/2} \|x + k.\theta\|^{1/2}\right] > c^d \kappa . \tag{4.10}$$

One has

$$\sum_{k\in\mathbb{Z}} \exp\left[-\left(\tfrac{\kappa\varepsilon |k|}{Cd}\right)^{1/2}\right] < \tfrac{Cd}{\kappa\varepsilon} \tag{4.11}$$

$$\sum_{|k|>k_0} \exp\left[-\left(\tfrac{\kappa\varepsilon |k|}{Cd}\right)^{1/2}\right] < \tfrac{Cd}{\kappa\varepsilon} \exp\left[-\tfrac{1}{2}\left(\tfrac{\kappa\varepsilon k_0}{Cd}\right)^{1/2}\right] . \tag{4.12}$$

Split the sum in (4.10) as

$$\sum_{|k_j|<k_0} + \sum_{\max |k_j|>k_0} = (I) + (II) . \tag{4.13}$$

Then, by (4.11)

$$(I) < \left(\tfrac{Cd}{\kappa\varepsilon}\right)^d \max_{|k_j|<k_0} \exp\left(-\left[\tfrac{\kappa N}{Cd}\|x + k\theta\|\right]^{1/2}\right) \tag{4.14}$$

and by (4.12)

$$(II) < d\left(\tfrac{Cd}{\kappa\varepsilon}\right)^d \exp -\tfrac{1}{2}\left(\tfrac{\kappa\varepsilon k_0}{Cd}\right)^{1/2} . \tag{4.15}$$

Take thus

$$k_0 > \tfrac{Cd}{\kappa\varepsilon} d^2 \left(\log \tfrac{Cd}{\kappa\varepsilon}\right)^2 \tag{4.16}$$

to insure that

$$(II) < \tfrac{1}{2} c^d \kappa\,. \tag{4.17}$$

Hence, by (4.10), (4.13), (4.14), (4.17) we get for some $k \in \mathbb{Z}^d$

$$|k_j| < k_0 \qquad (1 \le j \le d) \tag{4.18}$$

that

$$\exp - \left[\tfrac{\kappa N}{Cd} \|x + k\theta\|\right]^{1/2} > \tfrac{1}{2} \left(\tfrac{\kappa\varepsilon}{Cd}\right)^d c^d \kappa \tag{4.19}$$

$$\begin{aligned} \|x + k\theta\| &< \tfrac{Cd}{\kappa N} d^2 \left[\log \tfrac{Cd}{\kappa\varepsilon}\right]^2 \\ &< \tfrac{Cd^4}{\kappa^2 N} \left(\log \tfrac{1}{\varepsilon}\right)^2 . \end{aligned} \tag{4.20}$$

From (4.18), (4.16), (4.20), the conclusion (4.2), (4.3) in Lemma 4.0 clearly follows.

5 Density

Let $A \subset \{1, \dots, N\}$ satisfying

$$|A| > \delta N\,. \tag{5.1}$$

For λ a probability measure on $\mathbb{Z}$, define

$$\lambda(A) = \sum_{n \in A} \lambda_n\,. \tag{5.2}$$

Starting from $\lambda_0 = \frac{1}{2N+1} \mathbb{1}_{\{-N,\dots,N\}}$ and assuming A does not contain a nontrivial triple in progression, we will construct a sequence of probability measures λ of the form $\lambda = \lambda_{\theta,\varepsilon,M}$ for varying $d, \theta \in \mathbb{T}^d, \varepsilon$ and M, such that at each step $\lambda(A')$ will increase by at least $c\lambda(A')^2$ for some translate A' of A. Thus, by (5.1), this leads to a contradiction after at most $\sim \delta^{-1}$ steps.

We agree, when introducing measures of the form $\lambda_{\theta,\varepsilon,M}$, to always assume (ε, M) regular.

The main issue in the argument is then how $d, \theta, \varepsilon, M$ will evolve along the iteration.

Assume for some translate A' of A

$$\lambda(A') = \delta_1 \ge \delta \tag{5.3}$$

where $\lambda = \lambda_{\theta,\varepsilon,M}$.

