# COMBINATORIAL NUMBER THEORY: RESULTS OF HILBERT, SCHUR, FOLKMAN, AND HINDMAN

by

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# ABSTRACT

Let $r, m, a$ and $a_i$ be integers with $r > 0, m > 0, a \geq 0$ and $a_i > 0$ for $1 \leq i \leq m$. Let $\Delta$ be an $r$-coloring of $\mathbf{N}$. Hilbert's Cube Lemma guarantees that there exists a monochromatic $m$-cube of the form $Q_m(a, a_1, ..., a_m) = \{a + \sum_{i=1}^{m} \epsilon_i a_i : \epsilon_i = 0 \text{ or } 1\}$. Three different proofs of this lemma are given.

Hilbert Cube Lemma can be generalized in two different aspects. First, we can give a criterion where to look for a monochromatic $m$-cube. Szemerédi's Cube Lemma gives that criterion. Secondly, we can give more information about the $m$-cube. This was done by Schur. Schur's Theorem guarantees the existence of a monochromatic set of the form $Q_2(0, a_1, a_2)\backslash\{0\}$ in every finite coloring of $[1, N]$ if $N$ is big enough.

Schur's result can be extended by replacing $Q_2(0, a_1, a_2)\backslash\{0\}$ with $Q_m(0, a_1, ..., a_m)\backslash\{0\}$, where $m$ is finite and $a_i \neq a_j$ for $i \neq j$. This is called the Rado-Sanders-Folkman's Theorem or simply Folkman's Theorem. Folkman's Theorem states that for any finite coloring of $\mathbf{N}$ there exists an arbitrarily large finite sets $S = \{a_1, ..., a_k\}$ of positive integers such that $\{\sum_{i \in I} a_i : \emptyset \neq I \subseteq [1, k]\}$ is monochromatic. The finite form for Folkman's Theorem is: For all positive integers and $k$ there is $M = M(, k)$ such that for every -c oloring of $[1, M]$ there exist distinct $a_1, ..., a_k \in [1, M]$ with all $a_i$ are distinct such that $\sum_{i=1}^{k} a_i \leq M$ and $\{\sum_{i \in I} a_i : \emptyset \neq I \subseteq [1, k]\}$ is monochromatic.

The Finite Unions Theorem, an analogue to Folkman's Theorem, is also studied. In its finite form, the Finite Unions Theorem guarantees that for all positive integers and $k$ there exists $F = F(, k)$ such that for any $n \geq F$, if $P_n$, the set of all non-empty subsets of $[1, n]$, is - colored then there is a pairwise disjoint collection $\mathcal{D} \subseteq P_n$ with $| \mathcal{D} | = k$ such that $FU(\mathcal{D})$, the set of all unions of elements of $\mathcal{D}$, is monochromatic.

Then, we give upper bounds for $M$ and $F$, where $M$ and $F$ are as in Folkman's Theorem and the Finite Unions Theorem respectively.

Finally, we prove Hindman's Theorem that guarantees the existence of a monochromatic 'infinite cube' in every finite coloring of $\mathbf{N}$. We also prove this theorem by using methods of Topological Dynamics.

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# Chapter 1

# INTRODUCTION

Ramsey Theory is a part of Discrete Mathematics that has its root in the works of D. Hilbert (1892), I. Schur (1916), B.L. van der Waerden (1927) and F.P. Ramsey (1930). These four results were established for different reasons, not being aware of each other. In the thirties, R. Rado unified and extended the results of Hilbert, Schur and van der Waerden and since that time many mathematicians, led by the great P. Erdös, have been involved in the development of Ramsey Theory.

In this thesis two aspects of the generalization of the earliest result from above will be studied.

Hilbert's Cube Lemma guarantees that for any finite coloring of $\mathbf{N}$ and for any $m \in \mathbf{N}$ there exists a monochromatic structure that we call an '$m$-cube' of the form

$$Q_m(a, a\ , ..., a_m) = \{a + \sum_{i}^{m} \epsilon_i a_i : \epsilon_i = 0 \text{ or } 1, 1 \leq i \leq m\}.$$

In Chapter 2 we give three different proofs of Hilbert's Cube Lemma. In this chapter we also discuss Szemerédi's Cube Lemma. This lemma generalizes Hilbert's statement. It tells us where to look for a monochromatic '$m$-cube'.

Another way to generalize Hilbert's Cube Lemma is to give more information about the '$m$-cube'.

In Chapter 3 we discuss Schur's Theorem. Schur's Theorem guarantees that for any $r \in \mathbf{N}$ there exists $n \in \mathbf{N}$ so that for any $r$-coloring of $[1, n]$ there is a monochromatic set of the form $Q\ (0, a\ , a\ )\backslash\{0\}$. In this chapter, two problems related to Schur's Theorem are discussed in detail.

In Chapter 4 we go one step further. In this chapter we give two different ways to prove the fact that for any finite coloring of $\mathbf{N}$ and for any $m \in \mathbf{N}$ there is a monochromatic set of the form $Q_m(0, a\ , ..., a_m)\backslash\{0\}$ such that all the $a_i$'s are different. This statement is known as the Rado-Sanders-Folkman Theorem or simply Folkman's Theorem.

In Set Theory there is an analogue to Folkman's Theorem. It is called the Finite Unions Theorem. In Chapter 5 we give that analogy. Also, we prove the Finite Unions Theorem independently.

Both Folkman's Theorem and the Finite Unions Theorem have their finite forms, i.e., both of them can be stated in the form where the phrase "*...for any finite coloring of* $\mathbf{N}$*...*" is replaced by the phrase "*...for any* $r \in \mathbf{N}$ *there is* $n \in \mathbf{N}$ *such that for any* $r$*-coloring of* $[1, n]$*...*".

In Chapter 6 we discuss the upper bound for such $n$ in both Folkman's Theorem and the Finite Unions Theorem.

As a result of our discussion in Chapter 4, Chapter 5 and Chapter 6, we have the following fact. Let $m, r \in \mathbf{N}$ and let $n \geq r^{3^{r^{\cdot^{\cdot^{\cdot^{3}}}}}}$, where $r^{3^{r^{\cdot^{\cdot^{\cdot^{3}}}}}}$ is a tower of height $2r(m-1)$. Then for any $r$-coloring of $[1, 2^n]$ there is a monochromatic set of the form $Q_m(0, a_1, ..., a_m)\backslash\{0\}$ such that all the $a_i$'s are different.

Our last step is to prove Hindman's Theorem. Hindman's Theorem guarantees the existence of a monochromatic structure that we call an *'infinite cube'* of the form $Q(0, a_1, a_2, ...) = \{\sum_{i \in F} a_i : F \subseteq \mathbf{N} \text{ and } 1 \leq \mid F \mid < \infty\}$. We give two different proofs of this fact. In Chapter 7 we give a proof due to Baumgartner and in Chapter 8 we give a proof of Hindman's Theorem by using the methods of Topological Dynamics. Also, in Chapter 8 we prove van der Waerden's Theorem by using Topological Dynamics.

This thesis has two objectives. The first objective is to point out the steps in the development of Hilbert's result. Secondly, detailed proofs of the results mentioned before are given.

Generally, this thesis follows the book *"Ramsey Theory"* by R. Graham, B. Rothchild and J. Spencer [16]. Some relatively new papers related to the first objective are also presented.

In this introduction we provide some background material such as notations, definitions and theorems which will be used in the next chapters.

## 1.1 Notations and Definitions.

- In this thesis, $\mathbf{N}$ denotes the set of natural numbers, $\mathbf{Z}$ denotes the set of integers, $\mathbf{Q}$ denotes the set of rational numbers, and $\mathbf{R}$ denotes the set of real numbers.

- If $m, n \in \mathbf{N}$ with $m \leq n$, then $[m, n]$ denotes $\{i \in \mathbf{N} : m \leq i \leq n\}$.

- If $n \in \mathbf{N}$ then $\mathbf{Z}^n$ denotes $\{(x_1, ..., x_n) : x_i \in \mathbf{Z}\}$.

- If $p$ is a prime number then the set of non-zero integers modulo $p$ is denoted by $\mathbf{Z}_p^* = \{\bar{1}, ..., \overline{p-1}\}$.

- If $X$ is a set then we define $\mid X \mid$ to be the number of elements of $X$ for finite sets $X$ and we write $\mid X \mid = \infty$ if $X$ is an infinite set. The notation $\mid X \mid < \infty$ indicates that $X$ is a finite set.

- If $X$ is a set and $m \in \mathbf{N}$ then we define $P(X)$ to be the set $\{Y : \emptyset \neq Y \subseteq X \text{ and } \mid Y \mid < \infty\}$ and we define $[X]^m$ to be the set $\{Y : Y \subseteq X$ and

$| Y |= m\}$. For convenience we write $P([1, n])$ as $P_n$, and $[[1, n]]^m$ as $[1, n]^m$, for every $n \in \mathbf{N}$.

- If $X$ is a subset of N then we define $\mathcal{P}(X)$ to be the set $\{\sum_{x \in F} x : F \in P(X)\}$.
- For integers $m, a, a_i$ with $m \geq 1, a \geq 0$ and $a_i \geq 1$, $1 \leq i \leq m$, we define the *$m$-cube* $Q_m(a, a_1, ..., a_m)$ to be the set $\{a + \sum_{i=1}^{m} \epsilon_i a_i : \epsilon_i = 0$ or $1$ for $1 \leq i \leq m\}$.
- If $S$ is a set and $r$ is a natural number then an $r$-coloring of $S$ is a function from $S$ into $[1, r]$. If $\Delta$ is an $r$-coloring of $S$ and $T \subseteq S$, then we say that $T$ is *$\Delta$-monochromatic* or simply *monochromatic* if $\Delta(x)$ is constant for $x \in T$.
- If $k \in \mathbf{N}$, then an *arithmetic progression of length $k$* is a set of the form $\{a + id : 0 \leq i \leq k - 1\}$ with $a, d \in N$.

## 1.2 Background Theorems

We introduce two background theorems where the proof of each theorem can be found in [16].

**Theorem 1.1 (Van der Waerden, 1927)** *Let $c, k \in \mathbf{N}$. Then there exists a positive integer $W = W(c, k)$ with the property: If $n \geq W$ and $[1, n]$ is c-colored then $[1, n]$ contains a monochromatic arithmetic progression of length $k$.*

**Theorem 1.2 (Ramsey, 1930)** *Let $c, k, s \in \mathbf{N}$. Then there exists a positive integer $R = R(c, k, s)$ with the property: If $n \geq R$ and $[1, n]^k$ is c-colored then there exists a set $X \in [1, n]^s$ such that $[X]^k$ is monochromatic.*

We say that $R(c, k, s)$ is a Ramsey number. If $k = 2$ we write $R(c, 2, s)$ as $R(c, s)$.

(Theorem 1.1 will be used in the proofs of Lemma 4.1 and lemma 4.2. Theorem 1.2 will be used in the proofs of Theorem 3.1, Theorem 3.2, and Theorem 5.1.)

# Chapter 2

# HILBERT'S CUBE LEMMA

In this chapter we will discuss Hilbert's Cube Lemma. Besides the pigeon hole principle, this lemma can be considered as the earliest partition theorem. This lemma was established in 1892 in connection with Hilbert's investigation of the irreducibility of rational functions.

**Definition 2.1** *Let $m, a, a_i$ be integers with $a \geq 0, m \geq 1, a_i \geq 1, 1 \leq i \leq m$. We define the $m$-cube $Q_m(a, a_1, ..., a_m)$ to be the set*

$$\{a + \sum_{i=1}^{m} \epsilon_i a_i : \epsilon_i = 0 \text{ or } 1, 1 \leq i \leq m\}.$$

For example, the 2-cube $Q_2(a, a_1, a_2)$ is the set $\{a, a + a_1, a + a_2, a + a_1 + a_2\}$.

**Definition 2.2** *Let $n, r$ be positive integers. An $r$-coloring of $[1, n]$ is a function from $[1, n]$ into $[1, r]$. If*

$$\Delta : [1, n] \longrightarrow [1, r]$$

*is an $r$-coloring of $[1, n]$ and $A \subseteq [1, n]$, then $A$ is monochromatic if there exists $i \in [1, r]$ such that $\Delta(x) = i$ for every $x \in A$.*

**Theorem 2.1 (Hilbert's Cube Lemma)** *Let $m, r$ be positive integers. Then there exists a positive integer $n = H(m, r)$ such that for every $r$-coloring of $[1, n]$ there is a monochromatic $m$-cube*

$$Q_m(a, a_1, ..., a_m) = \{a + \sum_{i=1}^{m} \epsilon_i a_i : \epsilon_i = 0 \text{ or } 1, 1 \leq i \leq m\}$$

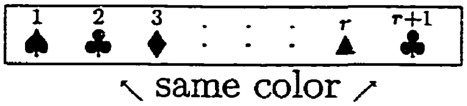
*contained in $[1, n]$.*

We will give three different proofs, each of which illustrates a different technique. Proof 1 is Hilbert's original proof. Proof 2 proves a density version of Hilbert's Cube Lemma. This density version is often called Szemerédi's Cube Lemma. Proof 3 is somewhat similar to proof 2.

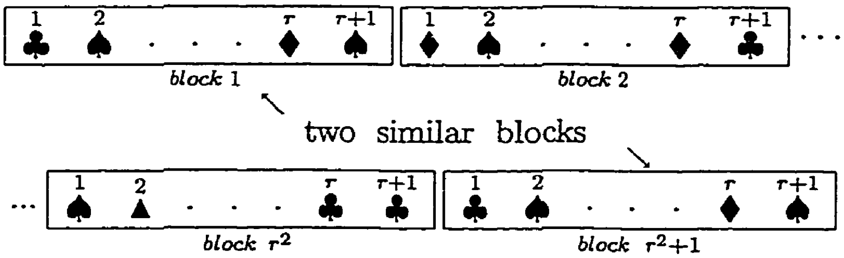
**Proof 1 (Hilbert, 1892).** We want to find $n = h_m$ such that if $[1, h_m]$ is $r$-colored ($r$ is fixed), there is a monochromatic $m$-cube contained in $[1, h_m]$.

We will make use of the Fibonacci sequence $\{F_n\}_{n \in \mathbf{N}}$ which is defined by $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, $n \geq 0$. This sequence starts off $0, 1, 1, 2, 3, 5, 8, 13, ...$ so that $F_2 = 1, F_4 = 3, F_6 = 8, ...$ .

For $m = 1$, a 1-cube is $\{a, a + a_1\}$, i.e., a two-element set. Clearly, if $[1, r + 1]$ is $r$-colored then there are at least two elements with the same color. In other words, there is a monochromatic 1-cube. Therefore $h_1 \leq r + 1$.




Furthermore, since there are $r$ different values for $a_1$ and r different colors then there are $r^2$ different types of 1-cubes possibly occurring in an $r$-coloring of $[1, r + 1]$. Hence, if $r^2 + 1$ consecutive blocks each of length $r + 1$ are $r$-colored, there are at least two blocks, where each of these blocks has a monochromatic 1-cube of the same type.




Consider that the union of these two 1-cubes forms a monochromatic 2-cube. Hence, $h_2 \leq (r + 1)(r^2 + 1) < (r + 1)^3$. Now, since there are $r$ different colors, $r$ possible values of $a_1$ and at most $(r+1)(r^2+1)-2$ possible values of $a_2$, then there are at most $r \cdot r \cdot ((r+1)(r^2+1)-2) < (r+1)^5$ different types of 2-cubes. Therefore if $(r + 1)^5$ consecutive blocks each of length $(r + 1)(r^2 + 1)$ are $r$-colored, there are at least two blocks where each of these blocks has a monochromatic 2−cube of the same type. Again, the union of these two 2-cubes forms a monochromatic 3-cube. Hence, $h_3 \leq (r+1)(r^2+1)(r+1)^5 < (r+1)^8$. Since there are $r$ different colors, $r$ possible values of $a_1$, at most $(r + 1)(r^2 + 1) - 2$ possible values of $a_2$, at most $(r + 1)(r^2 + 1)(r + 1)^5 - 3$ possible values of $a_3$, then there are at most $r \cdot r \cdot ((r+1)(r^2+1) - 2)((r+1)(r^2+1)(r+1)^5 - 3) < (r+1)^{13}$ different types of 3-cubes which can occur.

Continuing the process we have

| $m$ | $h_m$ | # types which can occur |
|---|---|---|
| 1 | $(r+1)^1$ | $r^2 < (r+1)^2$ |
| 2 | $(r+1)(r^2+1) < (r+1)^3$ | $< (r+1)^5$ |
| 3 | $< (r+1)^8$ | $< (r+1)^{13}$ |
| . | . | . |
| . | . | . |
| . | . | . |

We will show by induction that $h_m \le (r+1)^{F_{2m}}$, and that when $[1, (r+1)^{F_{2m}}]$ is $r$-colored, there is a monochromatic $m$-cube of one of less than $(r+1)^{F_{2m+1}}$ different types.

We have seen that this is true for $m = 1$.

Suppose that it is true for $m-1$, i.e., $h_{m-1} \le (r+1)^{F_{2(m-1)}}$ and there are less than $(r+1)^{F_{2(m-1)+1}} = (r+1)^{F_{2m-1}}$ different types of $(m-1)$-cubes. Therefore if $(r+1)^{F_{2m-1}}$ consecutive blocks each of length $(r+1)^{F_{2(m-1)}}$ are $r$-colored, there are at least two blocks where each of these blocks has a monochromatic $(m-1)$-cube of the same type. Again, the union of these two $(m-1)$-cubes forms a monochromatic $m$-cube, hence $h_m \le (r+1)^{F_{2(m-1)}} \cdot (r+1)^{F_{2m-1}} = (r+1)^{F_{2m}}$ , and there are less than $(r+1)^{F_{2m+1}}$ different types of $m$-cubes. ■

**Proof 2.** Fix $m$ and $r$ and pick $n$ such that $n \ge \frac{1}{3}(3r)^{2^m}$. Let $A \subseteq [1, n]$ with $|A| \ge \frac{n}{r}$. There are at least $\binom{\frac{n}{r}}{2}$ elements of the form $b - a$, with $a, b \in A$ and $a < b$, each of which is in $[1, n]$. Therefore some difference occurs at least $\binom{\frac{n}{r}}{2} \cdot 1/n \ge \frac{n}{3r^2}$ times. Let $a_1 = y_i - x_i, 1 \le i \le \frac{n}{3r^2}$, $x_i, y_i \in A$ with $x_i < y_i$. Let $A_1 = \{a \in A : a_1 + a \in A\}$. Then $|A_1| \ge \frac{n}{3r^2}$ .

Now, for $1 \le k \le m$, let $s = \frac{1}{3}(3r)^{2^{k-1}}$, then $3s^2 = \frac{1}{3}(3r)^{2^k}$ . Then,

$$\frac{1}{n}\binom{\frac{n}{s}}{2} = \frac{1}{2}\cdot\frac{1}{n}\cdot\frac{n}{s}\cdot\left(\frac{n}{s}-1\right) = \frac{1}{2}\cdot\frac{n}{s^2} - \frac{1}{2s} \ge \frac{1}{3s^2},$$

since $\frac{1}{6}\cdot\frac{n}{s^2} \ge \frac{1}{2s}$ , or $n \ge 3s$, which is true since $n \ge \frac{1}{3}(3r)^{2m}$ and $s = \frac{1}{3}(3r)^{2^k}$ , $k \le m$.

Therefore,

$$\frac{1}{n}\cdot\binom{\frac{n}{\frac{1}{3}(3r)^{2^{k-1}}}}{2} \ge \frac{n}{\frac{1}{3}(3r)^{2^k}},$$

for $1 \le k \le m$.

By taking $k = 1$, we get

$$\frac{1}{n}\cdot\binom{\frac{n}{r}}{2} \ge \frac{n}{\frac{1}{3}(3r)^2} = \frac{n}{3r^2},$$

and from this we get

$$a_1 \in [1,\ n],\ A_1 = \{a \in A : a_1 + a \in A\}, |A_1| \ge \frac{n}{\frac{1}{3}(3r)^2}.$$

Starting with $A_1$, by the same argument we get

$$a_2 \in [1,n],\ A_2 = \{a \in A_1 : a_2 + a \in A_1\}, |\ A_2\ | \geq \frac{n}{\frac{1}{3}(3r)^{2^2}}.$$

Since $a_1 + A_1 \subseteq A$ then $\epsilon_1 a_1 + A_1 \subseteq A$, where $\epsilon_1 = 0$ or 1. Similarly, since $a_2 + A_2 \subseteq A_1$ then $\epsilon_2 a_2 + A_2 \subseteq A_1$, where $\epsilon_2 = 0$ or 1 so $\epsilon_1 a_1 + \epsilon_2 a_2 + A_2 \subseteq A$, for $\epsilon_1, \epsilon_2 = 0$ or 1.

By continuing this process we get

$$a_1, ..., a_m \in [1,n]$$

with

$$\epsilon_1 a_1 + ... + \epsilon_m a_m + A_m \subseteq A$$

and

$$|\ A_m\ | \geq \frac{n}{\frac{1}{3}(3r)^{2^m}} \geq 1.$$

Since $|\ A_m\ | \geq 1$ we can find an $a \in A_m$ such that $\{a + \sum_{k=1}^{m} \epsilon_k a_k : \epsilon_k = 0$ or $1\} \subseteq A$. ■

Let us observe that Proof 2 indeed proves the claim of Hilbert's Cube Lemma. Let $\mathbf{N} = C_1 \cup ... \cup C_r$ with $C_i \cap C_j = \emptyset$ for $i \neq j$, and let for $n \in \mathbf{N}$ and $i \in [1,r]$

$$C_i^{(n)} = C_i \cap [1,n].$$

Clearly,

$$[1,n] = \bigcup_{i=1}^{n} C_i^{(n)}.$$

If for all $i \in [1,r]$

$$|\ C_i^{(n)}\ | < \frac{n}{r}$$

then

$$n = \sum |\ C_i^{(n)}\ | < n.$$

Thus, there is $i \in [1,r]$ with

$$|\ C_i^{(n)}\ | \geq \frac{n}{r}.$$

In other words, for any $r$-coloring of $[1,n]$ there are at least $\frac{n}{r}$ elements of the same color and Proof 2 shows that there is an $m$-cube in that color, if $n$ is large enough.