Fix $\kappa > 0$, to be specified, and define

$$\lambda' = \lambda_{\theta, \frac{c\kappa}{d}\varepsilon, \frac{c\kappa}{d} M} \tag{5.4}$$

$$\lambda'' = \lambda_{\theta, (\frac{c\kappa}{d})^2 \varepsilon, (\frac{c\kappa}{d})^2 M}\,. \tag{5.5}$$

Let λ''' denote the measure

$$\begin{aligned} \lambda'''_n &= \lambda''_{n/2} \quad \text{if} \quad n \in 2\mathbb{Z} \\ &= 0 \qquad \text{otherwise}. \end{aligned} \tag{5.6}$$

Thus

$$\lambda''' = \lambda_{\tilde{\theta}, (\frac{\epsilon\kappa}{d})^2 \varepsilon, 2(\frac{\epsilon\kappa}{d})^2 M} \tag{5.7}$$

where

$$\tilde{\theta} = \tfrac{\theta}{2} \cup \left\{\tfrac{1}{2}\right\} . \tag{5.8}$$

Observe that

$$\Lambda_{\tilde{\theta}, \varepsilon', M'} \subset \Lambda_{\theta, 2\varepsilon', M'} . \tag{5.9}$$

According to Lemma 3.16 and preceding regularity assumption, it follows that

$$\big\|\lambda - (\lambda * \lambda')\big\|_1 < \kappa \tag{5.10}$$

$$\big\|\lambda' - (\lambda' * \lambda'')\big\|_1 < \kappa \tag{5.11}$$

$$\big\|\lambda' - (\lambda' * \lambda''')\big\|_1 < \kappa \tag{5.12}$$

(for appropriate choice of constants c in (5.4), (5.5), (5.7)).

Assume for each $m \in \mathbb{Z}$

$$\big|\lambda'(A' + m) - \lambda(A')\big| > 10\kappa \quad \text{or} \quad |\lambda''(A' + m) - \lambda(A')| > 10\kappa . \tag{5.13}$$

Then, clearly, for either $\lambda^1 = \lambda'$ or $\lambda^1 = \lambda''$

$$\sum \lambda_m \big|\lambda^1(A' - m) - \lambda(A')\big| > 5\kappa . \tag{5.14}$$

Since, by (5.10), (5.11), also

$$\begin{aligned} \Big|\sum \lambda_m \big[\lambda^1(A' - m) - \lambda(A')\big]\Big| &= \big|(\lambda * \lambda^1)(A') - \lambda(A')\big| \\ &< \big\|(\lambda * \lambda^1) - \lambda\big\|_1 \\ &< 3\kappa \end{aligned} \tag{5.15}$$

it follows that for some m

$$\lambda^1(A' + m) > \lambda(A') + \kappa . \tag{5.16}$$

Hence, there is either some translate $A'' = A' + m$ of A satisfying

$$\big|\lambda'(A'') - \lambda(A')\big| < 10\kappa ; \quad \big|\lambda''(A'') - \lambda(A')\big| < 10\kappa , \tag{5.17}$$

or, for some translate $A'' = A' + m$, there is a density increment

$$\lambda'(A'') > \lambda(A') + \kappa \quad \text{or} \quad \lambda''(A'') > \lambda(A') + \kappa . \tag{5.18}$$

In the preceding, we let

$$\kappa = 10^{-8} \delta_1^2 . \tag{5.19}$$

6 Comparison of the Integrals

Assume (5.17) for some translate A'' of A. Following the circle method, consider the sums

$$S' = \sum \lambda'_n\, e^{2\pi inx} \tag{6.1}$$

$$S'_A = \sum_{n\in A''} \lambda'_n\, e^{2\pi inx} \tag{6.2}$$

$$S'' = \sum \lambda''_n e^{2\pi inx} \tag{6.3}$$

$$S''_A = \sum_{n\in A''} \lambda''_n\, e^{2\pi inx} \tag{6.4}$$

$$S''' = \sum \lambda'''_n\, e^{2\pi inx}. \tag{6.4$'$}$$

Since A hence A'' does not contain a nontrivial triple in progression

$$\begin{aligned} I_1 &\equiv \int_{\mathbb{T}} S'_A(x)^2 S''_A(-2x)dx \\ &= \sum_{n\in A''} (\lambda'_n)^2 \lambda''_n\,. \end{aligned} \tag{6.5}$$