This leads us to the following important statement, known as Szemerédi's Cube Lemma. Szemerédi's Cube Lemma was a part of Szemerédi's proof of Roth's Theorem.

In 1936 P. Erdös and P. Turán gave the following question.

If $A \subseteq \mathbf{N}$ is a set such that

$$\limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n} > 0,$$

does $A$ contain arbitrarily long arithmetic progressions? In 1952, K. F. Roth proved that $A$ must contain at least an arithmetic progression of length three. Using the statement that now we call Szemerédi's Cube Lemma, in 1969, Szemerédi gave a different proof of Roth's result and showed that if $A$ is as above then $A$ must contain at least an arithmetic progression of length four. In 1972, Szemerédi gave a positive answer to the Erdös-Turán question.

**Corollary 2.1 (Szemerédi's Cube Lemma)** *Let* $c$ be fixed, $0 < c < 1$. Let $n \geq \left(\frac{3}{c}\right)^2$. *Let* $A \subseteq [1,n]$ *with* $|A| > cn$. *Then* $A$ *contains an* $m$*-cube, for* $m \geq \log\log n - C$*, where* $C$ *is a constant depending only on* $c$.

**Proof.** Given a positive integer $n$ and $0 < c < 1$. We can find an integer $m$ such that

$$\left(\frac{3}{c}\right)^{2^m} \leq n < \left(\frac{3}{c}\right)^{2^{m+1}}.$$

Pick a positive integer $r$ such that $r \leq \frac{1}{c}$. From the first inequality we have

$$n \geq \left(\frac{3}{c}\right)^{2^m} \geq \frac{1}{3}\left(\frac{3}{c}\right)^{2^m} \geq \frac{1}{3}(3r)^{2^m}.$$

Moreover,

$$|A| > cn \geq \frac{n}{r}.$$

From Proof 2, $A$ contains an $m$-cube. Taking the logarithms of the above inequalities we have:

$$2^m \log\frac{3}{c} \leq \log n \leq 2^{m+1}\log\frac{3}{c}$$

$$\iff m\log 2 + \log\log\frac{3}{c} \leq \log\log n < (m+1)\log 2 + \log\log\frac{3}{c}.$$

The second inequality gives:

$$m > \frac{\log\log n}{\log 2} - \frac{\log\log\frac{3}{c}}{\log 2} - 1 > \log\log n - C. \blacksquare$$

**Proof 3.** Let $n$ be a fixed positive integer and for each positive integer $d$, let $t_d = 4n^{1-\frac{1}{2^d}}$. Let $A \subseteq [1,n]$ with $|A| \geq t_d$. It will be shown that there exist $a, a_1, ..., a_d \in [1,n]$ with $Q_d(a, a_1, ..., a_d) = \{a + \sum_{i=1}^{d} \epsilon_i a_i : \epsilon_i = 0 \text{ or } 1\} \subseteq A$.

For $d = 1$, we have $t_1 = 4n^{1/2}$. Since a 1-cube is a two element set, we need just $4n^{1/2} \geq 2$, which is true since $n \geq 1$.

Let $d \geq 2$ and assume the result is true for $d-1$. Let $A = \{a_1, ..., a_t\} \subseteq [1,n]$ with $|A| = t \geq t_d$.

Consider that

$$\binom{t}{2} \geq \binom{t_d}{2} = \frac{t_d(t_d-1)}{2} \geq \frac{t_d^2}{4} = 4n^{2-\frac{2}{2^d}} = n.4n^{1-\frac{1}{2^{d-1}}} = t_{d-1}.n$$

Since $|A| = t \leq n$, at least $t_{d-1}$ of the different $a_i - a_j$, $i > j$, are equal, say

$$\{a_{i_k} - a_{j_k} : 1 \leq k \leq t_{d-1}\} = \{w\}.$$

Let $A' = \{a_{j_k} : 1 \leq k \leq t_{d-1}\}$. Since $|A'| = t_{d-1}$, by the induction hypothesis, there exist $a, a_1, ..., a_{t-1} \in [1,n]$ with $Q_{t-1}(a, a_1, ..., a_{t-1}) \subseteq A'$.

But, if $x \in A'$ then $x + w \in A$. Hence, $Q_t(a, a_1, ..., a_{t-1}, w) \subseteq A'$.

Now let $0 < c < 1$ be given and $d$ be the largest integer with

$$d \leq \frac{\log\log n - \log\log\left(\frac{4}{c}\right)}{\log 2},$$

then $t_d \leq cn$, if we take $t_d = 4n^{1-\frac{1}{2^d}}$.

Thus we have shown that if $c$ is given, $0 < c < 1$, and $A \subseteq [1,n]$, $|A| \geq cn$, and $d \leq \frac{\log\log n - \log\log\left(\frac{4}{c}\right)}{\log 2}$, then $|A| \geq t_d$, so $A$ contains a $d$-cube. $\blacksquare$

# Chapter 3

# SCHUR'S THEOREM

Our goal now is to prove a strengthened form of Hilbert's Cube Lemma. Recall that Hilbert's Cube Lemma guarantees that for any $m \in \mathbf{N}$ and any finite coloring of $\mathbf{N}$ there are a non-negative integer $a_0$ and positive integers $a_1, ..., a_m$ such that $Q_m(a_0, a_1, ..., a_m)$ is monochromatic. Two natural questions are:

1. Can we say anything about $a_0$?
2. Can we choose $a_0, a_1, ..., a_m$ in such a way that $a_i \neq a_j$ if $i \neq j$?

The answers to both of these questions are: yes, we can. Actually, in the next chapter we will see that for any $m \in \mathbf{N}$ and any finite coloring of $\mathbf{N}$ we can find arbitrarily many monochromatic sets $Q_m(0, a_1, ..., a_m)\backslash\{0\}$ with $a_i \neq a_j$ for $i \neq j$. This statement follows from the result proved independently by Rado (1969), Sanders (1969) and Folkman (1970).

In this chapter we discuss Schur's Theorem, the first result that partially answered the first question above. Schur's Theorem guarantees that for any finite coloring of $\mathbf{N}$ there exists a monochromatic set of the form $Q_2(0, a_1, a_2)\backslash\{0\}$.

Let us note that Schur's original paper from 1916 was motivated by the famous Fermat's Last Theorem. Some authors (see [16], Ch.3) consider Schur's Theorem as the earliest result in Ramsey Theory.

**Theorem 3.1 (Schur's Theorem, 1916)** *For all $r \geq 1$ there exists $n = n(r) \in \mathbf{N}$ such that for every $r$-coloring*

$$\Delta : [1, n] \longrightarrow [1, r],$$

*there exist $x, y, x + y \in [1, n]$ such that $\{x, y, x + y\}$ is monochromatic.*

**Proof.** To prove this theorem we apply Ramsey's Theorem. Let $N$ be the Ramsey number $R(r, 3)$, i.e., the minimal $n$ such that for every $r$-coloring of $[1, n]^2 = \{A \subseteq [1, n] : | A |= 2\}$ there exists a set $T \subseteq [1, n]$ with $| T |= 3$ so that $[T]^2 = \{B \subset T : | B |= 2\}$ is monochromatic. Let

$$\Delta : [1, N - 1] \longrightarrow [1, r]$$

be an $r$-coloring of $[1, N - 1]$. This coloring will induce an $r$-coloring $\Delta^*$ of the edges of the complete graph $K_N$ on the vertex set $\{0, 1, ..., N - 1\}$ by

$$\Delta^*(\{i, j\}) = \Delta(| i - j |), i \neq j.$$

By the definition of $N, K_N$ must contain a triangle with the vertices $\{i, j, k\}$ with $\Delta^*(\{i,j\}) = \Delta^*(\{i,k\}) = \Delta^*(\{j,k\})$. Without lost of generality we can assume $i > j > k$. Let

$$x = i - j \text{ and } y = i - k.$$

Then

$$\Delta(x) = \Delta(y) = \Delta(x+y). \blacksquare$$

We note that Hilbert's Cube Lemma only gives a monochromatic set $\{a + x, a+y, a+x+y\}$. Also, Hilbert's Cube Lemma is a density result, but Schur's Theorem is not (take the odd integers).

Until here we only prove the existence of $n$. An illustration below shows how to get more information about the number $n$.

*I* . Let $r = 5$ and take $n_0 \geq 327$. Let

$$\Delta : [1, n_0] \longrightarrow [1, 5]$$

be a 5-coloring of $[1, n_0]$ with no monochromatic $x, y, x+y$. Let $i_1 < 5$ be the most frequently occurring color in $[1, n_0]$, and let $x_0 < x_1 < \ldots < x_{n_1-1}$ have color $i_1$. Let $N_1 = \{x_i - x_0, 1 \leq i \leq n_1 - 1\}$. Then $N_1$ misses color $i_1$ (otherwise $x_i - x_0, x_0$ and $(x_i - x_0) + x_0$ have color $i_1$), and $n_1 \geq \frac{n_0}{5}$ or $n_0 \leq 5n_1$ (or $n_1 \geq 66$). Now let $i_2$ be the most frequently occurring color in $N_1$ and let

$$\Delta^{-1}(i_2) \cap N_1 = \{y_0 < y_1 < y_2 < \ldots < y_{n_2-1}\}.$$

Let

$$N_2 = \{y_i - y_0 : 1 \leq i \leq n_2 - 1\}.$$

Then $N_2$ misses $i_2$ and $i_1$. Also $n_2 \geq \frac{n_1-1}{4}$ or $n_1 \leq 4n_2 + 1$ (or $n_2 \geq 17$). Now let $i_3$ be the most frequent color in $N_2$ and let

$$\Delta^{-1}(i_3) \cap N_2 = \{z_0 < z_1 < z_2 < \ldots < z_{n_3-1}\}$$

Let

$$N_3 = \{z_i - z_0 : 1 \leq i \leq n_3 - 1\}$$

Then,

(i) $N_3$ misses color $i_3$, since $z_0, z_1, z_2, \ldots, z_{n_3-1}$ all have color $i_3$.

(ii) $N_3$ misses color $i_2$, since otherwise

$z_i - z_0 = (y_k - y_0) - (y_j - y_0) = y_k - y_j$ has color $i_2$.

(iii) $N_3$ misses color $i_1$, since otherwise

$z_i - z_0 = y_k - y_j = (x_\alpha - x_0) - (x_\beta - x_0) = x_\alpha - x_\beta$ has color $i_1$.

Also $n_3 \geq \frac{n_2-1}{3}$ or $n_2 \leq 3n_3 + 1$ (or $n_3 \geq 6$).

Let $i_4$ = most frequent color in $N_3$ and let

$$\Delta^{-1}(i_4) \cap N_3 = \{w_0 < w_1 < w_2 < ... < w_{n_4-1}\}.$$

Let

$$N_4 = \{w_i - w_0 : 1 \leq i \leq n_4 - 1\}$$

Then, $n_4 \geq \frac{n_3-1}{2}$ or $n_3 \leq 2n_4 + 1$ (or $n_4 \geq 3$) and $N_4$ misses colors :

- $i_4$, since otherwise $w_i - w_0$, $w_0, w_i$ are monochromatic.
- $i_3$, since otherwise $w_i - w_0 = (z_k - z_0) - (z_j - z_0) = z_k - z_j$ has color $i_3$, but then $z_k - z_j, z_j, z_k$ are monochromatic.
- $i_2$, since otherwise $w_i - w_0 = z_k - z_j = (y_\alpha - y_0) - (y_\beta - y_0) = y_\alpha - y_\beta$ has color $i_2$, but then $y_\alpha - y_\beta, y_\beta, y_\alpha$ are monochromatic.
- $i_1$, since otherwise $w_i - w_0 = y_\alpha - y_\beta = (x_s - x_0) - (x_t - x_0) = x_s - x_t$ has color $i_1$, but then $x_s - x_t, x_s, x_t$ are monochromatic.

Thus $N_4$ is monochromatic in color $i_5$.

Since $(w_2 - w_1) + (w_1 - w_0) = (w_2 - w_0)$, and $w_1 - w_0, w_2 - w_0$ have color $i_5$, $w_2 - w_1$ can not have color $i_5$. But, checking as before, $w_2 - w_1$ must also avoid colors $i_4, i_3, i_2, i_1$. This is impossible. Therefore there must be a monochromatic set $\{x, y, x + y\}$ in $[1, n_0]$.

This argument can be re-stated as follows :

Let

$$\Delta : [1, n_0] \longrightarrow [1, 5]$$

be a 5-coloring of $[1, n_0]$ such that there is no monochromatic set $\{x, y, x + y\}$ in $[1, n_0]$. Then by the above arguments we get

$$n_0 \leq 5n_1$$

$$n_1 \leq 4n_2 + 1$$

$$n_2 \leq 3n_3 + 1$$

$$n_3 \leq 2n_4 + 1$$

$$n_4 \leq 1 \cdot 1 + 1.$$

Therefore,

$$n_4 \leq 1 + 1$$

$$n_3 \leq 2 + 2 + 1$$

$$n_2 \leq 3 \cdot 2 + 3 \cdot 2 + 3 + 1$$

$$n_1 \leq 4! + 4! + 4 \cdot 3 + 1$$

$$n_0 \leq 5! + 5! + 5 \cdot 4 \cdot 3 + 5 \cdot 4 + 5 = 5!(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}) < 5!e. \blacksquare$$

We define $S(r)$ to be the minimum of $n(r)$ where $n(r)$ satisfies Schur's Theorem. One can use the method of the illustration to show that

$$S(r) \leq \lfloor r!e \rfloor + 1$$

for all $r \geq 1$.

However one can show directly that if $m = \lfloor r!e \rfloor + 1$, then every $r$-coloring of the edges of $K_m$ gives a monochromatic $K_3$. Therefore by the proof of Schur's Theorem given previously one gets $S(r) \leq \lfloor r!e \rfloor + 1$. We note that so far, the best known upper bound is $S(r) \leq r!(e - \frac{1}{24})$.

From Schur's theorem we can derive the following corollary.

**Corollary 3.1** *For all $m \geq 1$, there exists $N = N(m)$ such that*

$$x^m + y^m \equiv z^m (mod\, p)$$

*always has a non-trivial solution for each prime $p \geq N$.*

**Proof.** Fix $m \geq 1$ and let $p$ be a prime with $p > S(m)$, where $S(m)$ is as in Theorem 3.1. Then every $m$-coloring of $[1, p-1]$ gives monochromatic $u, v, w$ with $u + v = w$. Let

$$\mathbf{Z}_p^* = \{\overline{1}, \overline{2}, \overline{3}, ..., \overline{p-1}\},$$

the set of non-zero integers modulo $p$. This forms a cyclic group under multiplication modulo $p$. Say

$$\mathbf{Z}_p^* = \{g_1, ..., g_{p-1}\},$$

for some $g \in \mathbf{Z}_p^*$.

Let $t \in \mathbf{Z}_p^*$, then $t = g^k$ and $t^m = g^{km}$, for some $k$ with $1 \leq k \leq p-1$. Hence, the $m$th powers in $\mathbf{Z}_p^*$ are exactly

$$g^m, g^{2m}, g^{3m}, ...,$$

so these $m$th powers form a subgroup generated by $g^m$. This subgroup has order

$$\frac{p-1}{\gcd(m, p-1)}.$$

Now let

$$H = \{t^m : t \in \mathbf{Z}_p^*\}. \qquad (i)$$

$H$ is a subgroup of $\mathbf{Z}_p^*$ of index $n = gcd(m, p-1) \leq m$. Therefore $\mathbf{Z}_p^*$ is decomposed into cosets

$$t_1 H, t_2 H, ..., t_n H,$$

where $t_j \in \mathbf{Z}_p^*, 1 \leq j \leq n$.

Since $a, b$ belong to the same coset if and only if $ab^{-1} \in H$, we see that the cosets of $\mathbf{Z}_p^*$ (relatively to $H$) define an $n$-coloring $\Delta$ of $\mathbf{Z}_p^*$ with

$$\Delta(a) = \Delta(b) \text{ if and only if } ab^{-1} \in H. \qquad (ii)$$

Since $n \leq m$, by Schur's Theorem there exist $a, b, c \in [1, p-1]$ with

$$\Delta(a) = \Delta(b) = \Delta(c) \text{ and } a + b = c.$$

From $\Delta(a) = \Delta(b) = \Delta(c)$, by $(ii)$ we have $1, a^{-1}b$ and $a^{-1}c \in H$.
Therefore in $\mathbf{Z}_p^*$ we have

$$a^{-1}a + a^{-1}b = a^{-1}c$$

or

$$1 + a^{-1}b = a^{-1}c,$$

with $1, a^{-1}b$ and $a^{-1}c \in H$. By $(i)$, there exist $x, y, z \in \mathbf{Z}_p^*$ with $1 = x^m$, $a^{-1}b = y^m$ and $a^{-1}c = z^m$, i.e., there exist $x, y, z$ in $[1, p-1]$ such that $x^m + y^m \equiv z^m$ (modulo $p$). ■

Now, we finish this chapter with two problems related to Schur's Theorem.
First, let

$$f : \mathbf{N} \longrightarrow \mathbf{N}$$

be the function defined in the following way. For $r \in \mathbf{N}$, let $f(r)$ be such that

(i). There is $\Delta : [1, f(r)] \longrightarrow [1, r]$ such that for any $x, y \in [1, f(r)]$

$$\Delta(x) = \Delta(y) \Longrightarrow (\Delta(x+y) \neq \Delta(x) \text{ or } x+y > f(r))$$

(ii). For any $\Delta : [1, f(r)+1] \longrightarrow [1, r]$ there are $x, y \in [1, f(r)+1]$ with

$$\Delta(x) = \Delta(y) = \Delta(x+y).$$

In other words, $f(r)$ is the maximum of all $n$ with the property that we can find an $r$-coloring of $[1, n]$ with no monochromatic solution to the equation $x + y = z$.

**Theorem 3.2** *For all $r \in \mathbf{N}$ we have that*

$$\frac{3^r - 1}{2} \leq f(r) \leq R(r, 3) - 2.$$

**Proof.** From the proof of Schur's Theorem we have that if $[1, R(r,3) - 1]$ is $r$-colored there is a monochromatic set $\{x, y, x+y\}$. Thus

$$R(r, 3) - 1 > f(r)$$

or

$$f(r) \leq R(r, 3) - 2.$$

Let $n \in \mathbf{N}$ and let $\Delta : [1, n] \longrightarrow [1, r]$. Let

$$\Delta' : [1, 3n+1] \longrightarrow [1, r+1]$$

be defined in the following way.

$$\Delta'(x) = \begin{cases} \Delta(x), & \text{for } x \in [1, n] \\ r+1, & \text{for } x \in [n+1, 2n+1] \\ \Delta(x - (2n+1)), & \text{for } x \in [2n+2, 3n+1]. \end{cases}$$

Suppose that $\Delta$ is such that for any $x, y \in [1, n]$,

$$\Delta(x) = \Delta(y) \Longrightarrow (\Delta(x+y) \neq \Delta(x) \text{ or } x+y > n)$$

and let $u, v \in [1, 3n+1]$ be such that $\Delta'(u) = \Delta'(v)$ and $u + v \leq 3n + 1$.

Clearly, there are four cases.

The first case is that $u, v \in [1, n]$ and $u + v \leq n$. Then

$$\Delta'(u) = \Delta(u) \neq \Delta(u+v) = \Delta'(u+v)$$

and we have that $\{u, v, u+v\}$ is not $\Delta'$-monochromatic.

The second case is that $u, v \in [1, n]$ and $u + v \in [n+1, 2n+1]$. In this case,

$$\Delta'(u) = \Delta(u) \in [1, r] \text{ and } \Delta'(u+v) = r+1$$

and again $\{u, v, u+v\}$ is not $\Delta'$-monochromatic.

The third case is that $u, v \in [n+1, 2n+1]$. Now we have that

$$\Delta'(u) = \Delta'(v) = r+1$$

and, since

$$u + v > 2n+1, \text{ then } \Delta'(u+v) \in [1, r].$$

Thus, $\{u, v, u+v\}$ is not $\Delta'$-monochromatic.

The fourth case is that $u \in [1, n]$ and $v \in [2n+2, 3n+1]$. Since $u+v \leq 3n+1$ we have that $u + v - (2n+1) \leq n$ and

$$\Delta'(u+v) = \Delta(u+v-(2n+1)) = \Delta'(u+(v-2n-1)).$$

From

$$\Delta(u) = \Delta'(u) = \Delta'(v) = \Delta(v-2n-1)$$

it follows that

$$\Delta'(u+v) \neq \Delta'(u).$$

Hence, $\{u, v, u+v\}$ is not $\Delta'$-monochromatic.

Therefore, if $\Delta$ is such that there is no monochromatic solution for $x + y = z$ in $[1, n]$ then $\Delta'$ is such that there is no monochromatic solution for $x + y = z$ in $[1, 3n+1]$. In other words, if $n \leq f(r)$ then $3n+1 \leq f(r+1)$.

Clearly, $f(1) = 1 = \frac{1}{2}(3^1 - 1)$. Let $r \geq 2$ be such that

$$f(r-1) \geq \frac{1}{2}(3^{r-1} - 1).$$

Then

$$f(r) \geq 3 \cdot \frac{1}{2}(3^{r-1} - 1) + 1 = \frac{1}{2}(3^r - 3 + 2) = \frac{1}{2}(3^r - 1).$$

Therefore by the principle of mathematical induction we have that for all $r \in \mathbf{N}$

$$\frac{1}{2}(3^r - 1) \leq f(r). \blacksquare$$

A *Schur number* is any element of the range of the function $f$. So far the only known exact values of $f(r)$ are $f(1) = 1, f(2) = 4, f(3) = 13$ and $f(4) = 44$. The best bounds for $f(5)$ are $160 \leq f(5) \leq 321$. The lower bound of $f(5)$ was proved by G. Exoo in [8] and the upper bound was proved by E.G. Whitehead in [26].

For the second problem related to Schur's Theorem that we discuss, let us introduce the following notation.

For $n \in \mathbf{N}$ let $\mathcal{S}_n$ be the family of non-empty subsets of $[1, n]$ such that $S \in \mathcal{S}_n$ if and only if

$$x, y \in S \Longrightarrow x + y \notin S.$$

Let $N = \mathrm{a\quad x}\{| S |: S \in \mathcal{S}_n\}$ and let

$$\mathcal{S}_n^* = \{S \in \mathcal{S}_n : | S |= N\}.$$

Note that for any $a, b \in [1, n]$

$$\{a, b\} \in \mathcal{S}_n.$$

Thus, $\mathcal{S}_n \neq \emptyset$ and $\mathcal{S}_n^* \neq \emptyset$.

It is easy to see that for any $n \in \mathbf{N}$

$$O_n^* = \{2k - 1 : k \in [1, \lceil\frac{n}{2}\rceil]\} \in \mathcal{S}_n$$

and

$$T_n^* = [\lfloor\frac{n}{2}\rfloor + 1, n] \in \mathcal{S}_n.$$

Here, for $a \in \mathbf{R}^+$,

$$\lceil a \rceil \text{ denotes } \quad \mathrm{in}\{k \in \mathbf{Z} : k \geq a\}$$

and

$$\lfloor a \rfloor \text{ denotes } \quad \mathrm{ax}\{k \in \mathbf{Z} : k \leq a\}.$$

We prove the following.

**Theorem 3.3** $S \in \mathcal{S}_n^* \Longrightarrow | S |= \lceil\frac{n}{2}\rceil$.

**Proof.** Let $S \in \mathcal{S}_n^*$ and let $m = \quad \mathrm{ax}\{x : x \in S\}$. Let

$$g : S \longrightarrow [0, \lceil\frac{m-2}{2}\rceil]$$

be defined in the following way. For $a \in S$ and $i \in [0, \lceil \frac{m-2}{2} \rceil]$ let $g(a) = i$ if and only if $a \in \{i, m-i\}$. Since

$$[0, m] = [0, \lceil \frac{m-2}{2} \rceil] \cup [\lceil \frac{m}{2} \rceil, m]$$

we have that

$$S \cap [0, m] = (S \cap [0, \lceil \frac{m-2}{2} \rceil]) \cup (S \cap [\lceil \frac{m}{2} \rceil, m].$$

Note that if $m$ is an even number then $\lceil \frac{m}{2} \rceil = \frac{m}{2} \notin S$ (otherwise $\frac{m}{2} + \frac{m}{2} = m$), and if $m$ is an odd number then $\lceil \frac{m}{2} \rceil = m - \lceil \frac{m-2}{2} \rceil$.

Therefore

$$S = (S \cap [0, \lceil \frac{m-2}{2} \rceil]) \cup (S \cap \{m - i : i \in \lceil \frac{m-2}{2} \rceil\}).$$

Hence, $g$ is well defined. Furthermore, if $a, b \in S$ are such that $g(a) = g(b) = i$ then $\{a, b\} \subseteq \{i, m-i\}$. If $a \neq b$ then $\{a, b\} = \{i, m-i\}$ and $a + b = m \in S$. This is not possible and therefore $a = b$. Thus, $g$ is $1-1$. This means that

$$| S | = | g(S) | \leq \lceil \frac{m-2}{2} \rceil + 1 = \lceil \frac{m}{2} \rceil \leq \lceil \frac{n}{2} \rceil.$$

On the other hand since $O_n^* \in \mathcal{S}_n$ and since $| O_n^* | = \lceil \frac{n}{2} \rceil$, we have that $| S | \geq \lceil \frac{n}{2} \rceil$. Therefore

$$| S | = \lceil \frac{n}{2} \rceil. \blacksquare$$

**Remark 3.1** *Let $n = 8$ and let us consider the set $A = \{1, 4, 6\}$. Clearly, $A \in \mathcal{S}_8$. Let us note the following two facts. First, $| A | = 3 < \lceil \frac{8}{2} \rceil = 4$. Secondly, for any $x \in [1, 8] \backslash A$, $A \cup \{x\} \notin \mathcal{S}_8$. This means that there are $n$ such that*

$$\mathcal{S}_n^* \subsetneqq \{S \in \mathcal{S}_n : S \cup \{x\} \notin \mathcal{S}_n \text{ for all } x \in [1, n] \backslash S\}.$$

Next, we discuss the following problem.

**Problem.** *For given $n$, find all elements of $\mathcal{S}_n^*$.*

**Theorem 3.4** *Let $k \in \mathbf{N}$. Then*

$$\mathcal{S}_{2k+1}^* = \{O_{2k+1}^*, T_{2k+1}^*\}.$$

**Proof.** Clearly,

$$\mathcal{S}_3^* = \{\{1, 3\}, \{2, 3\}\} = \{O_3^*, T_3^*\}$$

and

$$\mathcal{S}_5^* = \{\{1,3,5\},\{3,4,5\}\} = \{O_5^*, T_5^*\}$$

so that the claim is true for $k = 1$ and $k = 2$.

Let $k \geq 3$ be the smallest integer such that there exist a set $S \in \mathcal{S}_{2k+1}^*$ with

$$O_{2k+1}^* \neq S \neq T_{2k+1}^*. \qquad (*)$$

Then $\{2k, 2k+1\} \subseteq S$. Indeed. If $2k+1 \notin S$ then $S \in \mathcal{S}_{2k}^*$. This implies, by Theorem 3.3, that

$$k = \lceil \frac{2k}{2} \rceil = \mid S \mid = \lceil \frac{2k+1}{2} \rceil = k+1,$$

which is impossible. If $2k \notin S$, then

$$S \backslash \{2k+1\} \in \mathcal{S}_{2(k-1)+1}^* = \mathcal{S}_{2k-1}^*.$$

Since, by our choice of $k$,

$$\mathcal{S}_{2(k-1)+1}^* = \mathcal{S}_{2k-1}^* = \{O_{2k-1}^*, T_{2k-1}^*\}$$

in the case if $2k \notin S$ we would have that

$$S = T_{2k-1}^* \cup \{2k+1\} = [k, 2k-1] \cup \{2k+1\}.$$

Since $k \geq 3$ we have that $k+1 \in [k, 2k-1]$. Thus, if $2k \notin S$ then $\{k, k+1, 2k+1\} \subseteq S$. this contradicts the fact that $S \in \mathcal{S}_{2k+1}^*$. Therefore $\{2k, 2k+1\} \subseteq S$. Hence, $1, k \notin S$.

Now, let

$$S' = S \backslash \{2k, 2k+1\}, \;\; B = [2, k-1] \cup [k+1, 2k-1],$$

then $S' \subseteq B$ and

$$\mid S' \mid = (k+1) - 2 = k-1.$$

Let

$$S_1' = S' \cap [2, k-1]$$

and

$$S_2' = S' \cap [k+1, 2k-1].$$

Then we can express $S'$ as a disjoint union of $S_1'$ and $S_2'$, i.e.,

$$S' = S_1' \cup S_2'.$$

If $S_1' = \emptyset$ then $S' = [k+1, 2k-1]$. It implies

$$S = [k+1, 2k+1] = T_{2k+1}^*$$

which contradicts $(*)$. If $S'_2 = \emptyset$ then $S' = S'_1$. It implies

$$k-1 = |S'| = |S| \leq |[2,k-1]| = (k-1)-1 = k-2.$$

Impossible. Therefore

$$S' = S'_1 \cup S'_2$$

with

$$S'_1 \neq \emptyset \neq S'_2.$$

Let $S'_1 = \{a_1, ..., a_p\}$ where $2 \leq a_1 < ... < a_p \leq k-1$. Since for $1 \leq i \leq p$, we have $a_i + 2k - a_i = 2k \in S$ and $a_i + 2k + 1 - a_i = 2k+1 \in S$ then

$$C = \{2k - a_1, 2k+1-a_1, ..., 2k-a_p, 2k+1-a_p\} \subseteq [k+1, 2k-1] \backslash S'_2.$$

Note that $|C| \geq p+1$. Hence
$k-1 = |S'| = |S'_1| + |S'_2| \leq p + \{(k-1)-(p+1)\} = k-2$. Impossible.
Conclusion: $\mathcal{S}^*_{2k+1} = \{O^*_{2k+1}, T^*_{2k+1}\}$ for all $k \in \mathbf{N}$. ■

Now, we want to find $\mathcal{S}^*_k$ if $k$ is even. It is easy to check that

$$\mathcal{S}^*_4 = \{\{1,3\},\{2,3\},\{3,4\},\{1,4\}\} = \{O^*_3, T^*_3, T^*_4\} \cup \{\{1,4\}\},$$

$$\mathcal{S}^*_6 = \{\{1,3,5\},\{3,4,5\},\{4,5,6\},\{2,5,6\},\{1,4,6\} = \{O^*_5, T^*_5, T^*_6\} \cup \{\{2,5,6\},\{1,4,6\}\}$$

and

$$\mathcal{S}^*_8 = \{\{1,3,5,7\},\{4,5,6,7\},\{5,6,7,8\},\{2,3,7,8\}\} = \{O^*_7, T^*_7, T^*_8\} \cup \{\{2,3,7,8\}\}.$$

**Theorem 3.5** *For* $k \geq 5$,

$$\mathcal{S}^*_{2k} = \{T^*_{2k-1}, O^*_{2k-1}, T^*_{2k}\}.$$

**Proof.** Let $k \geq 5$. Clearly, $\{T^*_{2k-1}, O^*_{2k-1}, T^*_{2k}\} \subseteq \mathcal{S}^*_{2k}$. It is enough to show that if $T \in \mathcal{S}^*_{2k}$ with $2k \in T$ then $T = T^*_{2k}$. Let $n \geq 5$ be the smallest integer such that $2n \in T \in \mathcal{S}^*_{2n} \backslash \{T^*_{2n}\}$.

Suppose that $2n-1 \notin T$. Then $|T \backslash \{2n\}| = n-1$ and $T \backslash \{2n\} \in \mathcal{S}^*_{2n-2}$.

If $T\backslash\{2n\} = T^*_{2n-3} = [n-1, 2n-3]$, then $\{n, 2n\} \subseteq T$. This contradicts the fact that $T \in \mathcal{S}^*_{2n}$.

If $T\backslash\{2n\} = O^*_{2n-3}$, then $\{3, 2n-3\} \subseteq T\backslash\{2n\}$. This implies $\{3, 2n-3, 2n\} \subseteq T$ which contradicts the fact that $T \in \mathcal{S}^*_{2n}$.

If $T\backslash\{2n\} = T^*_{2n-2} = [n, 2n-2]$, then $\{n, 2n\} \subseteq T$. This contradicts the fact that $T \in \mathcal{S}^*_{2n}$. Therefore, by the choice of $n$, the only possibility is $n = 5$. In this case

$$T\backslash\{2n\} \in \mathcal{S}^*_8 = \{O^*_7, T^*_7, T^*_8\} \cup \{\{2,3,7,8\}\}.$$

Since $T\backslash\{2n\} \neq O_7^*, T\backslash\{2n\} \neq T_7^*$ and $T\backslash\{2n\} \neq T_8^*$ then $T\backslash\{2n\} = \{2,3,7,8\}$. This implies $\{2,8,10\} \in T$ which contradicts the fact that $T \in \mathcal{S}_{10}^*$.

Hence, $2n-1 \in T$ if $2n \in T \in \mathcal{S}_{2n}^*$. Note that in this case $1, n \notin T$.

Let $T' = T\backslash\{2n-1, 2n\}$. Then $| T' |= n-2$ and $T' \subseteq [2, n-1] \cup [n+1, 2n-2]$. Furthermore, we can express $T'$ as

$$T' = T_1' \cup T_2'$$

where

$$T_1' = T' \cap [2, n-1]$$

and

$$T_2' = T' \cap [n+1, 2n-2].$$

If $T_1' = \emptyset$ then $T' = T_2' = [n+1, 2n-2]$. This implies $T = [n+1, 2n] = T_{2n}^*$, contradicting the fact that $T \in \mathcal{S}_{2n}^*\backslash\{T_{2n}^*\}$.

If $T_2' = \emptyset$ then $T' = T_1' = [2, n-1]$. Since $n \geq 5$ then $\{2,4\} \subseteq [2, n-1] = T' \subseteq T$, contradicting the fact that $T \in \mathcal{S}_{2n}^*$.

Therefore

$$T_1' \neq \emptyset \neq T_2'.$$

Let

$$T_1' = \{a_1, ..., a_p\}$$

with

$$2 \leq a_1 < ... < a_p \leq n-1.$$

There are two possibilities: either $2n-2 \in T_2'$ or $2n-2 \notin T_2'$.

If $2n-2 \in T_2'$ then $a_p < n-1$ (otherwise $\{n-1, 2n-2\} \subseteq T' \subseteq T$ which is impossible since $T \in \mathcal{S}_{2n}^*$). Therefore

$$D = \{2n-1-a_i : 1 \leq i \leq p\} \cup \{2n - a_i : 1 \leq i \leq p\} \subseteq [n+1, 2n-2]\backslash T_2'.$$

Note that $| D |\geq p+1$. Hence,

$n-2 =| T' |=| T_1' | + | T_2' |\leq p + \{(n-2)-(p+1)\} = n-3$. Impossible.

Thus $2n-2 \notin T_2'$. Let

$$T_2' = \{b_1, ..., b_q\}$$

where

$$k+1 \leq b_1 \leq ... \leq b_q < 2n-2.$$

Therefore

$$E = \{2n-1-b_j : 1 \leq j \leq q\} \cup \{2n - b_1 : 1 \leq j \leq q\} \subseteq [2, n-1]\backslash T_1'.$$

Note that $| E |\geq q+1$. Hence,

$$n-2 =| T' |=| T_1' | + | T_2' |\leq \{(n-2)-(q+1)\} + q = n-3.$$

This contradiction completes the proof. ■

# Chapter 4

# GENERALIZATION OF SCHUR'S THEOREM

In this chapter we will see two closely related generalizations of Schur's Theorem. They are Rado's Theorem and Folkman's Theorem.

Since Folkman's Theorem can be derived from Rado's Theorem, we will state and prove Rado's Theorem first. To do this we need the definition of the regularity of a system of equations.

Let $S = S(x_1, ..., x_n)$ denote a system of equations in the variables $x_1, ..., x_n$.

**Definition 4.1** *Let $T$ be a set of real numbers. We call $S$ to be $r$-regular on $T$ if for every $r$-coloring of $T$ we can find $a_1, ..., a_n \in T$ which are not necessarily distinct such that $\{a_1, ..., a_n\}$ is monochromatic and $(a_1, ..., a_n)$ is a solution to the system $S$. Furthermore, if $S$ is $r$-regular on $T$ for every positive integer $r$, then we say that $S$ is regular on $T$.*

According to Schur's Theorem, for each positive integer $r$ and any $r$-coloring of $\mathbf{N}$ we can find $a_1, a_2, a_3$ such that $a_1+a_2-a_3 = 0$ and $a_1, a_2, a_3$ are monochromatic, hence the equation $x_1 + x_2 - x_3 = 0$ is regular on $\mathbf{N}$.

Theorem 4.2 below gives the sufficient and necessary conditions of regularity of a system of linear homogeneous equations. We prove this for the special case of a single linear homogeneous equation.

**Theorem 4.1** *Let $S : c_1x_1 + ... + c_nx_n = 0, c \in \mathbf{Z}$ be a linear homogeneous equation. Then, $S$ is regular on $\mathbf{N}$ if and only if there exists a positive integer $k$ with $0 < k \leq n$ such that .*

$$\sum_{j=1}^{k} c_j = 0$$

*with not all $c_j = 0$.*

To prove the theorem, we need a lemma :

**Lemma 4.1** *Let $r, k, s \geq 1$. Then there exists $n = n(r, k, s)$ so that if $[1, n]$ is $r$-colored, there exist $a, d > 0$ with*

$$\{a, a+d, a+2d, ..., a+kd\} \cup \{sd\} \qquad (*)$$

*monochromatic.*

**Proof.** Induction on $r$. Let $r = 1$. There are two possibilities, either $k < s$ or $k \geq s$ :

- If $k < s$, then by taking $n = s$ and $a = d = 1$ we have

$$\{a, a + d, a + 2d, ..., a + kd\} \cup \{sd\} = \{1, 2, ..., 1 + k\} \cup \{s\},$$

a monochromatic set in $[1, n]$.

- If $k \geq s$, then by taking $n = k + 1$ and $a = d = 1$ we have

$$\{a, a + d, a + 2d, ..., a + kd\} \cup \{sd\} = \{1, 2, ..., 1 + k\},$$

a monochromatic set in $[1, n]$.

Therefore we can take $n(k, 1, s) = \max\{k + 1, s\}$.

Using van der Waerden's Theorem, let $W(r, t)$ be the minimal $W$ such that if $[1, W]$ is $r$-colored there exists a monochromatic arithmetic progression of length $t + 1$.

Let $r, k, s$ be given. Assume that $r \geq 2$ and that $n(r - 1, k, s)$ exists. Take

$$n = sW(r, kn(r - 1, k, s))$$

and let

$$\chi : [1, n] \longrightarrow [1, r]$$

be an $r$-coloring of $[1, n]$.

Let us consider the restriction of $\chi$ on $[1, W(r, kn(r - 1, k, s))]$. By van der Waerden's Theorem, among $[1, W(r, kn(r-1, k, s))]$ we can find a monochromatic arithmetic progression

$$a', a + d', a' + 2d', ..., a' + kn(r - 1, k, s)d' \qquad (**)$$

for some positive integers $a'$ and $d'$.

Consider two possibilities :

(i). There exists $j \in [1, n(r - 1, k, s)]$ such that $sd'j$ has the same color as $(**)$.

(ii). For every $j \in [1, n(r - 1, k, s)], sd'j$ and $(**)$ have different color.

If (i) happens then we also have

$$\{a', a' + jd', a' + 2jd', ..., a' + kjd'\} \subseteq \{a', a' + d', a' + 2d', ...., a' + kn(r - 1, k, s)d'\},$$

since $kjd' \leq kn(r - 1, k, s)d'$.

By taking $a = a'$ and $d = jd'$ we conclude that $(*)$ has the same color as $(**)$, and we are done.

If (ii) happens then $\{sd'j : 1 \leq j \leq n(r - 1, k, s)\}$ is $(r - 1)$-colored. Let

$$f : \{sd', 2sd', 3sd', ..., n(r - 1, k, s)sd\} \longrightarrow \{1, 2, ..., n(r - 1, k, s)\}$$

with

$$f(x) = \frac{x}{sd}.$$

Then $f$ is $1-1$. So an $(r-1)$-coloring of $\{sd', 2sd', 3sd', ..., n(r-1,k,s)sd'\}$ will result in an $(r-1)$-coloring of $[1, n(r-1,k,s)]$ and vice versa. Therefore the $(r-1)$-coloring $\chi$ of $\{sd', 2sd', 3sd', ..., n(r-1,k,s)sd'\}$ will result in an $(r-1)$-coloring $\chi'$ of $[1, n(r-1,k,s)]$. (Here, we take $\chi'(x) = \chi(sd'x)$.) By the induction hypothesis, there exist $a'', d'' > 0$ such that

$$\{a'', a''+d'', ..., a''+kd'\} \cup \{sd''\}$$

is a monochromatic set under $\chi'$ in $[1, n(r-1,k,s)]$. Therefore

$$\{sd'a'', sd'(a''+d''), ..., sd'(a''+kd'')\} \cup \{sd'sd''\}$$

is monochromatic under $\chi$ in

$$\{sd', 2sd', 3sd', ..., n(r-1,k,s)sd'\}.$$

By taking $a = sd'a''$ and $d = sd'd''$, then $(*)$ is monochromatic in $[1, n]$. ■

Let us note that Lemma 4.1 generalizes both van der Waerden's Theorem and Schur's Theorem. Clearly Lemma 4.1 implies van der Waerden's Theorem. If in Lemma 4.1 we take $s = 1$ and $k = 1$, then we have Schur's Theorem.

**Corollary 4.1** *Let $r, k, s \geq 1$. Then there exists $n = n(r,k,s)$ so that if $[1, n]$ is $r$-colored, there exist $a, d > 0$ such that*

$$\{a + \lambda d : |\lambda| \leq k) \cup \{sd\}$$

*is monochromatic.*

The proof of this Corollary is simply from Lemma 4.1 by taking $k' = 2k$ to find $a', d'$ such that $\{a', a'+d', a'+2d', ..., a'+2kd') \cup \{sd'\}$ is monochromatic. Then by taking $d = d'$ and $a = a' + kd'$, the Corollary is proved.

**Proof of Theorem 4.1** ($\Longleftarrow$) Suppose $0 < k \leq n$ and $c_{i_1} + ... + c_{i_k} = 0$ with not all $c_{i_j} = 0$ for $1 \leq j \leq k$. If $k = n$ then $(x_1, ...x_n)$ with $x_1 = ... = x_n = m$ satisfies the equation for every $m \in \mathbf{N}$. Therefore we can assume $0 < k < n$. Re-numbering if necessary, assume that

$$c_1 + c_2 ... + c_k = 0.$$

Let

$$B = c_{k+1} + ... + c_n.$$

If $B = 0$ then $(x_1, ..., x_n)$ with $x_1 = ... = x_k = m$ and $x_{k+1} = ... = x_n = p$ satisfies the equation for every $m, p \in \mathbf{N}$. Hence we may assume $0 < k < n$ and $c_{k+1} + ... + c_n \neq 0$. Let

$$A = gcd(c_1, ..., c_k)$$

and

$$s = \frac{A}{w},$$

where $w = gcd(A, B)$.