On the other hand

$$\begin{aligned} I_2 &\equiv \int_{\mathbb{T}} \big[\lambda'(A'')S'(x)\big]^2 \big[\lambda''(A'')S''(-2x)\big]dx \\ &= \lambda'(A'')^2\lambda''(A'') \sum_{n_1+n_2=2m} \lambda'_{n_1}\lambda'_{n_2}\lambda''_m \\ &= \lambda'(A'')^2\lambda''(A'') \sum_{n,m} \lambda'_n\lambda'_{n-2m}\lambda''_m\,. \end{aligned} \tag{6.6}$$

By construction of $\lambda', \lambda'', \lambda'''$, cf. (5.6), (5.12) we have

$$\sum_n \Big|\lambda'_n - \Big(\sum_m \lambda'_{n-2m}\lambda''_m\Big)\Big| < \big\|\lambda' - (\lambda' * \lambda''')\big\|_1 < \kappa \tag{6.7}$$

$$\Big(\sum_n \Big|\lambda'_n - \Big(\sum_m \lambda'_{n-2m}\lambda''_m\Big)\Big|^2\Big)^{1/2} < \kappa^{1/2}\|\lambda'\|_\infty^{1/2} = \kappa^{1/2}\|\lambda'\|_2\,. \tag{6.8}$$

Hence, from (5.17), (6.8)

$$(6.6) > (\delta_1 - 10\kappa)^3(1-\kappa^{1/2})\|\lambda'\|_2^2 \tag{1}$$

$$\overset{(5.19)}{>} \tfrac{1}{2}\delta_1^3\|\lambda'\|_2^2\,. \tag{6.9}$$

We will assume that throughout the construction of the measures $\lambda = \lambda_{\theta,\varepsilon,M}$, $\theta \in \mathbb{T}^d$, the condition

$$\log M \gg d\left(\log\tfrac{1}{\varepsilon} + \log\tfrac{1}{\delta} + \log d\right) \tag{6.10}$$

is fulfilled.

Thus

$$(6.5) < \frac{1}{|\Lambda''|}\sum(\lambda_n')^2 < \frac{1}{(\frac{c\kappa}{d})^{2(d+1)}\varepsilon^d M}\|\lambda'\|_2^2 < M^{-1/2}\|\lambda'\|_2^2 \tag{6.11}$$

and from (6.6), (6.9), (6.11)

$$|I_1 - I_2| > \tfrac{1}{3}\ \delta_1^3\|\lambda'\|_2^2\,. \tag{6.12}$$

Estimate

$$|I_1 - I_2| \le \Big[\int_{\mathbb{T}} |S_A'(x)|^2 dx\Big]\big\|S_A'' - \lambda''(A'')S''\big\|_\infty \tag{6.13}$$

$$+\int_{\mathbb{T}}\big|S_A'(x)^2 - [\lambda'(A'')S'(x)]^2\big|\,\lambda''(A'')|S''(-2x)\big|\,. \tag{6.14}$$

Write by (6.2), (5.12)

$$\begin{aligned}(6.13) &= \tfrac{1}{|\Lambda'|^2}|\Lambda'\cap A''|\big\|S_A'' - \lambda''(A)S''\big\|_\infty\\ &= \lambda'(A'')\|\lambda'\|_2^2\big\|S_A'' - \lambda''(A)S''\big\|_\infty\\ &< 2\delta_1\|\lambda'\|_2^2\big\|S_A'' - \lambda''(A)S''\big\|_\infty\,.\end{aligned} \tag{6.15}$$

Thus, if $(6.13) > \frac{1}{6}\delta_1^3\|\lambda'\|_2^2$, it follows from (6.15) that

$$\big\|S_A'' - \lambda''(A)S''\big\|_\infty > \tfrac{1}{12}\delta_1^2\,. \tag{6.16}$$