Since $w \mid B$ we find $v \in \mathbf{Z}$ such that $B = vw$. Hence,

$$w = \frac{A}{s} = \frac{B}{v},$$

or

$$Av - Bs = 0.$$

Taking $t = -v$, we have

$$At + Bs = 0. \qquad (i)$$

Furthermore, since $A = gcd(c_1, ..., c_k)$, we find $\alpha_1, ..., \alpha_k \in \mathbf{Z}$ such that $c_1\alpha_1 + ... + c_k\alpha_k = A$. Letting $\lambda_i = \alpha_i t$, we have

$$c_1\lambda_1 + ... + c_k\lambda_k = At. \qquad (ii)$$

From $(i)$ and $(ii)$, we have

$$c_1\lambda_1 + ... + c_k\lambda_k + (c_{k+1} + ... + c_n)s = 0.$$

Let $\chi : \mathbf{N} \longrightarrow [1, r]$ be an $r$-coloring of $\mathbf{N}$ and let $M = max\{\mid \lambda_i \mid : 1 \leq i \leq k\}$. By Corollary 4.1 there are $a, d > 0$ such that

$$X = \{a + \lambda d : \mid \lambda \mid \leq M\} \cup \{sd\}$$

is monochromatic. Note that $a + \lambda_i d \in X$, $1 \leq i \leq k$. Now, let $(x_1, ..., x_n) \in \mathbf{Z}^n$ be such that

$$x_i = a + \lambda_i d, \text{ for } 1 \leq i \leq k,$$

and

$$x_i = sd, \text{ for } k + 1 \leq i \leq n.$$

Then $\{x_1, ..., x_n\} \subseteq X$. Thus $\{x_1, ..., x_n\}$ is monochromatic. Furthermore,

$$c_1x_1 + ... + c_nx_n = \sum_{i=1}^{n} c_ix_i = \sum_{i=1}^{k} c_i(a + \lambda_i d) + sd \sum_{i=k+1}^{n} c_i$$

$$= a\sum_{i=1}^{k} c_i + d\left[\sum_{i=1}^{k} c_i\lambda_i + s\sum_{i=k+1}^{n} c_i\right] = 0 + d(At + Bs) = 0.$$

Thus, $(x_1, ..., x_n)$ is a monochromatic solution of $S$. Therefore $S$ is regular.

($\Longrightarrow$). We will prove the contrapositive.

Let $\{c_1, ..., c_n\} \subseteq \mathbf{Z}$ be such that if $\emptyset \neq F \subseteq [1, n]$ then $\sum_{i\in F} c_i \neq 0$. Let $p$ be a prime such that $p > \max\{|\sum_{i\in F} c_i| : \emptyset \neq F \subseteq [1, n]\}$. Define $F_p$, the $(p-1)$-coloring on $\mathbf{N}$ in the following way. If $a \in \mathbf{N}$ and $b \in [1, p-1]$, then $F_p(a) = b$ if and only if there are a non-negative integer $i$ and an integer $m$ such that

$$a = p^i(mp + b).$$

Thus, $F_p(a) = b$ if and only if there is a non-negative $i$ such that

$$\frac{a}{p^i} \equiv b (\text{mod } p)$$

For example, $F_5(25) = F_5(5^2(0 \cdot 5 + 1)) = 1$ and $F_5(37) = F_5(5^0(7 \cdot 5 + 2)) = 2$.
Note that, for $\alpha, x, y \in \mathbf{N}$

$$F_p(\alpha x) = F_p(\alpha y) \Longrightarrow F_p(x) = F_p(y).$$

Indeed, if $\alpha = p^i(m_\alpha p + b_\alpha), x = p^j(m_x p + b_x), y = p^k(m_y p + b_y)$ with $b_\alpha, b_x, b_y \in [1, p-1]$, then

$$\alpha x = p^{i+j}\{(m_\alpha m_x p + m_\alpha b_x + m_x b_\alpha)p + b_\alpha b_x\}$$

and

$$\alpha y = p^{i+k}\{(m_\alpha m_y p + m_\alpha b_y + m_y b_\alpha)p + b_\alpha b_y\}.$$

Since $F_p(\alpha x) = F_p(\alpha y)$ then $b_\alpha b_x \equiv b_\alpha b_y (\text{mod } p)$.
If, let us say, $b_x > b_y$, then there is $q \in \mathbf{N}$ such that

$$b_\alpha(b_x - b_y) = b_\alpha b_x - b_\alpha b_y = qp.$$

This contradicts the fact that $p$ is prime and $b_\alpha, b_x - b_y \in [1, p-1]$ so $p \nmid b_\alpha(b_x - b_y)$. Thus,

$$F_p(x) = b_x = b_y = F_p(y).$$

Now, let us consider the equation $S$:

$$\sum_{i=i}^{n} c_i x_i = 0, \qquad (*)$$

and suppose that $(a_1, ..., a_n)$ is an $F_p$-monochromatic solution of $(*)$. Let $a = gcd(a_1, ..., a_n)$ and let $a_i = aa'_i$, $i = 1, ..., n$. By our note above, we have that $(a'_1, ..., a'_n)$ is an $F_p$-monochromatic solution of $(*)$ where $gcd(a'_1, ..., a'_n) = 1$.

Hence,

$$F = \{i \in [1, n] : p \nmid a'_i\} \neq \emptyset.$$

Since $(a'_1, ..., a'_n)$ is an $F_p$-monochromatic solution of $(*)$, there exists $b \in [1, p-1]$ such that

$$F_p(a'_i) = b, 1 \leq i \leq n.$$

For $i \in F$, let $m_i \in \mathbf{Z}$ be such that

$$a'_i = m_i p + b,$$

and let $m \in \mathbf{Z}$ be such that

$$\sum_{i \in [1,n] \setminus F} c_i a'_i = mp.$$

Thus,

$$0 = \sum_{i=i}^{n} c_i a'_i = \sum_{i \in F} c_i a'_i + \sum_{i \in [1,n] \setminus F} c_i a'_i$$

$$= \sum_{i \in F} c_i (m_i p + b) + mp$$

$$= p \sum_{i \in F} c_i m_i + b \sum_{i \in F} c_i + mp$$

$$= (m + \sum_{i \in F} c_i m_i) p + b \sum_{i \in F} c_i.$$

Hence,

$$b\sum_{i\in F} c_i = -(m + \sum_{i\in F} c_i m_i)p.$$

From the facts that $b < p$ and $p$ is prime

$$p \mid \sum_{i\in F} c_i.$$

This contradicts the fact that $p >\mid \sum_{i\in F} c_i \mid\neq 0$.

Therefore $S$ is not regular. This completes the proof. ■

Theorem 4.1 characterizes the regularity of a single linear homogeneous equation. It is usually named as Rado's Theorem-Abridged. This next theorem gives the necessary and sufficient conditions for the regularity of an arbitrary system of linear homogeneous equations. Before doing this we need the definition of the columns condition of a matrix.

**Definition 4.2** *A matrix* $\mathbf{C} = c_{ij}$ *with entries from* $\mathbf{Z}$ *is said to satisfy the Columns condition if we can order the column vectors* $c_1, ..., c_n$ *so that there exist* $0 = k_0 < k_1 < k_2 < ... < k_t = n$ *such that if we define columns* $A_1, A_2, ..., A_t$ by

$$A_i = \sum_{j=k_{i-1}+1}^{k_i} c_j,$$

*then* $A_1 = 0$ *and*

$$A_i = \sum_{j=1}^{k_{i-1}} a_{ij} c_j$$

*for* $1 < i \leq t$ *and* $a_{ij} \in \mathbf{Q}$.

Now we are ready to state the generalization of Theorem 4.1.

**Theorem 4.2 (Rado's Theorem Complete).** *The system of linear homogeneous equations (with coefficients from* $\mathbf{Z}$*)* $\mathbf{Cx} = 0$ *is regular on* $\mathbf{N}$ *if and only if* $\mathbf{C}$ *satisfies the Columns condition.*

It turns out that this Theorem is equivalent to:

*The system of linear homogeneous equation* $\mathbf{Cx} = \mathbf{0}$ *is regular on* $\mathbf{N}$ *if and only if for every prime number p the system has a monochromatic solution with the coloring* $F_p$ *defined in the proof of Theorem 4.1.*

The proof of this Theorem can be found in [16].

A special case of Rado's Theorem is Folkman's Theorem.

**Definition 4.3** *Let $S \subseteq \mathbf{N}$. We define $\mathcal{P}(S)$, the sum-set of $S$ to be* $\mathcal{P}(S) = \{\sum_{s \in S} \epsilon_s s$, *where* $\epsilon_s = 0, 1$ *and* $\epsilon_s = 1$ *for a finite non-zero number of* $s\}$.

**Theorem 4.3 (Folkman's Theorem).** *If* $\mathbf{N}$ *is finitely colored then there exist arbitrarily large finite sets* $S$ *such that* $\mathcal{P}(S)$ *is monochromatic.*

Folkman's Theorem is a generalization of Schur's Theorem. To see that, it is enough to ask that the set $S$ in Folkman's Theorem has two elements. If $S = \{a, b\}$ then $\mathcal{P}(S) = \{a, b, a + b\}$, and for such $S$ Folkman's Theorem gives us the claim of Schur's Theorem.

Here is a way to connect Folkman's Theorem and Rado's Theorem.

Let $k \in \mathbf{N}$ and $P_k = \{T : \emptyset \neq T \subseteq [1, k]\}$. Let us consider the system $\mathcal{S}_k$ of $2^k - 1$ equations with $2^k - 1$ unknowns, given by:

$$x_T = \sum_{i \in T} x_{\{i\}},\ T \in P_k.$$

If the claim of Folkman's Theorem is true, then for any finite coloring of $\mathbf{N}$ there exists $S \subseteq \mathbf{N}$ such that $\mid S \mid = k$ and $\mathcal{P}(S)$ is monochromatic. Let $S = \{a_1, ..., a_k\}$. For $T \in P_k$, let $a_T = \sum_{i \in T} a_i$. Note that for $i \in [1, k]$, $a_i = a_{\{i\}}$. Also, $a_T \in \mathcal{P}(S)$ for all $T \in P_k$. Thus, $\{a_T : T \in P_k\}$ is a monochromatic solution of the system $\mathcal{S}_k$.

Hence, Folkman's Theorem implies that $\mathcal{S}_k$ is a regular system for any $k \in \mathbf{N}$.

Now, assume this system $\mathcal{S}_k$ is a regular system for every $k \in \mathbf{N}$. Let $l$ be given. It is not difficult to check that there are distinct $\lambda_T$, $T \in P_l$, such that $\{ \lambda_T : T \in P_l \}$ is the set of solutions of $\mathcal{S}_l$. By Corollary $8\frac{1}{2}$ in [16] pp.62, since the system $\mathcal{S}_l$ is regular, then for any finite coloring of N, the system $\mathcal{S}_l$ has a monochromatic solution $L = \{a_T : T \in P_l\}$ such that $T \neq T' \Longrightarrow a_T \neq a_{T'}$.

Let $S = \{a_{\{1\}}, ..., a_{\{l\}}\}$. Then $\mid S \mid = l$ and $P(S) = L$ is monochromatic. Hence if $\mathcal{S}_k$ is regular for every $k \in \mathbf{N}$, then the claim of Folkman's Theorem is true.

Therefore, Folkman's Theorem is equivalent to the regularity of the system $\mathcal{S}_k$, $k \in \mathbf{N}$.

It can be shown that $\mathcal{S}_k$ satisfies the Columns condition. Hence Folkman's Theorem follows from Rado's Theorem.

The statement that is now generally called Folkman's Theorem was independently proved by Rado, Sanders and Folkman, so sometime it is called the Rado-Sanders-Folkman Theorem.

However, we will prove Folkman's Theorem without using Rado's Theorem. Here, we shall prove the finite form of Folkman's Theorem.

Let $\{a_i\}$ be a sequence and $I$ be a finite non-empty set. We define $a(I)$ to be $\sum_{i \in I} a_i$.

Then, using a standard compactness argument Folkman's Theorem can be restated as :

**Theorem 4.3′** *For all $c$ and $k$, there exists $M = M(c,k)$ such that for every $c$-coloring of $[1,M]$ there exist $a_1, ..., a_k \in [1,M]$ with all $a_i$ distinct, such that all $a(I)$ are monochromatic, $I \in P_k$.*

To prove Theorem 4.3′ we use the following lemma :

**Lemma 4.2** *For all positive integers $c$ and $k$ there exists $n = n(c,k)$ so that if $[1,n]$ is $c$-colored there exist $a_1 < a_2 < ... < a_k$ with all $a(I) \leq n$ so that the color of $a(I)$ depends only on $max(I), I \in P_k$.*

**Proof.** The proof of Lemma 4.2 is based on van der Waerden's Theorem.

Fix $c$. We show the existence of $n(c,k)$ by induction on $k$. Clearly, it is trivial for $k = 1$. Suppose that it is true for $k$. Let

$$n = n(c, k+1) = 2W(c, n(c,k)),$$

where $W(c,n(c,k))$ denotes the minimal $W$ such that if $[1,W]$ is $c$-colored there exists a monochromatic arithmetic progression of length $n(c,k)+1$. Now, let

$$\chi : [1,n] \longrightarrow [1,c]$$

be a $c$-coloring of

$$[1,n] = \{1, ..., W(c,n(c,k)), W(c,n(c,k))+1, ..., 2W(c,n(c,k))\}.$$

Consider the second half of the $c$-coloring of $[1,n]$, i.e. the $c$-coloring of

$$\{\frac{n}{2}+1, ..., n\} = \{W(c,n(c,k))+1, ..., 2W(c,n(c,k))\}.$$

By van der Waerden's Theorem and the fact that

$$\{W(c,n(c,k))+1, ..., 2W(c,n(c,k))\}$$

is a translate of the set

$$\{1, 2, ...., W(c,n(c,k))\},$$

among the set

$$\{\frac{n}{2}+1, ..., n\} = \{W(c,n(c,k))+1, ..., 2W(c,n(c,k))\}$$

we can find positive integers $a_{k+1}$ and $d$ so that $\{a_{k+1} + \lambda d : 0 \leq \lambda \leq n(c,k)\}$ is monochromatic in $\{W(c,n(c,k))+1, ..., 2W(c,n(c,k))\}$. Here, $a_{k+1} > \frac{n}{2} > n(c,k)d$.

Now consider the $c$-coloring of $\{d, 2d, ..., n(c,k)d\}$. Since

$$\{d, 2d, ..., n(c,k)d\} \text{ and } \{1, 2, ..., n(c,k)\}$$

are equivalent, by the induction hypothesis we can find $a_1 < a_2 < ... < a_k$ in $\{d, 2d, ..., n(c,k)d\}$with all

$$\sum_{i \in I_1 \subseteq [1,k]} a_i \leq n(c,k)d$$

so that the color of $\sum_{i=1}^{k} a_i$ depends only on $\max(I_1)$.

Let

$$A = \{a_1, a_2, ..., a_{k+1}\}.$$

If $j < k+1$, then by the induction hypothesis, the $a(I)$ where $\max(I) = j$ are monochromatic. If $\max(I) = k+1$ then

$$a(I) = a_{k+1} + \lambda d,$$

with $0 \leq \lambda \leq n(c,k)$, so all of the $a(I)$ with $\max(I) = k+1$ have the same color. Therefore the lemma is proved. ■

**Proof of Theorem 4.3′** Fix $c$ and $k$ and take

$$M = M(c,k) = n(c, c(k-1)+1).$$

Perform a $c$-coloring of $[1, M]$. By Lemma 4.2 there exist

$$a_1 < ... < a_{c(k-1)+1}$$

with all $a(I) \leq M$ so that the color of $a(I)$ depends only on $\max(I)$. Now, color the indices of $a_i$'s, i.e.,

$$[1, c(k-1)+1]$$

by coloring $i$ with the color of all $a(I)$, with $\max(I) = i$. Since we color a set of $c(k-1)+1$ members with at most $c$ colors, the pigeon hole principle guarantees that we can find a subset

$$S \subseteq [1, c(k-1)+1]$$

with $\mid S \mid = k$ such that $S$ is monochromatic. Let

$$A = \{a_i : i \in S\}.$$

Then $\mathcal{P}(A)$ is monochromatic. ■

# Chapter 5

# THE FINITE UNIONS THEOREM

In Set Theory, we have a theorem analogue to Folkman's Theorem. It is called the Finite Unions Theorem. In this chapter first we show that analogy. Then, we also prove the Finite Unions Theorem independently.

For a non-empty set $X$, let $P(X)$ be *the set of all non-empty finite subsets of* $X$. For convenience, we write $P([1, n])$ as $P_n$.

**Definition 5.1** *Let $X$ be a non-empty set and let $I$ be a non-empty index set. A family of sets $\mathcal{D} = \{D_i : i \in I\} \subseteq P(X)$ is called a disjoint collection if for any $i, j \in I, i \neq j$, we have $D_i \cap D_j = \emptyset$. Next, we define $FU(\mathcal{D})$, the family of finite unions of $\mathcal{D}$, to be the set $\{\bigcup_{i \in T} D_i : T \in P(I)\}$.*

In the rest of this chapter we assume that $I$ is a finite set.

**Theorem 5.1 (Finite Unions Theorem).** *If $P(\mathrm{N})$ is finitely colored then there exist arbitrarily large disjoint collections $\mathcal{D} \subseteq P(\mathrm{N})$ such that $FU(\mathcal{D})$ is monochromatic.*

As in the case of Folkman's Theorem, we prove Theorem 5.1 in its finite form.

**Theorem 5.1′** *For all $c, k \in \mathrm{N}$, there exists $F = F(c, k)$ such that for any $n \geq F$, if $P_n$ is $c$-colored then there exists a disjoint collection $\mathcal{D} \subseteq P_n$ with $|\mathcal{D}| = k$ such that $FU(\mathcal{D})$ is monochromatic.*

There is a natural correspondence between $P(\mathrm{N})$ and $\mathrm{N}$. This correspondence is given in the following way.

Let $\Phi : P(\mathrm{N}) \longrightarrow \mathrm{N}$ be defined by

$$\Phi(I) = \sum_{i \in I} 2^{i+1}.$$

We show that $I = J$ if and only if $\Phi(I) = \Phi(J)$.

Let $I, J \in P(\mathrm{N})$ with $I \neq J$. We can express $I$ and $J$ as

$$I = A \cup C \text{ and } J = B \cup C,$$

where

$$C = I \cap J \text{ and } A \cap B = \emptyset.$$

Note that at least one of $A$ and $B$ is not empty. Without loss of generality assume $A \neq \emptyset \neq B$.

If $\Phi(I) = \Phi(J)$ then

$$\sum_{i \in A} 2^{i+1} + \sum_{i \in C} 2^{i+1} = \sum_{i \in B} 2^{i+1} + \sum_{i \in C} 2^{i+1}$$

or

$$\sum_{i \in A} 2^{i+1} = \sum_{i \in B} 2^{i+1}.$$

Since $A \cap B = \emptyset$ then

$$\min\{2^{i+1} : i \in A\} \neq \min\{2^{j+1} : j \in B\}.$$

Without loss of generality assume that

$$2^{i_0+1} = \min\{2^{i+1} : i \in A\} < \min\{2^{j+1} : j \in B\} = 2^{j_0+1}.$$

So

$$2^{i_0+1} + \sum_{\substack{i \in A \\ i \neq i_0}} 2^{i+1} = \sum_{i \in B} 2^{i+1}.$$

Hence,

$$1 + \frac{1}{2^{i_0+1}} \cdot \sum_{\substack{i \in A \\ i \neq i_0}} 2^{i+1} = \frac{1}{2^{i_0+1}} \cdot \sum_{i \in B} 2^{i+1},$$

contradicting the fact that the left hand side is an odd number and the right hand side is an even number. Therefore $\Phi(I) \neq \Phi(J)$ if $I \neq J$. Since from the definition of $\Phi$ we have $\Phi(I) = \Phi(J)$ if $I = J$, then we have

$$I = J \text{ if and only if } \Phi(I) = \Phi(J).$$

To prove that Theorem 5.1′ and Theorem 4.3′ are equivalent, we first prove that Theorem 5.1′ implies Theorem 4.3′. To show this, let $c, k \in \mathbf{N}$ and let $\chi$ be a $c$-coloring of $[1, 2^{F(c,k)}]$. Next, define the $c$-coloring $\chi'$ of $P_{F(c,k)}$ by

$$\chi'(D) = \chi(\Phi(D)) = \chi(\sum_{i \in D} 2^{i+1}),$$

for $D \in P_{F(c,k)}$.

Let us note that $\chi'$ is well defined. Indeed, for $D \subseteq [1, F(c,k)]$ we have

$$\Phi(D) = \sum_{i \in D} 2^{i+1} \leq \sum_{i=1}^{F(c,k)} 2^{i+1} = \sum_{i=0}^{F(c,k)-1} 2^i = 2^{F(c,k)} - 1 \in [1, 2^{F(c,k)}].$$

Thus, if $D \in P_{F(c,k)}$ then $\Phi(D) \in [1, 2^{F(c,k)}]$ and $\chi'$ is well defined.

By Theorem 5.1′, there exists a disjoint collection $\mathcal{D} = \{D_i : 1 \leq i \leq k\} \subseteq P_{F(c,k)}$ such that $FU(\mathcal{D})$ is $\chi'$-monochromatic. Let $a_i = \Phi(D_i), 1 \leq i \leq k$. For $I \in P_k$ we have

$$\Phi(\cup_{i \in I} D_i) = \sum_{i \in I} (\sum_{j \in D_i} 2^{j+1}) = \sum_{i \in I} a_i.$$

Thus, for each $I, J \in P_k$ we have

$$\chi(\sum_{i \in I} a_i) = \chi'(\bigcup_{i \in I} D_i) = \chi'(\bigcup_{i \in J} D_i) = \chi(\sum_{i \in J} a_i).$$

Thus, $M(c,k) < 2^{F(c,k)}$, so that the claim of Theorem 4.3′ is true.