Estimate

$$\begin{aligned}(6.14) &\le \lambda''(A'')\big[\|S_A'\|_2 + \lambda'(A'')\|S'\|_2\big]\big\|\,|S_A'(x) - \lambda'(A'')S'(x)|\,|S''(-2x)|\,\big\|_2\\ &< 2\lambda''(A'')\lambda'(A'')^{1/2}\|\lambda'\|_2\big\|\,|S_A'(x) - \lambda'(A'')S'(x)|\,|S''(-2x)|\,\big\|_2\\ &< 8\delta_1^{3/2}\|\lambda'\|_2\big\|\,|S_A'(x) - \lambda'(A'')S'(x)|\ |S''(-2x)|\,\big\|_2\end{aligned} \tag{6.17}$$

and it follows that if $(6.14) > \frac{1}{6}\delta_1^3\|\lambda'\|_2^2$, then

$$\big\|\,|S_A'(x) - \lambda'(A'')S'(x)|\,|S''(-2x)|\,\big\|_2 > \tfrac{1}{48}\delta_1^{3/2}\|\lambda'\|_2\,. \tag{6.18}$$

We will show in the next 2 sections that both (6.16), (6.18) imply a density increment

$$\lambda_1(A_1) > \lambda(A') + 0(\delta_1^2) = \delta_1 + 0(\delta_1^2) \tag{6.19}$$

for some $\lambda_1 = \lambda_{\theta_1,\varepsilon_1,M_1}$ and translate A_1 of A.

7 Density Increment (1)

Assume (6.16)

Thus from some $x_0 \in \mathbb{T}$

$$\big|S_A''(x_0) - \lambda''(A'')S''(x_0)\big| > \tfrac{1}{12}\delta_1^2\,. \tag{7.1}$$

Recalling (5.5) where $\theta \in \mathbb{T}^d$, replace d by $d+1$, θ by $\tilde\theta = \theta \cup \{x_0\} \in \mathbb{T}^{d+1}$ and let

$$\tilde\lambda = \lambda_{\tilde\theta, (\frac{c\kappa}{d})^3 \varepsilon, (\frac{c\kappa}{d})^3 M} \, . \tag{7.2}$$

Then

$$\begin{aligned} S''_A(x_0) &= \sum_{n \in A''} \lambda''_n e^{2\pi i n x_0} \\ &= \sum_{m,n \in A''} \lambda''_m \tilde\lambda_{n-m} e^{2\pi i n x_0} + 0\big(\|\lambda'' - (\lambda'' * \tilde\lambda)\|_1\big) \qquad (7.3) \\ &\overset{(3.16)}{=} \sum \lambda''_m \, e^{2\pi i m x_0} \tilde\lambda(A'' - m) \\ &\quad + 0\Big(\sum \tilde\lambda_n |e^{2\pi i n x_0} - 1| + \kappa\Big) \qquad (7.4) \\ &= \sum \lambda''_m \tilde\lambda(A'' - m) e^{2\pi i m x_0} + 0\big(\big(\tfrac{c\kappa}{d}\big)^3 \varepsilon + \kappa\big) \qquad (7.5) \end{aligned}$$

where $\kappa = 10^{-8}\delta_1^2$, cf. (5.14).

Thus (7.1), (7.5) imply that

$$\Big| \sum_m \lambda''_m \big[\tilde\lambda(A'' - m) - \lambda''(A'')\big] e^{2\pi i m x_0} \Big| > \tfrac{1}{13}\delta_1^2 \tag{7.6}$$

$$\sum \lambda''_m \big|\tilde\lambda(A'' - m) - \lambda''(A'')\big| > \tfrac{1}{13}\delta_1^2 \, . \tag{7.7}$$

Again

$$\Big| \sum \lambda''_m \big[\tilde\lambda(A'' - m) - \lambda''(A'')\big] \Big| = \big|(\lambda'' * \tilde\lambda)(A'') - \lambda''(A'')\big| < \kappa \tag{7.8}$$

and (7.7), (7.8), (5.17) permit us to ensure that

$$\tilde\lambda(\tilde A) > \lambda''(A'') + \tfrac{1}{30}\delta_1^2 > \lambda(A') - 10\kappa + \tfrac{1}{30}\delta_1^2 > \delta_1 + \tfrac{1}{40}\delta_1^2 \tag{7.9}$$

for some translate $\tilde A = A'' - m$ of A.

Thus (7.9) produces the required density increment (6.19).

8 Density Increment (2)

Assume next (6.18).