To prove the converse we need a lemma :

**Lemma 5.1** *Let $k, n_1, ..., n_k$ and $t$ be positive integers. There is a positive integer $m'$ so that if $| S |= m > m'$ and $[S]^i = \{A \subseteq S : | A |= i\}$ is $n_i$ -colored for $1 \leq i \leq k$, then there exists $T \subseteq S$, with $| T |= t$ so that for each $i, 1 \leq i \leq k$, $[T]^i = \{B \subseteq T : | B |= i\}$ is monochromatic.*

**Proof.** Let $k, n_1, ..., n_k$ and $t$ be given. From Ramsey's Theorem we know that for every triple of positive integers $(c, k, s)$ there exists $n_0 = n_0(c, k, s)$ such that for every $c$-coloring of $[1, n_0]^k = \{A \subseteq [1, n_0] : | A |= k\}$ there exist $j, 1 \leq j \leq c$, and a set $T \subseteq [1, n_0]$ with $| T |= s$ so that $[T]^k = \{B \subseteq T : | B |= k\}$ is colored $j$.

Now, define a sequence $m_1, ..., m_k$ inductively with

$$m_1 = n_0(n_1, 1, t)$$

and

$$m_i = n_0(n_i, i, m_{i-1}), 2 \leq i \leq k.$$

Next, let $m' = m_k$ and $| S |= m > m'$. Perform an $n_i$-coloring of $[S]^i, 1 \leq i \leq k$. By the definition of $m_k$, there exist $j_k$, $1 \leq j_k \leq n_k$, and a set $S_{k-1} \subseteq S$ with $| S_{k-1} |= m_{k-1}$ so that $[S_{k-1}]^k = \{B \subseteq S_{k-1} : | B |= k\}$ is colored $j_k$. Then, by the definition of $m_{k-1}$, there exists a set $S_{k-2} \subseteq S_{k-1}$ with $| S_{k-2} |= m_{k-2}$ so that $[S_{k-2}]^{k-1} = \{B \subseteq S_{k-2} : | B |= k-1\}$ is colored $j_{k-1}$. Continuing this process we have a sequence $S = S_k \supseteq S_{k-1} \supseteq ... \supseteq S_1 \supseteq S_0$, where $| S_0 |= t$ so that for all $i$ with $1 \le i \le k$, $[S_0]^i = \{B \subseteq S_0 : \mid B \mid = i\}$ is colored $j_i$ and the lemma is proved. ■

Now we are ready to prove that Theorem 4.3′ implies Theorem 5.1′. Let $c$ and $k$ be fixed positive integers and let $t = M(c,k)$ be as in Theorem 4.3′. Let

$$n_1 = n_2 = \dots = n_t = c,$$

and let $m'$ be a positive integer whose existence is guaranteed by Lemma 5.1.

Let $m > m'$ and let $f : P_m \longrightarrow [1,c]$ be a $c$-coloring of $P_m$. We note that for $[1,m]^i = \{A \subseteq [1,m] : \mid A \mid = i\}$, $i = 1, \dots, t$,

$$f \mid_{[1,m]^i}$$

is a $c$-coloring of $[1,m]^i$. Since $m > m'$, by Lemma 5.1 there is $T \subseteq [1,m]$, $\mid T \mid = t = M(c,k)$ such that each $[T]^i = \{A \subseteq T : \mid A \mid = i\} \subseteq [1,m]^i$ is monochromatic. In other words, for each $i \in [1, M(c,k)]$ there is $j \in [1,c]$ such that for all $A \in [1,m]^i$,

$$f(A) = j.$$

Thus, we can define $g$, a $c$-coloring of $[1, M(c,k)]$ in the following way.

For $i \in [1, M(c,k)]$,

$$g(i) = j \text{ if and only if } f(A) = j,$$

for $A \in [T]^i$.

By Theorem 4.3′ there are $a_1, \dots, a_k \in [1, M(c,k)]$ and $p \in [1,c]$ such that for $r \in \mathcal{P}(\{a_1, \dots, a_k\})$,

$$g(r) = p.$$

Since

$$\sum_{i=1}^{k} a_i \le M(c,k)$$

we can find a disjoint collection

$$\mathcal{D} = \{D_i : 1 \le i \le k\} \subseteq P(T)$$

such that

$$\mid D_i \mid = a_i, 1 \le i \le k.$$

Since

$$D_i \cap D_j = \emptyset$$

for $i \ne j$ then for any $D_{i_1}, \dots, D_{i_s} \in \mathcal{D}$,

$$\mid D_{i_1} \cup \dots \cup D_{i_s} \mid = \mid D_{i_1} \mid + \dots + \mid D_{i_s} \mid = a_{i_1} + \dots + a_{i_s} = r \in \mathcal{P}(\{a_1, \dots, a_k\}).$$

From $g(r) = p$ and $D_{i_1} \cup ... \cup D_{i_s} \in [T]^r$ we have that

$$f(D_{i_1} \cup ... \cup D_{i_s}) = p.$$

Therefore $FU(\mathcal{D})$ is $f$-monochromatic. Hence, $F(c,k)$ exists and $F(c,k) \leq m'$ ■.

To prove the Finite Unions Theorem without using Folkman's Theorem, we need two lemmas :

**Lemma 5.2** *For each pair of positive integers $c$ and $k$ there is a positive integer $n = n(c,k)$ so that if*

$$\mathcal{V} = \{V_i : 1 \leq i \leq n\}$$

*is a disjoint collection and $\approx$ is an equivalence relation on $FU(\mathcal{V})$ with at most $c$ equivalence classes, then there exists a disjoint collection*

$$\mathcal{D} = \{D_1, ..., D_k\} \subseteq FU(\mathcal{V})$$

*so that*

$$D_1 \approx D_1 \cup S$$

*for every $S \in FU(\mathcal{D})$.*

**Proof.** Induction on $k$. For $k = 1$ we have $n(c,1) = 1$ for every positive integer $c$.

For$k > 1$ we show that

$$n(c,k) \leq (c+1) + n(\frac{1}{2}(c^2+c^3), k-1). \qquad (*)$$

To prove $(*)$, let $n = (c+1)+n(\frac{1}{2}(c^2+c^3), k-1)$, let $\mathcal{V} = \{V_i : 1 \leq i \leq n\}$ be a disjoint collection and let $\approx$ be an equivalence relation on $FU(\mathcal{V})$ with equivalence classes $A_1, ..., A_c$.

Let $T \in FU(\{V_{c+2}, ..., V_n\})$. Then the set $\{ \cup\{V_1, ..., V_j\} \cup T : 1 \leq j \leq c+1\}$ has $(c+1)$ elements. Since in $FU(\mathcal{V})$ there are at most $c$ equivalence classes, we can find a triple $p, q, i$ with $1 \leq p < q \leq c+1, 1 \leq i \leq c$ such that the sets $\cup\{V_1, ..., V_p\} \cup T$ and $\cup\{V_1, ..., V_q\} \cup T$ are both in $A_i$.

Now, for each $T \in FU(\{V_{c+2}, ..., V_n\})$ let us fix a triple $f(T) = (p,q,i)$ with the property as above. Next, on $FU(\{V_{c+2}, ..., V_n\})$ we define a binary relation $\equiv$ with

$$T_1 \equiv T_2 \text{ if and only if } f(T_1) = f(T_2).$$

Clearly, $\equiv$ is an equivalence relation on $FU(\{V_{c+2}, ..., V_n\})$.

Since the number of elements of the set $\{(p,q,i) : 1 \leq p < q \leq c+1, 1 \leq i \leq c\}$ is $\binom{c+1}{2} \cdot c = \frac{1}{2}(c^2 + c^3)$, then the number of elements of the set $\{f(T) : T \in$

$FU(\{V_{c+2},...,V_n\})\}$ is at most $\frac{1}{2}(c^2+c^3)$. Thus the number of equivalence classes with respect to the equivalence relation $\equiv$ is at most $\frac{1}{2}(c^2+c^3)$. By the induction hypothesis there exists a disjoint collection

$$\mathcal{T} = \{T_1,...,T_{k-1}\} \subseteq FU(\{V_{c+2},...,V_n\})$$

so that

$$T_1 \equiv T_1 \cup S'$$

for every $S' \in FU(\mathcal{T})$.

Let $f(T_1) = (p,q,i)$ with $1 \le p < q \le c+1$ and $1 \le i \le c$. Hence, for every $S' \in FU(\mathcal{T})$, we have

$$\left.\begin{array}{l} \cup\{V_1,...,V_p\}\cup T_1, \cup\{V_1,...,V_q\}\cup T_1, \cup\{V_1,...,V_p\}\cup T_1\cup S' \\ \text{and} \\ \cup\{V_1,...,V_q\}\cup T_1 \cup S' \\ \text{are all in } A_i. \end{array}\right\} (**)$$

Now define $D_1 = \cup\{V_1,...,V_p\} \cup T_1, D_i = T_i$, for $2 \le i \le k-1$, $D_k = \cup\{V_{p+1},...,V_q\}$ and let $\mathcal{D} = \{D_1,...,D_k\}$. Note that $\mathcal{D}$ is pairwise disjoint and that $D_1 \in A_i$.

Next, define $S' = S - \cup\{V_1,...,V_{c+1}\}$, for $S \in FU(\mathcal{D})$. Since $S \in FU(\mathcal{D})$, we can express:

$$S = B_1 \cup B_k \cup (\cup\{T_{j_1},...,T_{j_m}\}),$$

where

$$B_1 = \begin{cases} \cup\{V_1,...,V_p\}, & \text{if } D_1 \subseteq S \\ \emptyset, & \text{if } D_1 \not\subseteq S, \end{cases}$$

$$B_k = \begin{cases} \cup\{V_{p+1},...,V_q\}, & \text{if } D_k \subseteq S \\ \emptyset, & \text{if } D_k \not\subseteq S, \end{cases}$$

and $\cup\{T_{j_1},...,T_{j_m}\} \in FU(\mathcal{T})$.

Note that for $S$ as above,

$$S' = S - \cup\{V_1,...,V_{c+1}\} = \cup\{T_{j_1},...,T_{j_m}\} \in FU(\mathcal{T}).$$

Consider the two possibilities : either $D_k \not\subseteq S$ or $D_k \subseteq S$.

- If $D_k \not\subseteq S$ then $B_k = \emptyset$ and $S = B_1 \cup S$. Thus,

$$D_1 \cup S = (\cup\{V_1,...,V_p\} \cup T_1) \cup B_1 \cup S'$$

$$= (\cup\{V_1, ..., V_p\} \cup B_1) \cup T_1 \cup S'$$

$$= (\cup\{V_1, ..., V_p\}) \cup T_1 \cup S'.$$

Since $S' \in FU(\mathcal{T})$, by (**) we have that $D_1 \cup S \in A_i$.

- If $D_k \subseteq S$ then $B_k = \cup\{V_{p+1}, ..., V_q\}$ and $S = B_1 \cup (\cup\{V_{p+1}, ..., V_q\}) \cup S'$. Thus,

$$D_1 \cup S = (\cup\{V_1, ..., V_p\} \cup T_1) \cup B_1 \cup (\cup\{V_{p+1}, ..., V_q\}) \cup S'$$

$$= (\cup\{V_1, ..., V_p\} \cup B_1 \cup \{V_{p+1}, ..., V_q\}) \cup T_1 \cup S'$$

$$= (\cup\{V_1, ..., V_q\}) \cup T_1 \cup S, \text{and again by } (**), D_1 \cup S \in A_i.$$

Therefore in both cases, we have $D_1 \approx D_1 \cup S$ as required. ■

**Lemma 5.3** *Given positive integers $c$ and $k$, there exists a positive integer $r = r(c,k)$ so that if*

$$\mathcal{V} = \{V_1, ..., V_r\}$$

*is a disjoint collection and $\approx$ is an equivalence relation on $FU(\mathcal{V})$ with at most $c$ equivalence classes, then there is a disjoint collection*

$$\mathcal{E} = \{E_1, ..., E_k\} \subseteq FU(\mathcal{V})$$

*so that*

$$E_i \approx E_i \cup S$$

*for $1 \leq i \leq k$ and $S \in FU(\{E_i, ..., E_k\})$.*

**Proof.** Induction on $k$. Clearly, $r(c, 1) = 1$ for every $c$. Now let $k > 1$. We show that

$$r(c,k) \leq n(c, 1 + r(c, k-1)).$$

Take $r = n(c, 1 + r(c, k-1))$, where $n(c, 1 + r(c, k-1))$ refers to Lemma 5.2, let

$$\mathcal{V} = \{V_1, ..., V_r\}$$

be a disjoint collection and let $\approx$ be an equivalence relation on $FU(\mathcal{V})$ with at most $c$ equivalence classes. By Lemma 5.2 there exists a disjoint collection

$$\mathcal{D} = \{E_1, D_1, ..., D_{r(c,k-1)}\} \subseteq FU(\mathcal{V})$$

so that

$$E_1 \approx E_1 \cup S \qquad (***)$$

for every $S \in FU(\mathcal{D})$.

By applying the induction hypothesis to

$$\mathcal{W} = \{D_1, ..., D_{r(c,k-1)}\}$$

there exists a disjoint collection

$$\mathcal{E}' = \{E_2, ..., E_k\} \subseteq FU(\mathcal{W}) \subseteq FU(\mathcal{V})$$

so that

$$E_i \approx E_i \cup S$$

with $S \in FU(\{E_i, ..., E_k\})$ and $i = 2, ..., k$.

To prove our lemma it is enough to show that $E_1 \approx E_1 \cup S$ for $S \in FU(\{E_1, ..., E_k\})$.
Let $S \in FU(\{E_1, ..., E_k\})$. Then we can express:

$$S = B \cup S',$$

where

$$B = \begin{cases} E_1, \text{ if } E_1 \subseteq S, \\ \emptyset, \text{ if } E_1 \not\subseteq S, \end{cases}$$

and

$$S' \in FU(\{E_2, ..., E_k\}) \subseteq FU(\mathcal{D}).$$

Thus,

$$E_1 \cup S = E_1 \cup (B \cup S')$$

$$= (E_1 \cup B) \cup S'$$

$$= E_1 \cup S'.$$

Since $S' \in FU(\mathcal{D}), E_1 \cup S = E_1 \cup S' \approx E_1$(by $(***)$).

Now, let $\mathcal{E} = \{E_1, ..., E_k\}$. Then $\mathcal{E}$ is a disjoint collection, $\mathcal{E} \subseteq FU(\mathcal{V})$ and $E_i \approx E_i \cup S$ for $S \in FU(\{E_1, ..., E_k\})$ and $i = 1, ..., k$. ■

Now we are ready to prove the Finite Unions Theorem :

Take $n = r(c, c(k-1)+1)$ and let $A_1, ..., A_c$ be a partition of $P_n$. Let

$$\mathcal{V} = \{\{1\}, ..., \{n\}\}.$$

Then $\mathcal{V}$ is a disjoint collection with $n$ sets. On $FU(\mathcal{V})$ we define a binary relation $\approx$ by

$$S \approx T \text{ if and only if } S, T \in A_i$$

for some $i \in \{1, ..., c\}$. Then $\approx$ is an equivalence relation.

By Lemma 5.3 there exists a disjoint collection

$$\mathcal{E} = \{E_1, ..., E_{c(k-1)+1}\} \subseteq FU(\mathcal{V})$$

so that $E_i \approx E_i \cup S$ with $S \in FU(\{E_i, ..., E_{c(k-1)+1}\})$ and $i = 1, ..., c(k-1)+1$.

Let $\mathcal{A}_i = \{E \in \mathcal{E} : E \in A_i\}, 1 \leq i \leq c$. Clearly, $\mathcal{E} = \bigcup_{i=1}^{c} \mathcal{A}_i$ and $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for $i \neq j$. If for all $i \in [1, c]$ $|\mathcal{A}_i| \leq k-1$ then $c(k-1)+1 = |\mathcal{E}| \leq c(k-1)$. Thus, there is $j \in [1, c]$ such that $|\mathcal{A}_j| \geq k$. Let

$$\mathcal{D} = \{E_{i_1}, ..., E_{i_k}\} \subseteq \mathcal{A}_j \subseteq \mathcal{E}.$$

Since $E_{i_p} \in \mathcal{A}_j$ and $E_{i_q} \in \mathcal{A}_j$ imply that

$$E_{i_p} \cup E_{i_q} \in \mathcal{A}_j$$

for every $p, q$ with $1 \leq p, q \leq k$ then

$$FU(\mathcal{D}) \subseteq \mathcal{A}_j$$

and the theorem is proved. ■

# Chapter 6

# UPPER BOUNDS

For positive integers $c$ and $k$, let $M(c,k)$ denote the least integer $n$ so that if $[1,n]$ is $c$-colored, then there exist distinct $a_1, a_2, ..., a_k$ such that all $a(I)$ are monochromatic, and let $F(c,k)$ denote the least integer $n$ so that if $\mathcal{P}_n$ is $c$-colored there exists a disjoint collection $\mathcal{D}$ of cardinality $k$ such that $FU(\mathcal{D})$ is monochromatic. (Recall that for $\emptyset \neq I \subseteq \{1,2,...,k\}$, $a(I) = \sum_{i \in I} a_i$, and that $FU(\mathcal{D})$ is the set of all finite unions of elements of $\mathcal{D}$.) We recall that Folkman's Theorem and the Finite Unions Theorem guarantee the existence of $M(c,k)$ and $F(c,k)$, respectively. Now we will find upper bounds for $M(c,k)$ and $F(c,k)$.

At the beginning of Chapter 5, we have shown that

$$M(c,k) \leq 2^{F(c,k)}.$$

Thus $2^{F(c,k)}$ is an upper bound of $M(c,k)$.

To find an upper bound of $F(c,k)$, we will prove the following theorem:

**Theorem 6.1** *Let $c, k \geq 2$ be positive integers and let $F(c,k)$ be defined as above. Then*

$$F(c,k) \leq c^{3^{c^{\cdot^{\cdot^{\cdot^{3}}}}}},$$

*where $c^{3^{c^{\cdot^{\cdot^{\cdot^{3}}}}}}$ is a tower of height $2c(k-1)$.*

The following lemma is required to prove Theorem 6.1:

**Lemma 6.1** *Let $n(c,k)$ and $r(c,k)$ be the functions given in Lemma 5.2 and Lemma 5.3 respectively. Then for $c \geq 2$, $k > 2$,*

*(i). $2^{(k-2)} c^{(3^{k-1})} < c^{(3^k)}$,*

*(ii). $n(c,k) < 2^{(k-2)} c^{(3^{k-1})}$, and*

*(iii). $r(c,k) < c^{3^{c^{\cdot^{\cdot^{\cdot^{3}}}}}}$, where $c^{3^{c^{\cdot^{\cdot^{\cdot^{3}}}}}}$ is a tower of height $2(k-1)$.*

**Proof.** (i). Note that

$$c^{(3^k)} = c^{(3 \cdot 3^{k-1})} = c^{(3^{k-1})} c^{(3^{k-1})} c^{(3^{k-1})},$$

and since $c \geq 2$,

$$c^{(3^{k-1})} \geq 2^{(3^{k-1})}$$

For every real number $x$ we have $3^{x+1} > x$. Therefore $3^{k-1} > k-2$. Thus,

$$c^{(3^k)} > 2^{k-2} c^{(3^{k-1})}.$$

(ii). To show that $n(c,k) < 2^{(k-2)} c^{(3^{k-1})}$ , for $c, k \geq 2$, we use induction on $k$: From the proof of Lemma 5.2 we have

$$n(c,1) = 1$$

and for $k \geq 2$,

$$n(c,k) \leq (c+1) + n(\frac{1}{2}(c^2+c^3), k-1).$$

Therefore, for $c \geq 2$ and $k=2$ we have

$$n(c,2) \leq (c+1) + n(\frac{1}{2}(c^2+c^3), 1) = (c+1) + 1 < c^3 = 2^0 c^{(3^1)}.$$

Thus, the Lemma is true for $k=2$.

For the induction step, we need first to show that $n(c,k) > c$, for $c, k \geq 2$. To show that this inequality is true, let us observe the following special case. Let $c \geq 2$ be fixed and let $\mathcal{V} = \{\{1\}, ..., \{c\}\}$. We define a relation $\approx$ on $FU(\mathcal{V})$ by

$$S \approx T \text{ if and only if } \mid S \mid = \mid T \mid .$$

Then $\approx$ is an equivalence relation on $FU(\mathcal{V})$. Furthermore, for any $S \in FU(\mathcal{V})$ we have $S \neq \emptyset$ and $S \subseteq [1,c]$, so $\mid S \mid \in [1,c]$. Therefore, there are at most $c$ equivalence classes. (On the other hand we know that $\{1\}, [1,2], [1,3], ..., [1,c]$ are in different classes. Thus, there are exactly $c$ equivalence classes.)

Now, let $k \geq 2$ and $\mathcal{D} = \{D_1, ..., D_k\} \subseteq FU(\mathcal{V})$ be a disjoint collection. Therefore

$$D_1 \neq \emptyset \neq D_2 \text{ and } D_1 \cap D_2 = \emptyset.$$

Thus

$$\mid D_1 \mid < \mid D_1 \cup D_2 \mid,$$

so $D_1$ and $D_1 \cup D_2$ are in the different equivalence classes. Here, $D_2 \in FU(\mathcal{D})$. By the definition of $n(c,k)$ we have $n(c,k) > \mid \mathcal{V} \mid = c$. In particular we have

$$n(c^3, k-1) > c^3 > c+1,$$

for $c \geq 2$ and $k > 2$.

Hence, for $k > 2$

$$n(c,k) \leq (c+1) + n(\frac{1}{2}(c^2+c^3), k-1)$$

$$< n(c^3, k-1) + n(\frac{1}{2}(c^2+c^3), k-1)$$

$$\leq n(c^3, k-1) + n(c^3, k-1)$$

$$= 2n(c^3, k-1).$$

Thus, by the induction hypothesis

$$n(c,k) < 2 \cdot 2^{(k-3)}(c^3)^{3^{k-2}} = 2^{(k-2)}(c^3)^{3^{k-2}},$$

and our claim is proved.

(iii). To prove this inequality we use induction on $k$.

From the proof of Lemma 5.3 we have

$$r(c,1) = 1,$$

and

$$r(c,k) \leq n(c, 1 + r(c, k-1)).$$

Therefore,

$$r(c,2) \leq n(c, 1 + r(c,1)) = n(c,2).$$

By (ii), we have

$$r(c,2) < c^3.$$

Here, $c^3$ is a tower of height $2 = 2 \cdot (2-1)$. Thus, the Lemma is true for $k = 2$.

Now, let $m = c^{3^{c^{\cdot^{\cdot^{\cdot^{3}}}}}}$ be a tower of height $2(k-2)$ with $k > 2$. By the induction hypothesis,

$$1 + r(c, k-1) \leq m.$$

Let us observe that from the definition of $n(c,k)$ we have

$$n(c,k) \leq n(c,k+1).$$

Thus, for $k > 2$

$$r(c,k) \leq n(c, 1 + r(c,k-1)) \leq n(c,m),$$

and because of (ii) and (i),

$$r(c,k) < 2^{(m-2)}c^{(3^{m-1})} < c^{3^m}.$$

Since $c^{3^m}$ is a tower of height $2 + 2(k-2) = 2(k-1)$, our claim is proved. ■

**Proof of Theorem 6.1** From the definition of $F(c,k)$ and the proof of the Finite Unions Theorem on p. 41 we have

$$F(c,k) \leq r(c, ck - c + 1).$$

By Lemma 6.1(iii),

$$F(c,k) < c^{3^{c^{\cdot^{\cdot^{\cdot^{3}}}}}},$$

where $c^{3^{c^{\cdot^{\cdot^{\cdot^{3}}}}}}$ is a tower of height $2(ck - c + 1 - 1) = 2c(k-1)$, so the Theorem is proved. ■

# Chapter 7

# HINDMAN'S THEOREM

A natural question which arises from the Folkman's Theorem is: can we consider $S$ to be an infinite set? In fact, the answer is yes, we can. Therefore we have the following theorem :

**Theorem 7.1 (Hindman's Theorem)** *Let* N *be finitely colored. Then there is an infinite set* $x_1 < x_2 < ...$ *such that* $\{\sum_{i \in I} x_i : I \in P(\mathbf{N})\}$ *is monochromatic.*

We must note that Hindman's Theorem is not a corollary of Folkman's Theorem since the existence of finite arbitrarily large monochromatic structures does not imply the existence of infinite monochromatic structures. This theorem was proved by Neil Hindman in 1974. His proof is very long and needs several difficult preliminary lemmas. In the same year James E. Baumgartner gave a much shorter proof.

We can see that Hindman's Theorem is equivalent to:

**Theorem 7.2** *Let* $P(\mathbf{N})$ *be partitioned into sets* $H_1, ..., H_k$*. Then there exist* $i$ *with* $1 \leq i \leq k$ *and an infinite disjoint collection* $\mathcal{D}$ *such that* $FU(\mathcal{D}) \subseteq H_i$*.*

The equivalence is given in the same way as the analogy between Folkman's Theorem and The Finite Unions Theorem.

In the rest of this chapter, we assume that all disjoint collections are infinite.

We follow the proof of Theorem 7.2 due to Baumgartner [1].
We start with the following definition.

**Definition 7.1** *Let* $\mathcal{D}$ *be a disjoint collection. We say that* $X \subseteq P(\mathbf{N})$ *is large for* $\mathcal{D}$ *if for any disjoint collection* $\mathcal{D}' \subseteq FU(\mathcal{D})$ *we have that*

$$FU(\mathcal{D}') \cap X \neq \emptyset.$$

Let $\mathcal{D}^*$ be the family of all sets that are large for $\mathcal{D}$. Note that $\mathcal{D}^* \neq \emptyset$, since $\{P(\mathbf{N}), FU(\mathcal{D})\} \subseteq \mathcal{D}^*$.

Baumgartner's proof of Theorem 7.2 is based on the following four lemmas.

**Lemma 7.1** *a). If* $X \in \mathcal{D}^*$ *and if* $X = \bigcup_{i=1}^{k} X_k$ *for some* $k \in \mathbf{N}$*, then there is a disjoint collection* $\mathcal{D}' \subseteq FU(\mathcal{D})$ *and* $i_0 \in [1, k]$ *such that* $X_{i_0} \in \mathcal{D}'^*$*.*

*b). If $X \in \mathcal{D}^*$ then for every $n \in \mathbf{N}$,*

$$X(n) = \{A \in X : \min A > n\} \in \mathcal{D}^*.$$

**Proof.** a). It is enough to show that if $X \in \mathcal{D}^*$ and $X = X_1 \cup X_2$ then there is a disjoint collection $\mathcal{D}' \subseteq FU(\mathcal{D})$ such that either $X_1 \in \mathcal{D}'^*$ or $X_2 \in \mathcal{D}'^*$.

Let $X = X_1 \cup X_2 \in \mathcal{D}^*$. Suppose that our claim is not true. Let $\mathcal{D}' \subseteq FU(\mathcal{D})$.Then $X_1 \notin \mathcal{D}'^*$. By Definition 7.1 there is $\mathcal{D}'' \subseteq FU(\mathcal{D}')$ such that

$$FU(\mathcal{D}'') \cap X_1 = \emptyset.$$

The assumption that the claim is not true implies that $X_2 \notin \mathcal{D}''^*$. Hence, there is a disjoint collection $\mathcal{D}''' \subseteq FU(\mathcal{D}'') \subseteq FU(D)$ such that

$$FU(\mathcal{D}''') \cap X_2 = \emptyset.$$

Clearly,

$$FU(\mathcal{D}''') \cap X_1 = \emptyset.$$

Thus

$$FU(\mathcal{D}''') \cap X = FU(\mathcal{D}''') \cap (X_1 \cup X_2) = (FU(\mathcal{D}''') \cap X_1) \cup (FU(\mathcal{D}''') \cap X_2) = \emptyset.$$

This contradicts the fact that $X \in \mathcal{D}^*$.

b). Let $n \in \mathbf{N}$ and let $\mathcal{D}' \subseteq FU(\mathcal{D})$ be a disjoint collection. Since $\mathcal{D}'$ is infinite,

$$\mathcal{D}'(n) = \{B \in \mathcal{D}' : \min B > n\} \neq \emptyset$$

(otherwise, for all $B \in \mathcal{D}'$, $\min B \leq n$ and $|\ \mathcal{D}'\ | \leq n$). It follows that $D'(n)$ is infinite. Since $X \in \mathcal{D}^*$,

$$FU(\mathcal{D}'(n)) \cap X \neq \emptyset.$$

Let $A \in FU(\mathcal{D}'(n)) \cap X$. Since $A \in FU(\mathcal{D}'(n))$ we have that

$$\min A > n.$$

Thus,

$$A \in X(n).$$

Therefore

$$A \in FU(\mathcal{D}') \cap X(n)$$

and $X(n) \in \mathcal{D}^*$. ∎

**Lemma 7.2** *Let $X \in \mathcal{D}^*$. Then there exists $\mathcal{E} \subseteq FU(\mathcal{D})$, $|\ \mathcal{E}\ | < \infty$, such that if $A \in FU(\mathcal{D})$ with*

$$A \cap (\bigcup_{B \in \mathcal{E}} B) = \emptyset,$$

*then there is $D \in FU(\mathcal{E})$ with*

$$A \cup D \in X.$$

**Proof.** Suppose that the claim of Lemma 7.2 is not true. Let $A_0 \in FU(\mathcal{D})$. By our assumption that the claim of the lemma is not true, if we take $\mathcal{E}_0 = \{A_0\}$, there is $A_1 \in FU(\mathcal{D})$ with

$$A_0 \cap A_1 = \emptyset$$

and

$$A_0 \cup A_1 \notin X.$$

Let, for $k \geq 2$, $A_k$ be defined in the following way. Let $A_0, ..., A_{k-1}$ be defined and let

$$\mathcal{E}_{k-1} = \{A_0, ..., A_{k-1}\}.$$

Let $A_k \in FU(\mathcal{D})$ be such that

$$A_k \cap (\bigcup_{i=0}^{k-1} A_i) = \emptyset$$

and that for all $D \in FU(\mathcal{E}_{k-1})$,

$$A_k \cup D \notin X.$$

Let, for each $n \in \mathbf{N}$,

$$B_n = A_{2n} \cup A_{2n+1}$$

and let

$$\mathcal{D}' = \{B_n : n \geq 0\}.$$

Clearly, $\mathcal{D}'$ is a disjoint collection and $\mathcal{D}' \subseteq FU(\mathcal{D})$. If $D \in FU(\mathcal{D}') \cap X$ then there exists $F \subseteq P(\mathbf{N})$ with

$$D = \bigcup_{n \in F} B_n.$$

Let $m = \max F$. Then

$$D = A_{2m+1} \cup (A_{2m} \cup (\bigcup_{n \in F \setminus \{m\}} B_n)).$$

Clearly,

$$D' = A_{2m} \cup (\bigcup_{n \in F \setminus \{m\}} B_n) \in FU(\mathcal{E}_{2m}).$$

But

$$D = A_{2m+1} \cup D' \in X$$

contradicts our choice of $A_{2m+1}$. Thus,

$$FU(\mathcal{D}') \cap X = \emptyset,$$

and this contradicts the fact that $X \in \mathcal{D}^*$. Therefore the claim of Lemma 7.2 is true. ■

**Lemma 7.3** *Let $X \in \mathcal{D}^*$. Then there exists $D \in FU(\mathcal{D})$ such that*

$$X_D = \{A \in X : A \cup D \in X\} \in \mathcal{D}'^*$$

*for some $\mathcal{D}' \subseteq FU(\mathcal{D})$.*

**Proof.** Let $\mathcal{E}$ be as in Lemma 7.2. Since $\mathcal{E}$ is finite, there is a disjoint collection $\mathcal{D}' \subseteq FU(\mathcal{D})$ such that

$$A \cap (\bigcup_{B \in \mathcal{E}} B) = \emptyset$$

for all $A \in FU(\mathcal{D}')$. Let $\mathcal{D}'' \subseteq FU(\mathcal{D}')$ be a disjoint collection. Since $X \in \mathcal{D}^*$, we have that

$$FU(\mathcal{D}'') \cap X \neq \emptyset$$

and from $FU(\mathcal{D}'') \subseteq FU(\mathcal{D}')$ we have that

$$FU(\mathcal{D}'') \cap (X \cap FU(\mathcal{D}')) = (FU(\mathcal{D}') \cap FU(\mathcal{D}'')) \cap X = FU(\mathcal{D}'') \cap X \neq \emptyset.$$

Thus,

$$X \cap FU(\mathcal{D}') \in \mathcal{D}'^*.$$

By our choice of $\mathcal{E}$ and $\mathcal{D}'$, for any $A \in X \cap FU(\mathcal{D}')$ there is $D \in FU(\mathcal{E})$ such that

$$A \cup D \in X,$$

or in other words, for each $A \in X \cap FU(\mathcal{D}')$ there is $D \in FU(\mathcal{E})$ such that $A \in X_D$. Thus,

$$X \cap FU(\mathcal{D}') \subseteq \bigcup_{D \in \mathcal{E}} X_D.$$

Since $X \cap FU(\mathcal{D}') \in \mathcal{D}'^*$ we have that

$$\bigcup_{D \in \mathcal{E}} X_D \in \mathcal{D}'^*.$$

Since $\mathcal{E}$ is finite, by Lemma 7.1a), there is $D \in \mathcal{E}$ such that $X_D \in \mathcal{D}'^*$. ■

**Lemma 7.4** Let $X \in \mathcal{D}^*$. *Then there exists a disjoint collection $\mathcal{D}' \subseteq FU(\mathcal{D})$ such that*

$$FU(\mathcal{D}') \subseteq X.$$

**Proof.** Let $X \in \mathcal{D}^*$. Take $\mathcal{D}_0 = \mathcal{D}$ and $X_0 = X$. Let $D_0 \in FU(\mathcal{D}_0)$ and $\mathcal{D}_1 \subseteq FU(\mathcal{D}_0)$ be as in Lemma 7.3, i.e., $D_0$ and $\mathcal{D}_1$ are such that

$$X_1 = X_{D_0} \in \mathcal{D}_1^*.$$

Let $n \in \mathbf{N}$ and let $\{\mathcal{D}_i : 0 \leq i \leq n\}, \{D_i : 0 \leq i \leq n-1\}$ and $\{X_i : 0 \leq i \leq n\}$ be defined. Then we take $\mathcal{D}_{n+1} \subseteq \mathcal{D}_n, D_n \subseteq FU(\mathcal{D}_n)$ and $X_{n+1} \subseteq X_n$ to be as in Lemma 7.3, i.e.,

$$X_{n+1} = (X_n)_{D_n} \in \mathcal{D}^*_{n+1}.$$

Note that by the proof of Lemma 7.3 we can take $D_n$ to be such that

$$D_n \notin FU(\mathcal{D}_{n+1}).$$

Thus, we can choose $D_i$ in a way such that

$$i \neq j \Longrightarrow D_i \cap D_j = \emptyset.$$

Let

$$\mathcal{D}_\infty = \{D_n : n \geq 0\}.$$

Let $A_0 \in FU(\mathcal{D}_\infty) \cap X$. We define $\{A_i\}_{i=0}^{\infty}$ in the following way. Let $n \in \mathbf{N}$ and let $A_0, ..., A_{n-1} \in FU(\mathcal{D}_\infty)$ be defined. let us consider the set

$$C_n = \{k : D_k \subseteq \bigcup_{i=0}^{n-1} A_i\}.$$

Since $A_0, ..., A_{n-1} \in FU(\mathcal{D}_\infty)$ we have that $\bigcup_{i=0}^{n-1} A_i \in FU(\mathcal{D}_\infty)$ and therefore $C_n$ is a finite set. Let

$$k_n = \max C_n.$$

Let

$$\mathcal{D}_{\infty,n} = \mathcal{D}_\infty \backslash \{D_i : i \leq k_n\} = \{D_i : i \geq k_n + 1\}.$$

Since

$$i \geq k_n + 1 \Longrightarrow D_i \in FU(\mathcal{D}_{k_n+1})$$

we have that

$$\mathcal{D}_{\infty,n} \subseteq FU(\mathcal{D}_{k_n+1}).$$

Hence,

$$X_{k_n+1} \cap FU(\mathcal{D}_{\infty,n}) \neq \emptyset.$$

we take $A_n$ to be any element of $X_{k_n+1} \cap FU(\mathcal{D}_{\infty,n})$. Note that

$$i < n \Longrightarrow A_n \cap A_i = \emptyset.$$

Let

$$\mathcal{D}' = \{A_i : i \geq 0\}.$$

Clearly, $\mathcal{D}'$ is a disjoint collection. We claim that

$$FU(\mathcal{D}') \subseteq X.$$

Since, for any $i \geq 0$,

$$A_i \subseteq X_{k_i+1} \subseteq X$$

we have that

$$\mathcal{D}' \subseteq X.$$

Let $A \in FU(\mathcal{D}')$ and let $i_1 < ... < i_n < r$ be such that

$$A = A_r \cup (\bigcup_{j=1}^{n} A_{i_j}).$$

Let $D_{j_1}, ..., D_{j_m}$ be such that $j_1 < ... < j_m$ and $\bigcup_{j=1}^{n} A_{i_j} = \bigcup_{p=1}^{m} D_{j_p}$. Since $r > i_j$, for all $j \in [1, n]$ we have that

$$D_{j_m} \subseteq \bigcup_{i=0}^{r-1} A_i$$

and hence

$$j_m \in C_r.$$

Thus,

$$j_m \leq k_r.$$

From

$$A_r \in X_{k_r+1} \subseteq X_{j_m+1}$$

we have that

$$A_r \cup D_{j_m} \in X_{j_m}.$$

Since $j_m \geq j_{m-1} + 1$ we have that

$$X_{j_m} \subseteq X_{j_{m-1}+1}.$$

Thus,

$$A_r \cup D_{j_m} \in X_{j_{m-1}+1}$$

and

$$A_r \cup D_{j_m} \cup D_{j_{m-1}} \in X_{j_{m-1}}.$$

Clearly,

$$A_r \cup D_{j_m} \cup D_{j_{m-1}} \cup ... \cup D_{j_1} \in X_{j_1} \subseteq X.$$

Thus,

$$A = A_r \cup (\bigcup_{j=1}^{n} A_{i_j}) = A_r \cup (\bigcup_{p=1}^{m} D_{j_p}) \subseteq X.$$

Therefore,

$$FU(\mathcal{D}') \subseteq X. \blacksquare$$

**Proof of Theorem 7.2.** Since $P(\mathrm{N}) \in \mathcal{D}^*$, for any $\mathcal{D}$, by Lemma 7.1a) there are $i \in [1, k]$ and a disjoint collection $\mathcal{D}'$ such that

$$H_i \in \mathcal{D}'^*.$$

By Lemma 7.4 there is $\mathcal{D}'' \subseteq FU(\mathcal{D}')$ such that

$$FU(\mathcal{D}'') \subseteq H_i. \blacksquare$$

# Chapter 8

# TOPOLOGICAL DYNAMICS

In this chapter we study the application of Topological Dynamics to Ramsey Theory. We will use these methods to prove van der Waerden's Theorem and Hindman's Theorem. In this chapter generally we follow [17].

**Definition 8.1** *We say that $(X,T)$ is a dynamical system if $X=(X,\rho)$ is a compact metric space and $T: X \longrightarrow X$ is a homeomorphism.*

**Definition 8.2** *A dynamical system $(X,T)$ is minimal if for any non-empty closed subset $Y$ of $X$ we have that*

$$T(Y) \subseteq Y \Longrightarrow Y = X.$$

In other words, $(X,T)$ is a minimal dynamical system if the only non-empty closed $T$-invariant subset of $X$ is $X$ itself.

Our first step is to prove the following theorem.

**Theorem 8.1** *For any dynamical system $(X,T)$ there exists a non-empty closed $T$-invariant subset $X_0$ of $X$ such that $(X_0,T)$ is minimal.*

**Proof.** Let $\mathcal{T}(X)$ be the family of all non-empty closed $T$-invariant subsets of $X$. Note that $\mathcal{T}(X) \neq \emptyset$ since $X \in \mathcal{T}(X)$.

Let $\mathcal{C}$ be a chain in $\mathcal{T}(X)$. Thus, $\mathcal{C} \subseteq \mathcal{T}(X)$ and for any $Y, Z \in \mathcal{C}$ we have that

$$Y \subseteq Z \text{ or } Z \subseteq Y.$$

Since $X$ is compact we have that

$$\bigcap_{Y \in \mathcal{C}} Y \neq \emptyset.$$

Clearly,

$$\bigcap_{Y \in \mathcal{C}} Y \in \mathcal{T}(X).$$

Hence, any chain in $\mathcal{T}(X)$ has a lower bound that belongs to $\mathcal{T}(X)$. By Zorn's Lemma, there is $X_0 \in \mathcal{T}(X)$ such that for any $Y \in \mathcal{T}(X)$ ,

$$Y \subseteq X_0 \Longrightarrow Y = X_0. \qquad (*)$$

To check that $(X_0,T)$ is a minimal dynamical system it is enough to show that $T(X_0)=X_0$. Clearly, $T(X_0) \subseteq X_0$. Therefore, $T(T(X_0)) \subseteq T(X_0)$, i.e., $T(X_0) \in$

$\mathcal{T}(X)$. Thus, $T(X_0) \in \mathcal{T}(X)$ and $T(X_0) \subseteq X_0$. By (*), we have $T(X_0) = X_0$, so that $(X_0, T)$ is a minimal dynamical system. ■

Now we introduce so-called uniformly recurrent points. They are closely related to minimal dynamical systems and that relationship will be given in Theorem 8.2 and Theorem 8.3. Also, they play the major role in the proof of the Topological Hindman's Theorem.

**Definition 8.3** *We say that $x \in X$ is a uniformly recurrent point for the dynamical system $(X,T)$ if for any $\epsilon > 0$ there exists $N \in \mathbf{N}$ with the property: if $a \in \mathbf{N}$ then there is $n \in [a, a+N]$ such that*

$$\rho(x, T^n x) < \epsilon.$$

For $x \in X$ let

$$X_x = \overline{\{T^n x : n \in \mathbf{Z}\}}.$$

Hence, $X_x$ is the smallest closed set containing $\{T^n x : n \in \mathbf{Z}\}$. Since $T$ is a homeomorphism we have that $(X_x, T)$ is a dynamical system.

Let

$$X'_x = \overline{\{T^n x : n = 0, 1, \ldots\}}.$$

Note that $X'_x$ is $T$-invariant, but it is not clear if $T : X'_x \longrightarrow X'_x$ is onto. For example, from the definition of $X'_x$ we do not see if $T^{-1}x$ belongs to $X'_x$.