Since

$$\begin{aligned} \big\|S'_A - \lambda'(A'')S'\big\|_2 \le \|S'_A\|_2 + \lambda'(A'')\|S'\|_2 &< 2\lambda(A'')^{1/2}\|\lambda'\|_2 \\ &< 4\delta_1^{1/2}\|\lambda'\|_2 \qquad (8.1) \end{aligned}$$

it follows from (6.18) that

$$\big\|[S'_A - \lambda'(A'')S']\big|_{\mathcal F}\big\|_2 > \tfrac{1}{10^2}\delta_1^{3/2}\|\lambda'\|_2 \tag{8.2}$$

where

$$\mathcal F = \big\{x \in \mathbb{T} \;\big|\; |S''(-2x)| > 10^{-3}\delta_1\big\} = \big\{x \in \mathbb{T} \;\big|\; |S'''(-x)| > 10^{-3}\delta_1\big\} \, . \tag{8.3}$$

In order to specify $\mathcal{F}$, apply Lemma 4.0 with λ replaced by λ''' given by (5.7). Thus if $x \in \mathcal{F}$, there is $k \in \mathbb{Z}^d$ s.t.

$$\begin{aligned} |k_j| &< Cd^4\delta_1^{-2}\left(\log \tfrac{d^2}{\delta_1^4}\tfrac{1}{\varepsilon}\right)^2 \tfrac{d^2}{\delta_1^4}\tfrac{1}{\varepsilon} \\ &< C\tfrac{d^7}{\delta_1^7}\left(\log\tfrac{1}{\varepsilon}\right)^2\tfrac{1}{\varepsilon} \qquad (1 \le j \le d) \end{aligned} \tag{8.4}$$

and

$$\|x - k\tilde{\theta}\| < C\tfrac{d^7}{\delta_1^7}\left(\log\tfrac{1}{\varepsilon}\right)^2 \tfrac{1}{M}\,, \tag{8.5}$$

where $\tilde{\theta}$ is given by (5.8).

Thus if we let

$$\overset{\approx}{\Lambda} = \Lambda_{\tilde{\theta}, \overset{\approx}{\varepsilon}, \overset{\approx}{M}} \tag{8.6}$$

with

$$\overset{\approx}{\varepsilon} = c\tfrac{\delta_1^9}{d^8}\left(\log\tfrac{1}{\varepsilon}\right)^{-2}\varepsilon \tag{8.7}$$

$$\overset{\approx}{M} = c\tfrac{\delta_1^9}{d^7}\left(\log\tfrac{1}{\varepsilon}\right)^{-2} M \tag{8.8}$$

it follows from (8.4), (8.5) that (for an appropriate constant c)

$$\|nx\| < 10^{-3}\delta_1^2 \text{ for } x \in \mathcal{F}\,, \quad n \in \overset{\approx}{\Lambda}\,. \tag{8.9}$$

Recalling (5.4)

$$\lambda' = \lambda_{\theta, c\frac{\delta_1^2}{d}\varepsilon, c\frac{\delta_1^2}{d}M} \tag{8.10}$$

the multiplier $\overset{\approx}{\lambda}$ associated with $\overset{\approx}{\Lambda}$ will also satisfy

$$\left\|(\lambda' * \overset{\approx}{\lambda}) - \lambda'\right\|_1 < 10^{-6}\delta_1^7 \tag{8.11}$$

from (8.7), (8.8), (3.16). Hence

$$\left\|(\lambda' * \overset{\approx}{\lambda}) - \lambda'\right\|_2 < 10^{-3}\delta_1^{7/2}\|\lambda'\|_2\,. \tag{8.12}$$

Write

$$\begin{aligned} S'_A(x) &= \sum_{n \in A''} \lambda'_n e^{2\pi i n x} \\ &= \sum_{n \in A''} (\lambda' * \overset{\approx}{\lambda})_n e^{2\pi i n x} && (8.13) \\ &\quad + \sum_{n \in A''} \left(\lambda' - (\lambda' * \overset{\approx}{\lambda})\right)_n e^{2\pi i n x}\,. && (8.14) \end{aligned}$$

From (8.12)

$$\|(8.14)\|_2 \le \left\|\lambda' - (\lambda' * \overset{\approx}{\lambda})\right\|_2 < 10^{-3}\delta_1^{7/2}\|\lambda'\|_2\,. \tag{8.15}$$

Write

$$(8.13) = \sum_{m,n \in A''} \lambda'_m \overset{\approx}{\lambda}_{n-m}\, e^{2\pi i n x}$$

$$= \sum_m \lambda'_m e^{2\pi imx} \overset{\approx}{\lambda}(A'' - m) \tag{8.16}$$

$$+ \sum_{m,n\in A''} \lambda'_m \overset{\approx}{\lambda}_{n-m} (e^{2\pi inx} - e^{2\pi imx})\,. \tag{8.17}$$

One has for $x \in \mathcal{F}$, by (8.9)

$$\begin{aligned} |(8.17)| &= \Big| \sum_{m,n\in A''} \lambda'_{n-m} \overset{\approx}{\lambda}_m (e^{2\pi inx} - e^{2\pi i(n-m)x}) \Big| \\ &= \Big| \sum_m \overset{\approx}{\lambda}_m (e^{2\pi imx} - 1) \Big[\sum_{n\in A''} \lambda'_{n-m}\ e^{2\pi i(n-m)x} \Big] \Big| \\ &\le 10^{-3} \delta_1^2 \sum_m \overset{\approx}{\lambda}_m \Big| \sum_{k\in A''-m} \lambda'_k e^{2\pi ikx} \Big| \end{aligned} \tag{8.18}$$

hence

$$\big\| (8.17)\big|_{\mathcal{F}} \big\|_2 \le 10^{-3} \delta_1^2 \sum_m \overset{\approx}{\lambda}_m \|\lambda'\|_2 = 10^{-3} \delta_1^2 \|\lambda'\|_2\,. \tag{8.19}$$

Thus, from (8.15), (8.19)

$$\begin{aligned} \big\| [S'_A - \lambda'(A'')S']\big|_{\mathcal{F}} \big\|_2 &< \Big\| \sum_m \lambda'_m e^{2\pi imx} \overset{\approx}{\lambda}(A'' - m) - \lambda'(A'')S' \Big\|_2 \\ &\quad + \tfrac{1}{500}\, \delta_1^2 \|\lambda'\|_2 \\ &= \Big(\sum_m (\lambda'_m)^2 \big[\lambda'(A'') - \overset{\approx}{\lambda}(A'' - m) \big]^2 \Big)^{1/2} + \tfrac{\delta_1^2}{500} \|\lambda'\|_2\,. \end{aligned} \tag{8.20}$$

Consequently, (8.2), (8.20) give

$$\Big(\sum_m (\lambda'_m)^2 \big[\lambda'(A'') - \overset{\approx}{\lambda}(A'' - m) \big]^2 \Big)^{1/2} > \tfrac{\delta_1^{3/2}}{200} \|\lambda'\|_2 \tag{8.21}$$

$$\sum \lambda'_m \big[\lambda'(A'') - \overset{\approx}{\lambda}(A'' - m) \big]^2 > \tfrac{\delta_1^3}{4.10^4} \tag{8.22}$$

$$\big[\lambda'(A'') + \max_m \overset{\approx}{\lambda}(A'' - m) \big] \Big[\sum \lambda'_m \big| \lambda'(A'') - \overset{\approx}{\lambda}(A'' - m) \big| \Big] > \tfrac{\delta_1^3}{4.10^4}\,. \tag{8.23}$$

From (8.23), either for some m

$$\overset{\approx}{\lambda}(A'' - m) > \tfrac{4}{3} \delta_1 \tag{8.24}$$

or

$$\sum \lambda'_m |\lambda'(A'') - \overset{\approx}{\lambda}(A'' - m)| > \tfrac{\delta_1^2}{10^5}\,. \tag{8.25}$$

Since again

$$\Big|\sum_m \lambda'_m\big[\lambda'(A'') - \tilde{\tilde{\lambda}}(A''-m)\big]\Big| = \big|\lambda'(A'') - (\lambda' * \tilde{\tilde{\lambda}})(A'')\big| \overset{(8.11)}{<} 10^{-6}\delta_1^7\,. \tag{8.26}$$