**Theorem 8.2** *If a dynamical system $(X,T)$ is minimal then each $x \in X$ is a uniformly recurrent point.*

**Proof.** Let $(X,T)$ be a minimal dynamical system and let $\epsilon > 0$. Let $x \in X$ and let

$$V = \{y \in X : \rho(x,y) < \epsilon\}.$$

Let

$$U = \bigcup_{n=0}^{\infty} T^{-n} V$$

and let

$$X' = X \backslash U.$$

Since $U$ is open then $X'$ is a closed subset of $X$. Since $T$ is $1-1$ and

$$T(U) = \bigcup_{n=0}^{\infty} T^{-n+1} V = TV \cup U \supseteq U,$$

we have that

$$T(X') = T(X) \backslash T(U) = X \backslash T(U) \subseteq X \backslash U = X'.$$

Hence, $X'$ is a closed $T$-invariant subset of $X$. Since $X$ is minimal then we have that

$$X' = \emptyset.$$

Therefore

$$U = X$$

and $\{T^{-n}V : n = 0, 1, 2...\}$ is an open cover of $X$. Since $X$ is compact then there exists $N \in \mathbf{N}$ such that

$$X = \bigcup_{n=0}^{N} T^{-n}V.$$

Let $a \in \mathbf{N}$ and let

$$y = T^{a}x \in X.$$

There is $n \in [0, N]$ such that

$$y \in T^{-n}V.$$

Thus, there is $n \in [0, N]$ such that

$$T^{n}y = T^{a+n}x \in V.$$

Note that $N$ does not depend on the choice of $a$. Therefore, $x$ is a uniformly recurrent point. ■

**Theorem 8.3** *If $x \in X$ is a uniformly recurrent point for $(X, T)$ then $(X'_x, T)$ is a minimal dynamical system.*

**Proof.** Let $x \in X$ be a uniformly recurrent point for $(X, T)$. Let $Z \subsetneq X'_x$ be closed and $T$-invariant. Note that such $Z$ exists. The reason is that the family of all closed $T$-invariant subsets of $X'_x$ is not empty, since $X'_x$ belongs to this family.

If $x \in Z$ then since $Z$ is closed and $T$-invariant we have that

$$X'_x \subseteq Z.$$

Thus, if $x \in Z$ then $X'_x = Z$ and $X'_x$ is minimal.

Suppose that $x \notin Z$. Since $Z$ is closed then we have that

$$\epsilon = \rho(x, Z) = \min\{\rho(x, z) : z \in Z\} > 0.$$

Let

$$V = \{y \in X'_x : \rho(x, y) < \frac{\epsilon}{2}\}.$$

Clearly,

$$V \cap Z = \emptyset.$$

Let

$$V' = \{y \in X : \rho(x, y) < \frac{\epsilon}{2}\}.$$

Then

$$V' \cap Z = \emptyset.$$

Since $x$ is a uniformly recurrent point, there is $N \in \mathbf{N}$ such that for all $a \in \mathbf{N}$ there is $n \in [a, a+N]$ with

$$\rho(x, T^n x) < \frac{\epsilon}{2}.$$

Note that since $x \in V \subseteq X'_x$ and since $X'_x$ is $T$-invariant we have that for all $n \in \mathbf{N}$,

$$T^n x \in X'_x.$$

Thus, for any $a \in \mathbf{N}$ there is $n \in [0, N]$ with

$$T^{a+n} x \in V.$$

Let $z \in Z$. Since $Z \subseteq X'_x$ there is a sequence $\{a_k\}_{k \in \mathbf{N}}$ with

$$\lim_{k \to \infty} T^{a_k} x = z.$$

For each $a_k$, $k \in \mathbf{N}$, let $n_k \in [0, N]$ be such that

$$T^{a_k + n_k} x \in V.$$

Since $[0, N]$ is finite, there are $n \in [0, N]$ and a sequence $\{b_k\}_{k \in \mathbf{N}}$ such that, for all $k \in \mathbf{N}$

$$T^n(T^{b_k} x) = T^{n+b_k} x = T^{b_k+n} x \in V.$$

Since $\{b_k\}_{k \in \mathbf{N}}$ is a subsequence of $\{a_k\}_{k \in \mathbf{N}}$ we have that

$$\lim_{k \to \infty} T^{b_k} x = z.$$

Thus, since $T$ is continuous

$$\lim_{k \to \infty} T^n(T^{b_k} x) = T^n z.$$

Note that $T^n z \in Z$, since $Z$ is $T$-invariant. Thus,

$$\rho(x, T^n z) \geq \epsilon.$$

On the other hand, from

$$T^n(T^{b_k} x) \in V, k \in \mathbf{N}$$

we have that

$$T^n z \in \overline{V} = \{y \in X'_x : \rho(x, y) \leq \frac{\epsilon}{2}\}.$$

Hence,

$$0 < \epsilon \leq \rho(x, T^n z) \leq \frac{\epsilon}{2}.$$

This contradiction means that we must have that $x \in Z$.

Therefore, $X'_x$ is minimal. ■

One of our main goals is to prove the following theorem known as the Topological van der Waerden Theorem.

**Theorem 8.4 (the Topological van der Waerden Theorem)** *Let $(X,T)$ be a dynamical system. Then for any $r \in \mathbf{N}$ and any $\epsilon > 0$ there are $x \in X$ and $n \in \mathbf{N}$ such that*

$$\max\{\rho(x, T^{in}x) : i \in [1,r]\} < \epsilon.$$

From Theorem 8.1 we have that it is enough to consider the case when $(X,T)$ is a minimal dynamical system.

Also, it is clear that Theorem 8.4 is a special case of the following theorem.

**Theorem 8.5** *Let $r \in \mathbf{N}$. For any compact metric space $X$, for any set $\{T_1, ..., T_r\}$ of commuting homeomorphisms of $X$, and for any $\epsilon > 0$ there are $x \in X$ and $n \in \mathbf{N}$ such that*

$$\max\{\rho(x, T_i^n x) : i \in [1,r]\} < \epsilon.$$

**Proof.** We prove Theorem 8.5 by induction on $r$.

Let $r = 1$ and let $(X,T)$ be a minimal dynamical system. By Theorem 8.2 and Theorem 8.3, if $x \in X$ then $X = X'_x$ and for given $\epsilon > 0$ there is $n \in \mathbf{N}$ such that

$$\rho(x, T^n x) < \epsilon.$$

Therefore the claim of Theorem 8.5 is true for $r = 1$.

Let $r > 1$ be such that the claim of the theorem is true for $r - 1$. Let $X$ be a compact metric space and let $\{T_1, ..., T_r\}$ be a set of commuting homeomorphisms of $X$.

Let

$$X^r = X \times ... \times X$$

and let $\rho^{(r)} : X^r \times X^r \longrightarrow \mathbf{R}$ be given by

$$\rho^{(r)}((x_1, ..., x_r), (y_1, ..., y_r)) = \sqrt{\sum_{i=1}^{r} (\rho(x_i, y_i))^2}.$$

Let

$$T : X^r \longrightarrow X^r$$

be given by

$$T(x_1, ..., x_r) = (T_1 x_1, ..., T_r x_r).$$

It is not difficult to see that $(X, T)$ is a dynamical system.

For $i \in [1, r-1]$, let
$$S_i = T_i T_r^{-1}.$$

From
$$S_i S_j = (T_i T_r^{-1})(T_j T_r^{-1}) = T_i T_j T_r^{-1} T_r^{-1} = T_j T_i T_r^{-1} T_r^{-1} = (T_j T_r^{-1})(T_i T_r^{-1}) = S_j S_i,$$
we have that $\{S_1, ..., S_{r-1}\}$ is a set of $r-1$ commuting homeomorphisms of $X$.

Let $\mu > 0$. By the induction hypothesis there are $x \in X$ and $n \in \mathbf{N}$ such that
$$\max\{\rho(x, S_i^n x) : i \in [1, r-1]\} < \frac{\mu}{\sqrt{r-1}}.$$

Now we have that
$$\rho^{(r)}((x, ..., x), T^n(T_r^{-n}x, ..., T_r^{-n}x)) = \rho^{(r)}((x, ..., x), (T_1^n T_r^{-n}x, ..., T_{r-1}^n T_r^{-n}x, x))$$
$$= \rho^{(r)}((x, ..., x), (S_1^n x, ..., S_{r-1}^n x, x)) = \sqrt{\sum_{i=1}^{r-1} (\rho(x, S_i^n x))^2} < \mu.$$

Let
$$\Delta^{(r)} = \{(x, ..., x) : x \in X\} \subseteq X^r.$$

We have seen that for any $\mu > 0$ there are $\alpha, \beta \in \Delta^{(r)}$ and $n \in \mathbf{N}$ such that
$$\rho^{(r)}(\alpha, T^n \beta) < \mu.$$

Let $\epsilon > 0$ and let $\epsilon_1 = \frac{\epsilon}{2}$. Since, by the Tychonoff Theorem, $X^r$ is compact, there are $m \in \mathbf{N}$ and $U_1, ..., U_m$ such that
$$X^r = \bigcup_{l=1}^{m} U_l$$
with, for all $l \in [1, m]$
$$\gamma, \delta \in U_l \Longrightarrow \rho^{(r)}(\gamma, \delta) < \frac{\epsilon}{2}.$$

Let $\alpha_0, \alpha_1 \in \Delta^{(r)}$ and $n_1$ be such that
$$\rho^{(r)}(T^{n_1}\alpha_1, \alpha_0) < \epsilon_1.$$

Since $T^{n_1}$ is continuous, there is $\epsilon_2 \in (0, \epsilon_1)$ such that for all $\alpha \in \Delta^{(r)}$
$$\rho^{(r)}(\alpha, \alpha_1) < \epsilon_2 \Longrightarrow \rho^{(r)}(T^{n_1}\alpha, \alpha_0) < \epsilon_1.$$

Let $\alpha_2 \in \Delta^{(r)}$ and $n_2 \in \mathbf{N}$ be such that
$$\rho^{(r)}(T^{n_2}\alpha_2, \alpha_1) < \epsilon_2.$$

Note that
$$\rho^{(r)}(T^{n_1+n_2}\alpha_2, \alpha_0) < \epsilon_1.$$

Now we proceed inductively and we find sets $\{\alpha_0, \alpha_1, ..., \alpha_m\} \subseteq \Delta^{(r)}$, $\{n_1, ..., n_m\} \subseteq \mathbf{N}$ and $\{\epsilon_1, ..., \epsilon_m\}$, $0 < \epsilon_{i+1} < \epsilon_i$ with

$$\rho^{(r)}(T^{n_i}\alpha_i, \alpha_{i-1}) < \epsilon_i, i \in [1, m]$$

and, for $\alpha \in \Delta^{(r)}$

$$\rho^{(r)}(\alpha, \alpha_i) < \epsilon_{i+1} \Longrightarrow \rho^{(r)}(T^{n_i}\alpha, \alpha_{i-1}) < \epsilon_i.$$

Note that since
$$\rho^{(r)}(T^{n_{i+1}}\alpha_{i+1}, \alpha_i) < \epsilon_{i+1}$$
we have that
$$\rho^{(r)}(T^{n_{i+1}+n_i}\alpha_{i+1}, \alpha_{i-1}) < \epsilon_i$$
and generally, for $i < j$

$$\rho^{(r)}(T^{n_j+n_{j-1}+...+n_{i+1}}\alpha_j, \alpha_i) < \epsilon_{i+1} < \frac{\epsilon}{2}.$$

Since
$$|\{U_1, ..., U_m\}| = m$$
and
$$|\{\alpha_0, \alpha_1, ..., \alpha_m\}| = m + 1,$$
we have that there are $i, j \in [0, m]$ and $l \in [1, m]$ with $i < j$ and $\alpha_i, \alpha_j \in U_l$.

Now we have, for $n = n_j + n_{j-1} + ... + n_{i+1}$,

$$\rho^{(r)}(T^n\alpha_j, \alpha_j) \leq \rho^{(r)}(T^n\alpha_j, \alpha_i) + \rho^{(r)}(\alpha_j, \alpha_i) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\alpha_j \in \Delta^{(r)}$, there is $x \in X$ such that $\alpha_j = (x, ..., x)$.

Hence,

$$\epsilon > \rho^{(r)}(T^n\alpha_j, \alpha_j) = \rho^{(r)}((T_1^n x, T_2^n x, ..., T_r^n x), (x, ..., x)) = \sqrt{\sum_{i=1}^{r} (\rho(T_i^n x, x))^2}$$

and consequently
$$\rho(T_i^n x, x) < \epsilon$$
for all $i \in [1, r]$. Thus, Theorem 8.5 is proved. ■

Note that Theorem 8.4 follows if we apply Theorem 8.5 for the set $\{T, T^2, ..., T^r\}$.

Before proving that the Topological van der Waerden Theorem and van der Waerden's Theorem are equivalent, let us introduce a special dynamical system that connects the set of all finite colorings of $\mathbf{Z}$ and Topological Dynamics.

Let $r \in \mathbf{N}$ and let $\mathcal{C}_r$ be the set of all colorings of $\mathbf{Z}$ with $r$ colors. Let

$$\Omega = [1, r]^{\mathbf{Z}} = \{\{a_i\}_{i \in \mathbf{Z}} : a_i \in [1, r]\}.$$

The function

$$\Phi : \mathcal{C}_r \longrightarrow \Omega$$

given by

$$\Phi(\phi) = \{a_i\}_{i \in \mathbf{Z}}$$

if and only if, for all $i \in \mathbf{Z}$,

$$\phi(i) = a_i$$

is bijective. We shall identify $\mathcal{C}_r$ and $\Omega$.

For $a = \{a_i\}_{i \in \mathbf{Z}}$ and $b = \{b_i\}_{i \in \mathbf{Z}}$, let

$$M(a, b) = \{k \in \mathbf{N} : | i | < k \Longrightarrow a_i = b_i\}.$$

Let $\rho : \Omega \times \Omega \longrightarrow \mathbf{R}$ be defined by

$$\rho(a, b) = \begin{cases} \inf\{\frac{1}{1+k} : k \in M(a, b)\}, & \text{if } M(a, b) \neq \emptyset \\ 1, & \text{if } M(a, b) = \emptyset. \end{cases}$$

If $l \in \mathbf{N}$ is such that

$$\rho(a, b) < \frac{1}{1 + l}$$

then there is $k \in M(a, b)$ with

$$\rho(a, b) \leq \frac{1}{1 + k} \leq \frac{1}{1 + l}$$

and consequently

$$l \leq k.$$

Therefore, if $l \in \mathbf{N}$ is such that

$$\rho(a, b) < \frac{1}{1 + l}$$

then

$$| i | < l \Longrightarrow a_i = b_i.$$

Also,

$$\rho(a, b) < 1 \Longrightarrow a_0 = b_0$$

and

$$\rho(a,b) = 1 \Longrightarrow a_0 \neq b_0.$$

It is not difficult to see that

$$\rho(a,b) = 0 \Longleftrightarrow a = b$$

and

$$\rho(a,b) = \rho(b,a).$$

Also, for any $a, b, c \in \Omega$ we have that

$$\rho(a,b) \leq \rho(a,c) + \rho(c,b).$$

Indeed. Let

$$a = \{a_i\}_{i\in\mathbf{Z}}, b = \{b_i\}_{i\in\mathbf{Z}} \text{ and } c = \{c_i\}_{i\in\mathbf{Z}}.$$

If

$$\rho(a,b) = 0$$

or

$$1 \in \{\rho(a,c), \rho(c,b)\},$$

the last inequality is obvious. Let us suppose that

$$\rho(a,b) > 0$$

and

$$1 \notin \{\rho(a,c), \rho(c,b)\}.$$

Note that since $1 \notin \{\rho(a,c), \rho(c,b)\}$, we have that $a_0 = b_0 = c_0$ and hence

$$\rho(a,b) < 1.$$

Let

$$\rho(a,b) = \frac{1}{1+k}.$$

Thus, $k = \max M(a,b)$. This means that

$$| i |< k \Longrightarrow a_i = b_i$$

and

$$a_k \neq b_k.$$

Let

$$\rho(a,c) = \frac{1}{1+l}.$$

If

$$\frac{1}{1+k} \leq \frac{1}{1+l}$$

we are done. Otherwise,

$$\frac{1}{1+k} > \frac{1}{1+l} \Longrightarrow k < l.$$

Thus,

$$| i | < k+1 \leq l \Longrightarrow a_i = c_i.$$

Consequently,

$$| i | < k \Longrightarrow b_i = c_i.$$

And

$$b_k \neq c_k.$$

Hence, in this case

$$\rho(b,c) = \frac{1}{1+k}$$

and again we have that

$$\rho(a,b) \leq \rho(a,c) + \rho(c,b).$$

Therefore, $(\Omega, \rho)$ is a metric space. By the Tychonoff Theorem, $(\Omega, \rho)$ is compact. (It is easy to check that the product topology and the metric topology are the same.)

Let $T : \Omega \longrightarrow \Omega$ be the shift operator, i.e., for $a = \{a_i\}_{i\in \mathbf{Z}} \in \Omega$ and $b = \{b_i\}_{i\in \mathbf{Z}} \in \Omega$ then

$$Ta = b \text{ if and only if } a_{i-1} = b_i, \; i \in \mathbf{Z}.$$

Clearly, $T$ is bijective.

Let $U$ be any non-empty open subset of $\Omega$. There are $a = \{a_i\}_{i\in \mathbf{Z}}$ and $n \in \mathbf{N}$ such that

$$U' = \{x \in \Omega : \rho(a,x) < \frac{1}{1+n}\} \subseteq U.$$

Let $b = \{b_i\}_{i\in \mathbf{Z}}$ be such that

$$b_i = a_{i+1}, \; i \in \mathbf{Z}$$

and let

$$V = \{y \in \Omega : \rho(b,y) < \frac{1}{1+(n+1)}\}.$$

Note that

$$Tb = a.$$

Let $y = \{y_i\}_{i\in \mathbf{Z}} \in V$ and let $x = \{x_i\}_{i\in \mathbf{Z}} \in \Omega$ be such that

$$Ty = x.$$

From

$$\rho(b,y) < \frac{1}{1+(n+1)}$$

we have that

$$-(n+1) < i < n+1 \Longrightarrow b_i = y_i \Longrightarrow a_{i+1} = x_{i+1}.$$

Hence,

$$|\, i \,| < n \Longrightarrow a_i = x_i.$$

In other words,

$$y \in V \Longrightarrow Ty \in U' \subseteq U,$$

and $T$ is continuous. In the same way we check that $T^{-1}$ is continuous. Therefore, $T$ is a homeomorphism and $(\Omega, T)$ is a dynamical system. If $X$ is any closed $T$-invariant subset of $\Omega$ we call $(X,T)$ a *symbolic dynamical system.*

**Theorem 8.6** *The Topological van der Waerden Theorem and van der Waerden's Theorem are equivalent.*

**Proof.** First we prove that van der Waerden's Theorem implies the Topological van der Waerden Theorem.

Let $(X,T)$ be a dynamical system. Let $\epsilon > 0$ and let $\{A_i : i \in [1,r]\}$ be a covering of $X$ with

$$x, y \in A_i \Longrightarrow \rho(x,y) < \epsilon$$

and

$$i \neq j \Longrightarrow A_i \cap A_j = \emptyset.$$

Let $y \in X$ and let

$$f : \mathbf{N} \longrightarrow [1,r]$$

be given by

$$f(n) = m \Longleftrightarrow T^n y \in A_m.$$

Clearly, $f$ is an $r$-coloring of $\mathbf{N}$ and

$$\mathbf{N} = \bigcup_{i=1}^{r} f^{-1}(i).$$

Let $l \in \mathbf{N}$. By van der Waerden's Theorem there are $i \in [1,r]$, $a \in \mathbf{N}$ and $n \in \mathbf{N}$ such that

$$a + jn \in f^{-1}(i),\ j \in [0,l].$$

Thus,

$$A_i \supseteq \{T^a y, T^{a+n} y, ..., T^{a+ln} y\} = \{T^a y, T^n(T^a y), ..., T^{ln}(T^a y)\}.$$

For $x = T^a y$, from

$$\{x, T^n x, ..., T^{ln} x\} \subseteq A_i$$

we have

$$\max\{\rho(x, T^{jn} x) : j \in [0,l]\} < \epsilon,$$

i.e., we have the claim of the Topological van der Waerden Theorem.

Now, we prove that the Topological van der Waerden Theorem implies van der Waerden's Theorem.

Let $(\Omega, T)$ be the symbolic dynamical system discussed above. Let

$$x = \{x_i\}_{i\in\mathbf{Z}} \in \Omega$$

and let

$$X_x = \overline{\{T^n x : n \in \mathbf{Z}\}}.$$

Then $(X_x, T)$ is a symbolic dynamical system.

Let $l \in \mathbf{N}$. By the Topological van der Waerden Theorem there are $y = \{y_i\}_{i\in\mathbf{Z}} \in X_x$ and $n \in \mathbf{N}$ such that

$$\max\{\rho(y, T^{in}y) : i \in [0, l]\} < 1.$$

Let, for $i \in [0, l]$, $T^{in}y = \{y_j^{(in)}\}_{j\in\mathbf{Z}}$. Then

$$y_0 = y_0^{(in)}, i \in [0, l].$$

On the other hand, for all $i \in [0, l]$ and all $j \in \mathbf{Z}$

$$y_j^{(in)} = y_{j+in}.$$

Hence,

$$y_0 = y_n = y_{2n} = ... = y_{ln}.$$

Since $y \in X_x$, there is $m \in \mathbf{Z}$ such that

$$\rho(y, T^m x) < \frac{1}{1 + ln + 1}.$$

Thus, for

$$T^m x = \{x_i^{(m)}\}_{i\in\mathbf{Z}}$$

we have, for all $i \in \mathbf{Z}$,

$$i < ln + 1 \Longrightarrow y_i = x_i^{(m)} = x_{m+i}.$$

Hence,

$$i \in [0, l] \Longrightarrow y_{in} = x_{m+in}$$

or equivalently

$$x_m = x_{m+n} = ... = x_{m+ln}.$$

In other words,

$$\{m, m + n, ...m + ln\}$$

is a monochromatic arithmetic progression of length $(l+1)$, and we have the claim of van der Waerden's Theorem. ■

Note that we have proved that the Topological van der Waerden Theorem implies that for any finite coloring of $\mathbf{Z}$ we have an arbitrarily long monochromatic arithmetic progression with its common difference in $\mathbf{N}$. The following observation shows that this fact guarantees an arbitrarily long monochromatic arithmetic progression for any finite coloring of $\mathbf{N}$.

Let $r \in \mathbf{N}$ and let

$$f : \mathbf{N} \longrightarrow [1, r].$$

Let

$$F : \mathbf{Z} \longrightarrow [1, 2r+1]$$

be defined in the following way

$$F(n) = \begin{cases} f(n), & \text{for } n > 0 \\ f(-n) + r, & \text{for } n < 0 \\ 2r+1, & \text{for } n = 0. \end{cases}$$

Let $k \in \mathbf{N}$. Let $a \in \mathbf{Z}, d \in \mathbf{N}$ and $i \in [1, 2r+1]$ be such that

$$F(a) = F(a+n) = \ldots = F(a+kn) = i.$$

If $i \in [1, r]$ we have that $a > 0$ and

$$f(a) = f(a+n) = \ldots = f(a+kn).$$

If $i \in [r+1, 2r]$ then

$$a + jn < 0, \; j \in [0, k].$$

Let

$$b = -(a+kn).$$

Then

$$i = F(a+kn) = f(b) + r \Longrightarrow f(b) = i - r.$$

For any $j \in [1, k]$ we have that

$$b + jn = -(a+kn) + jn = -(a + (k-j)n)$$

and

$$i = F(a + (k-j)n) = f(b+jn) + r \Longrightarrow f(b+jn) = i - r.$$

Thus,

$$f(b) = f(b+n) = \ldots = f(b+kn).$$

Our second main goal is to prove the Topological Hindman's Theorem.

To formulate and prove the Topological Hindman's Theorem we need two definitions and one lemma.

We consider a dynamical system $(X,T)$ and let $\rho$ denote the metric on $X$.

**Definition 8.4** *Two points $x, y \in X$ are proximal if there is a sequence $\{n_k\}_{k\in\mathbf{N}} \subseteq \mathbf{N}$ such that*

$$\lim_{k\to\infty} \rho(T^{n_k}x, T^{n_k}y) = 0.$$

Let $X^X$ be the space of all mappings of $X$ to itself with the product topology. One basis for this topology on $X^X$ is the family of all sets of the form

$$\{f \in X^X : f(x_i) \in U_i,\ i \in [1,k]\}$$

where $k \in \mathbf{N}$, $x_1, ..., x_k \in X$, and open subsets $U_1, ..., U_k$ of $X$ are given. The space $X^X$ is also a semigroup under composition.

Let $g \in X^X$ and let

$$R_g : X^X \longrightarrow X^X$$

be defined by

$$R_g(f) = fg.$$

We claim that $R_g$ is continuous.

To see this, let $x_1, ..., x_k \in X$ be given and let $\{U_1, ...U_k\}$ be a given set of open subsets of $X$. Let

$$x_i' = g(x_i),\ i \in [1,k].$$

Let

$$U = \{h \in X^X : h(x_i) \in U_i,\ i \in [1,k]\}$$

and

$$V = \{f \in X^X : f(x_i') \in U_i,\ i \in [1,k]\}.$$

Both $U$ and $V$ are open in $X^X$. For $f \in V$ we have that

$$(R_g(f))(x_i) = (fg)(x_i) = f(x_i') \in U_i$$

for all $i \in [1,k]$.

Thus,

$$R_g(V) \subseteq U$$

and $R_g$ is continuous.

For $g \in X^X$ let

$$L_g : X^X \longrightarrow X^X$$

be defined by

$$L_g(f) = gf.$$

We claim that if $g$ is continuous then $L_g$ is continuous too. To see this, let, as above, $x_1, ...x_k \in X$, $U_1, ...U_k \subseteq X$, $U_i$ open for each $i \in [1, k]$, and

$$U = \{h \in X^X : h(x_i) \in U_i,\ i \in [1, k]\}.$$

Since $g$ is continuous and $U_i$, $i \in [1, k]$ are open, for any $i \in [1, k]$ there is $V_i$, an open subset of $X$, such that

$$g(V_i) \subseteq U_i.$$

Thus,

$$V = \{f \in X^X : f(x_i) \in V_i,\ i \in [1, k]\}$$

is open in $X^X$. Clearly,

$$L_g(V) \subseteq U$$

and $L_g$ is continuous.

Let $E$ be the closure of $\{T^n : n \in \mathbf{Z}\}$ in $X^X$. Thus, $f \in E$ if and only if for any $x_1, ..., x_k \in X$ and any open $U_1, ..., U_k$ such that

$$f \in U = \{h \in X^X : h(x_i) \in U_i,\ i \in [1, k]\}$$

there is $n \in \mathbf{N}$ with $T^n \in U$. In other words, $f \in E$ if and only if for any $x_1, ..., x_k \in X$ and any open $U_1, ..., U_k$ there is $n \in \mathbf{N}$ with

$$f(x_i) \in U_i \Longrightarrow T^n(x_i) \in U_i.$$

for all $i \in [1, k]$.

We can reformulate the last statement in the following way:

$f \in E$ if and only if for any $x_1, ..., x_k \in X$ and any $\epsilon > 0$ there is $n \in \mathbf{N}$ with

$$\rho(f(x_i), T^n(x_i)) < \epsilon$$

for all $i \in [1, k]$.

We claim that $E$ is closed under composition. Indeed. Let $f, g \in E$, let $x_1, ..., x_k \in X$ and let $\epsilon > 0$. Since $f \in E$, there is $n \in \mathbf{N}$ such that

$$\rho(f(g(x_i)), T^n g(x_i)) < \frac{\epsilon}{2}$$

for all $i \in [1, k]$.

Since $T^n$ is continuous on $X$ and $X$ is compact, there is $\delta > 0$ such that

$$\rho(x, y) < \delta \Longrightarrow \rho(T^n x, T^n y) < \frac{\epsilon}{2}.$$

Let $m \in \mathbf{N}$ be such that

$$\rho(g(x_i), T^m x_i) < \delta.$$

Hence,

$$\rho(T^m g(x_i), T^{n+m} x_i) < \frac{\epsilon}{2}.$$

Thus,

$$\rho((fg)(x_i), T^{n+m} x_i) \leq \rho(f(g(x_i)), T^m g(x_i)) + \rho(T^m g(x_i), T^{n+m} x_i) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore,

$$f, g \in E \Longrightarrow fg \in E.$$

**Definition 8.5** *$E$ is called the enveloping semigroup of $(X, T)$.*

Note that $E$ is a compact semigroup and for any $g \in E$ the restriction of $R_g$ on $E$ is continuous.

**Lemma 8.1** *If $G$ is a compact semigroup such that for any $g \in G$, the function $R_g : G \longrightarrow G$ given by*

$$R_g(f) = fg$$

*is continuous, then there exists $f \in G$ with*

$$f^2 = f.$$

**Proof.** Let $\mathcal{A}$ be the family of all $A \subseteq G$ such that $A$ is a compact semigroup. Since $G \in \mathcal{A}$, we have that $\mathcal{A} \neq \emptyset$. Let $\mathcal{C} \subseteq \mathcal{A}$ be a chain. Since all elements of $\mathcal{C}$ are compact we have that

$$\bigcap_{A \in \mathcal{C}} A \neq \emptyset.$$

This fact together with

$$f, g \in \bigcap_{A \in \mathcal{C}} A \Longrightarrow fg \in \bigcap_{A \in \mathcal{C}} A$$

gives

$$\bigcap_{A \in \mathcal{C}} A \in \mathcal{A}.$$

Hence, any chain in $\mathcal{A}$ has its lower bound and that lower bound belongs to $\mathcal{A}$. By Zorn's Lemma, there is $A_0 \in \mathcal{A}$ such that for any $A \in \mathcal{A}$

$$A \subseteq A_0 \Longrightarrow A = A_0.$$

Let $g \in A_0$ and let us consider the set

$$A_0 g = \{fg : f \in A_0\}.$$

Clearly, $A_0g \subseteq A_0$. Since $A_0$ is compact and since $R_g$ is continuous we have that $A_0g = R_g(A_0)$ is compact. Secondly, for any $f, f' \in A_0$ we have that $fgf' \in A_0$ and consequently

$$(fg)(f'g) = (fgf')g \in A_0g.$$

Thus, $A_0g \in \mathcal{A}$ and $A_0g \subseteq A_0$. Hence, $A_0g = A_0$. This means that

$$A' = \{f \in A_0 : fg = g\} \neq \emptyset.$$

Since

$$A' = R_g^{-1}(\{g\}) \cap A_0$$

we have that $A'$ is compact. This fact together with, for any $f, f' \in A'$

$$(ff')g = f(f'g) = fg = g$$

means that $A' \in \mathcal{A}$. Thus, $A' = A_0$. Therefore, $g \in A'$ and $g^2 = g$. ■

The following result is often called the Topological Hindman's Theorem. It is due to J. Auslander(1960) and R. Ellis(1960). (Note that Hindman's Theorem was proved in 1974.)

**Theorem 8.7 (Topological Hindman's Theorem)** *If $(X, T)$ is a dynamical system and if $x$ is any point in $X$, there exists a uniformly recurrent point $y \in X$ such that $x$ and $y$ are proximal.*

**Proof.** Let $(X, T)$ be a dynamical system, let $x \in X$ and let $Y \subseteq X_x$ be minimal. Let $E$ be the enveloping semigroup of $(X_x, T)$ and let

$$F = \{f \in E : f(x) \in Y\}.$$

Let $x' \in Y$ and let $\{n_k\}_{k \in \mathbf{N}}$ be a sequence such that

$$\lim_{k \to \infty} \rho(T^{n_k}x, x') = 0$$

Since $E$ is compact and since $\{T^{n_k} : n \in \mathbf{N}\} \subseteq E$, there is $f \in E$ such that for any open $U \subseteq X_x^{X_x}$, $f \in U$, there is $n_k$ with

$$T^{n_k} \in U.$$

Let $\{n_{k_i}\}_{i \in \mathbf{N}} \subseteq \{n_k\}_{k \in \mathbf{N}}$ be a sequence such that

$$T^{n_{k_i}} \in \{h \in X^X : \rho(h(x), f(x)) < \frac{1}{i}\}.$$

From

$$\rho(T^{n_{k_i}}x, f(x)) < \frac{1}{i}$$

for all $i \in \mathbf{N}$, we have that

$$\lim_{i\to\infty} \rho(T^{n_{k_i}}x, f(x)) = 0.$$

Thus,

$$0 \leq \rho(f(x), x') \leq \rho(T^{n_{k_i}}x, f(x)) + \rho(T^{n_{k_i}}x, x') \underset{(i\to\infty)}{\longrightarrow} 0$$

which means that $f(x) = x'$.

Hence, $f \in F$ and $F \neq \emptyset$.

To see that $F$ is closed, let us consider $g \in E$ and a sequence $\{g_i\}_{i\in\mathbf{N}} \subseteq F$ with the property that for any open set $U \subseteq X_x^{X_x}$, $g \in U$, there is $g_i$ with $g_i \in U$.

Hence, for any $j \in \mathbf{N}$ there is $g_{i_j}$ such that

$$\rho(g(x), g_{i_j}(x)) < \frac{1}{j}.$$

Since $\{g_{i_j}(x)\}_{j\in\mathbf{N}} \subseteq Y$ and $Y$ is closed, there is $x' \in Y$ such that for infinitely many $j$'s we have that

$$\rho(g_{i_j}(x), x') < \frac{1}{j}.$$

Thus, for those $j$'s

$$0 \leq \rho(g(x), x') \leq \rho(g(x), g_{i_j}(x)) + \rho(g_{i_j}(x), x') < \frac{1}{j} + \frac{1}{j} = \frac{2}{j}$$

and consequently,

$$g(x) = x' \in Y.$$

Therefore, $g \in F$ and $F$ is closed, hence compact.

Let $f, f' \in F$. Then

$$(ff')(x) = f(f'(x)) \in f(Y).$$

Since $(ff')(x) \in X_x$, there is a sequence $\{m_i\}_{i\in\mathbf{N}}$ such that

$$\rho((ff')(x), T^{m_i}(f'(x))) < \frac{1}{i}.$$

Since $Y$ is closed and $T$-invariant, there is $x' \in Y$ and infinitely many $i$'s with

$$\rho(T^{m_i}(f'(x)), x') < \frac{1}{i}.$$

For those $i$'s we have that

$$0 \leq \rho(f(f'(x)), x') \leq \rho((ff')(x), T^{m_i}(f'(x))) + \rho(T^{m_i}(f'(x)), x') < \frac{1}{i} + \frac{1}{i} = \frac{2}{i}$$

which means that

$$f(f'(x)) = x' \in Y$$

Therefore, for any $f, f' \in F$ we have that

$$(ff')(x) \in Y.$$

This means that $F$ is a semigroup.

By Lemma 8.1, there is $h \in F$ with

$$h^2 = h.$$

Note that $y = h(x) \in Y$ and by Theorem 8.2, $y$ is a uniformly recurrent point. We claim $y$ is proximal to $x$.

Since $h \in E$, for any $i \in \mathbf{N}$ there is $p_i \in \mathbf{N}$ with

$$T^{p_i} \in \{f \in X_x^{X_x} : \rho(f(x), h(x)) < \frac{1}{i} \text{ and } \rho(f(h(x)), h(h(x)) < \frac{1}{i}\}.$$

Thus, since $h^2 = h$ we have

$$\rho(T^{p_i}x, y) < \frac{1}{i}$$

and

$$\rho(T^{p_i}y, y) < \frac{1}{i}.$$

Hence, $0 \leq \rho(T^{p_i}x, T^{p_i}y) < \frac{2}{i}$ for all $i \in \mathbf{N}$. This means $\lim_{i\to\infty} \rho(T^{p_i}x, T^{p_i}y) = 0$ and $x$ and $y$ are proximal. ■

Now, we will prove the following theorem.

**Theorem 8.8** *The Topological Hindman's Theorem implies Hindman's Theorem.*

**Proof.** Let $\chi$ be any $r$-coloring of $\mathbf{N}$ and $x$ be any element of $\Omega = [1, r]^{\mathbf{Z}}$ such that

$$x \mid_{\mathbf{N}} = \chi.$$

Let $(X_x, T)$ be a symbolic dynamical system as before. By the Topological Hindman's Theorem there exists a uniformly recurrent point $y \in X_x$ so that $x$ and $y$ are proximal.

Let $\delta > 0$. Since $y$ is a uniformly recurrent point, there is $N \in \mathbf{N}$ such that for any $n \in \mathbf{N}$ there is $i \in [1, N]$ with

$$\rho(T^{n+i}y, y) < \frac{\delta}{2}.$$

Since $T$ is continuous and $X_x$ is compact, for each $i \in [1, N]$ there is $\delta_i' > 0$ with

$$\rho(z', z'') < \delta_i' \Rightarrow \rho(T^i z', T^i z'') < \frac{\delta}{2}.$$

Let $n \in \mathbf{N}$ be such that

$$\rho(T^n x, T^n y) < \delta'.$$

Note that $n$ exists, since $x$ and $y$ are proximal. Let $i \in [1, N]$ be such that

$$\rho(T^{n+i} y, y) < \frac{\delta}{2}.$$

Then

$$\rho(T^{n+i} x, y) \leq \rho(T^{n+i} x, T^{n+i} y) + \rho(T^{n+i} y, y) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Thus, by taking $p = n + i$, we have proved that for any $\delta > 0$ there is $p \in \mathbf{N}$ with

$$\rho(T^p x, y) < \delta \text{ and } \rho(T^p y, y) < \delta.$$

Let $p_1 \in \mathbf{N}$ be such that

$$\rho(T^{p_1} x, y) < 1 \text{ and } \rho(T^{p_1} y, y) < 1.$$

Let $\epsilon_1 \in (0, 1)$ be such that

$$\rho(z, y) < \epsilon_1 \Longrightarrow \rho(T^{p_1} z, T^{p_1} y) < 1 - \rho(T^{p_1} y, y).$$

For such $\epsilon_1$ and $z \in X_x$ with $\rho(z, y) < \epsilon_1$, we have that

$$\rho(T^{p_1} z, y) < \rho(T^{p_1} z, T^{p_1} y) + \rho(T^{p_1} y, y) < 1 - \rho(T^{p_1} y, y) + \rho(T^{p_1} y, y) = 1.$$

Let $p_2 \in \mathbf{N}$ be such that

$$\rho(T^{p_2} x, y) < \epsilon_1 \text{ and } \rho(T^{p_2} y, y) < \epsilon_1.$$

Thus,

$$\rho(T^{p_2} x, y) < 1 \text{ and } \rho(T^{p_2} y, y) < 1$$

and

$$\rho(T^{p_2+p_1} x, y) < 1 \text{ and } \rho(T^{p_2+p_1} y, y) < 1.$$

Now we proceed inductively. Suppose that we have found $I_n = \{p_i\}_{i=1}^n$ so that for any $p \in P(I_n)$

$$\rho(T^p x, y) < 1 \text{ and } \rho(T^p y, y) < 1.$$

As above, for any $p \in P(I_n)$ there is $\epsilon_p \in (0, 1)$ with

$$\rho(z, y) < \epsilon_p \Longrightarrow \rho(T^p z, y) < 1.$$

Let $\epsilon_{n+1} = \min\{\epsilon_p \colon p \in P(I_n)\}$. Clearly, $\epsilon_{n+1} < 1$. Let $p_{n+1} \in \mathbf{N}$ be such that

$$\rho(T^{p_{n+1}} x, y) < \epsilon_{n+1} \text{ and } \rho(T^{p_{n+1}} y, y) < \epsilon_{n+1}.$$

Thus, for any $p \in P(I_n)$ we have

$$\rho(T^{p_{n+1}+p}x, y) < 1 \text{ and } \rho(T^{p_{n+1}+p}y, y) < 1$$

and consequently, for any $p \in P(I_{n+1})$

$$\rho(T^p x, y) < 1 \text{ and } \rho(T^p y, y) < 1.$$

Let $I = \{p_i\}_{i \in \mathbf{N}}$ and let $p \in P(I)$. From

$$\rho(T^p x, y) < 1$$

we have that

$$y(0) = T^p x(0) = x(p) = \chi(p).$$

Thus, $I = \{p_i\}_{i \in \mathbf{N}}$ is such that there is $j \in [1, r]$, $j = y(0)$, so that for any $i_1 < i_2 < \dots < i_k$

$$\chi(p_{i_1} + p_{i_2} + \dots + p_{i_k}) = j.$$

This is the claim of Hindman's Theorem. ■

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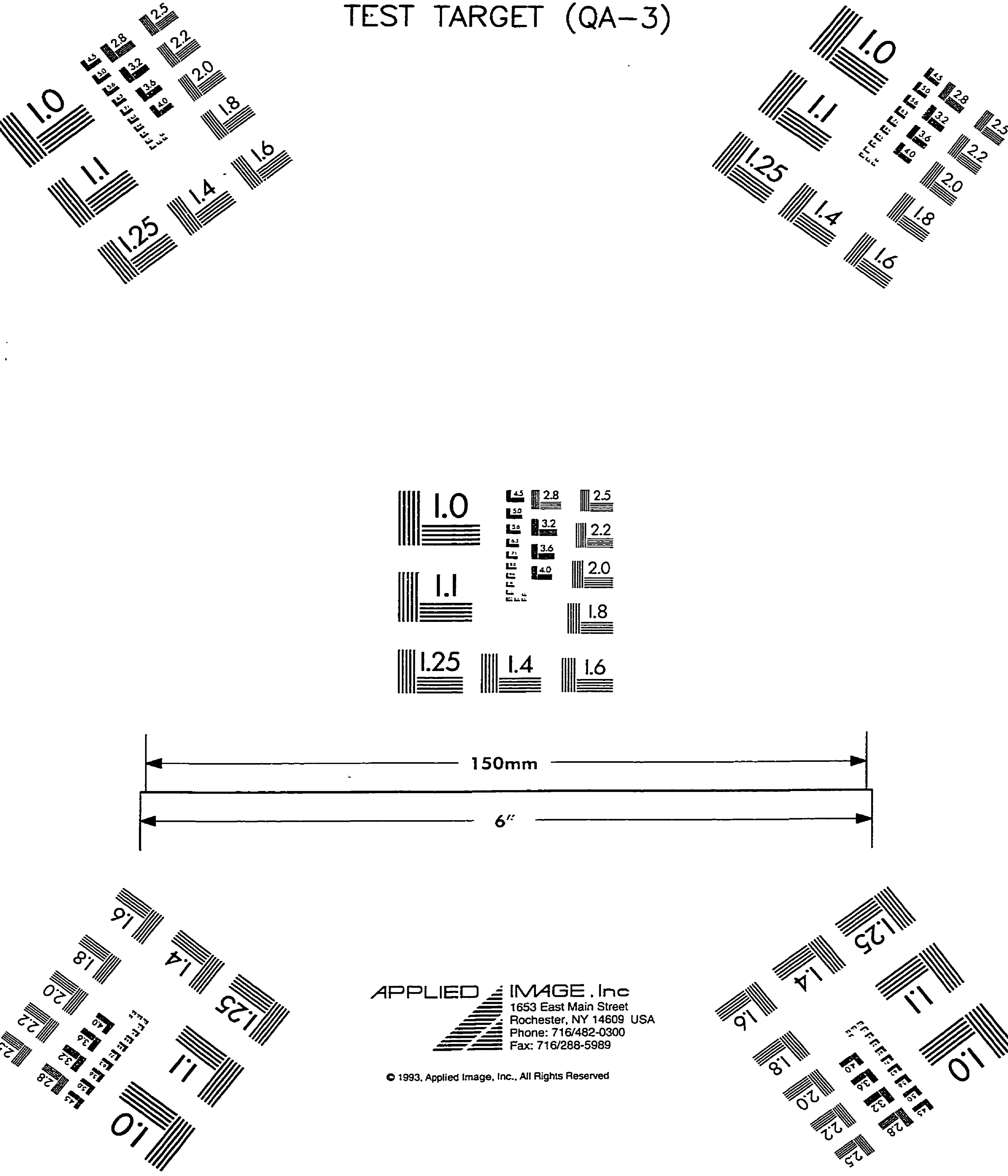
# INDEX

# IMAGE EVALUATION TEST TARGET (QA-3)

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