(8.25), (8.26) imply for some m

$$\tilde{\tilde{\lambda}}(A''-m) - \lambda'(A'') > \tfrac{1}{2}\left(\tfrac{\delta_1^2}{10^5} - \tfrac{\delta_1^7}{10^6}\right) > \tfrac{\delta_1^2}{10^6}$$

$$\tilde{\tilde{\lambda}}(A''-m) \overset{(5.17)}{>} \lambda(A') - 10\kappa + 10^{-6}\delta_1^2 > \delta_1 + 10^{-7}\delta_1^2 \tag{8.27}$$

for some m. Thus (8.24), (8.27) give again the increment

$$\tilde{\tilde{\lambda}}(\tilde{\tilde{A}}) > \delta_1 + 10^{-7}\delta_1^2 \tag{8.28}$$

for some translate

$$\tilde{\tilde{A}} = A'' - m \text{ of } A, \text{ i.e. } (6.19). \tag{8.29}$$

9 Conclusion

Taking into account (5.13), (7.9), (8.28), it follows that starting from $\lambda = \lambda_{\theta,\varepsilon,M}, \theta \in \mathbb{T}^d$ such that

$$\lambda(A') = \delta_1 > \delta \tag{9.1}$$

for some translate A' of A, one of the following holds

$$\lambda'(A_1) > \delta_1 + 10^{-8}\delta_1^2 \tag{9.2}$$

$$\lambda''(A_1) > \delta_1 + 10^{-8}\delta_1^2 \tag{9.3}$$

$$\tilde{\lambda}(A_1) > \delta_1 + \frac{1}{40}\delta_1^2 \tag{9.4}$$

$$\tilde{\tilde{\lambda}}(A_1) > \delta_1 + 10^{-7}\delta_1^2 \tag{9.5}$$

for some translate A_1 of A.

Here $\lambda', \lambda'', \tilde{\lambda}, \tilde{\tilde{\lambda}}$ are given by (5.4), (5.5), (7.2), (8.6)-(8.8) respectively.

Hence

$$\lambda_1(A_1) > \delta_1 + 10^{-8}\,\delta_1^2 \tag{9.6}$$

where λ_1 is of the form

$$\lambda_1 = \lambda_{\theta_1,\varepsilon_1,M_1} \tag{9.7}$$

with

$$\theta_1 \in \mathbb{T}^{d+1} \tag{9.8}$$

$$\varepsilon_1 > c\delta_1^9 d^{-8}\left(\log\tfrac{1}{\varepsilon}\right)^{-2}\varepsilon \tag{9.9}$$

$$M_1 > c\delta_1^9 d^{-8} \left(\log \tfrac{1}{\varepsilon}\right)^{-2} M\,. \tag{9.10}$$

Starting from $\lambda_0 = \frac{1}{2N+1}\mathbb{1}_{\{-N,\dots,N\}}$, $\lambda_0(A) > \delta$, it follows indeed from (9.6) that one needs at most $\sim 1/\delta$ iteration steps to reach a contradiction. Thus the number d is bounded by

$$d \leq C\delta^{-1} \tag{9.11}$$

(9.9) implies at each step α

$$\varepsilon_{\alpha+1} > c\delta^{17}\big(\log \tfrac{1}{\varepsilon_\alpha}\big)^{-2}\varepsilon_\alpha \tag{9.12}$$

hence

$$\varepsilon_\alpha > c\delta^{20\alpha} > \delta^{C\delta^{-1}}\,. \tag{9.13}$$

Similarly

$$M_{\alpha+1} > c\delta^{17}\left(\log \tfrac{1}{\varepsilon_\alpha}\right)^{-2} M_\alpha > c\delta^{20} M_\alpha \tag{9.14}$$

$$M_\alpha > \delta^{C\delta^{-1}} N\,. \tag{9.15}$$

Coming back to condition (6.10), we get from (9.11), (9.13), (9.15) the restriction

$$\log N \gg \delta^{-2}\log \tfrac{1}{\delta} \tag{9.16}$$

i.e.

$$\delta > C\left(\frac{\log\log N}{\log N}\right)^{1/2}. \tag{9.17}$$

Thus, if (9.17) holds, the set A must contain a nontrivial triple in progression.

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