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Arithmetic Progressions in Sequences with Bounded Gaps

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Let $G(k, r)$ denote the smallest positive integer g such that if $1 = a_1, a_2, ..., a_g$ is a strictly increasing sequence of integers with bounded gaps $a_{j+1} - a_j \leqslant r$, $1 \leqslant j \leqslant g-1$, then $\{a_1, a_2, ..., a_g\}$ contains a k-term arithmetic progression. It is shown that $G(k, 2) > \sqrt{(k-1)/2}\,(\frac{4}{3})^{(k-1)/2}$, $G(k, 3) > (2^{k-2}/ek)(1+o(1))$, $G(k, 2r-1) > (r^{k-2}/ek)(1+o(1))$, $r \geqslant 2$. © 1997 Academic Press

For positive integers k, r, the van der Waerden number $W(k, r)$ is the least integer such that if $w \geqslant W(k, r)$, then any partition of $[1, w]$ into r parts has a part that contains a k-term arithmetic progression. The celebrated theorem of van der Waerden [4] proves the existence of $W(k, r)$. The best known upper bound for $W(k, 2)$ is enormous, whereas the best known lower bound for $W(k, 2)$ (see [1]) is

$$W(k, 2) > \frac{2^k}{2ek}\,(1 + o(1)) \tag{1}$$

where e is the base of the natural logarithm.

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Let $G(k, r)$ denote the smallest positive integer g such that if $1 = a_1$, $a_2, ..., a_g$ is a strictly increasing sequence of integers with bounded gaps $a_{j+1} - a_j \leqslant r$, $1 \leqslant j \leqslant g-1$, then $\{a_1, a_2, ..., a_g\}$ contains a k-term arithmetic progression. In [3], Rabung notes that van der Waerden's theorem implies the existence of $G(k, r)$ for all k, r and conversely.

Nathanson makes the following quantitative connection between $W(k, r)$ and $G(k, r)$ [2, Theorem 4]:

$$G(k, r) \leqslant W(k, r) \leqslant G((k-1)r+1, 2r-1). \tag{2}$$

In particular, $W(k, 2) \leqslant G(2k-1, 3)$, which suggests that it is no easier to find a reasonable upper bound for $G(k, 3)$ than it is for $W(k, 2)$.

However, $G(k, 2)$ "escapes" Nathanson's inequalities in the sense that an upper bound for $G(k, 2)$ does not immediately give an upper bound for $W(k, 2)$.

Setting $r = 2$ and combining (1) and (2) gives

$$G(k, 3) > \frac{2^{(k+1)/2}}{e(k+1)} (1 + o(1)),$$

but again $G(k, 2)$ "escapes" in that no lower bound for $G(k, 2)$ can be deduced from Nathanson's inequalities.

In this note we obtain an exponential lower bound for $G(k, 2)$ and improved lower bounds for $G(k, r)$, $r > 2$. The Lovász local lemma is used when $r > 2$. However, when $r = 2$ this method fails, and elementary counting arguments are used.

THEOREM 1. *For all* $k \geqslant 3$,

$$G(k, 2) > \sqrt{(k-1)/2}\, (\tfrac{4}{3})^{(k-1)/2}.$$

Proof. We use the following notation. For each positive integer n, let

$$\Omega_n = \{\alpha = a_1, a_2, ..., a_n : a_1 = 1,\ 1 \leqslant a_{j+1} - a_j \leqslant 2,\ 1 \leqslant j \leqslant n-1\},$$

and let $\mathcal{S}_n$ be the set of all k-term arithmetic progressions contained in $[1, 2n-1]$.

Let $i \in [1, 2n-1]$ and $\alpha \in \Omega_n$. We say that i occurs in $\alpha = a_1, a_2, ..., a_n$ if $i \in \{a_1, a_2, ..., a_n\}$. Similarly, for any subset I of $[1, 2n-1]$, we say that I occurs in α if $I \subseteq \{a_1, a_2, ..., a_n\}$ and will write $I \subseteq \alpha$.

Let $k \geqslant 3$ be fixed and give Ω_n the uniform probability distribution. The idea of the proof is to show that for any k-term arithmetic progression $S \in \mathcal{S}_n$, $\Pr(S \subseteq \alpha) \leqslant (\tfrac{3}{4})^{k-1}$.

For each i, $1 \leqslant i \leqslant 2n-1$, let $A_i = \{\alpha \in \Omega_n : i \text{ occurs in } \alpha\}$. Then $\Pr(A_1) = 1$, $\Pr(A_2) = \frac{1}{2}$, and $\Pr(A_3) = \frac{3}{4}$.

To show that $\Pr(A_i) \leqslant \frac{3}{4}$ for $i > 3$, partition A_i so that it is the disjoint union $A_i = A_i^0 \cup A_i^1 \cup A_i^2$, where $A_i^0 = \{\alpha \in A_i : a_n = i\}$ and for $m = 1, 2$, $A_i^m = \{\alpha \in A_i : a_j = i \Rightarrow a_{j+1} = i + m\}$. Now $|A_i^1| = |A_i^2|$ and so $\Pr(A_i{}^m) \leqslant \frac{1}{2}\Pr(A_i)$, $1 \leqslant m \leqslant 2$. Moreover, A_{i+2} is the disjoint union of A_{i+1}^1 and A_i^2, and thus

$$\begin{aligned}\Pr(A_{i+2}) &= \Pr(A_{i+1}^1) + \Pr(A_i^2) \\ &\leqslant \tfrac{1}{2}(\Pr(A_{i+1}) + \Pr(A_i)).\end{aligned}$$

It follows by induction that for $i = 2, 3, ..., 2n-1$,

$$\Pr(A_i) \leqslant \tfrac{3}{4}. \tag{3}$$

Note that inequality (3) is independent of n. That is, for every $n \geqslant 1$ and every $i = 2, 3, ..., 2n-1$,

$$|\{\alpha \in \Omega_n : i \text{ occurs in } \alpha\}| \leqslant \tfrac{3}{4}\,|\Omega_n| = \tfrac{3}{4} \cdot 2^{n-1}. \tag{4}$$

Let n be fixed and let I be a nonempty subset of $\{2, 3, ..., 2n-1\}$. Let m be the largest element of I and define $A_I = \bigcap_{i \in I} A_i$. We proceed to show that $\Pr(A_I) \leqslant (\frac{3}{4})^{|I|}$.

Define $\tilde{A}_I = \{\tilde{\alpha} = a_1, a_2, ..., a_s : a_1 = 1, a_s = m, s \leqslant n, I \text{ occurs in } \alpha\}$ and for each $\tilde{\alpha} \in \tilde{A}_I$, $\tilde{\alpha} = a_1, a_2, ..., a_s$, define $B_{\tilde{\alpha}}$ to be the set of all 2^{n-s} "continuations" of $a_1, a_2, ..., a_s$. That is, let $B_{\tilde{\alpha}} = \{\alpha \in \Omega_n : \alpha = a_1, a_2, ..., a_s, b_{s+1}, ..., b_n\}$. Then A_I is the disjoint union

$$A_I = \bigcup_{\tilde{\alpha} \in \tilde{A}_I} B_{\tilde{\alpha}} \tag{5}$$

Let j be such that $m < j \leqslant 2n-1$. We now want to estimate the number of sequences in $B_{\tilde{\alpha}}$ in which j occurs. For each $\tilde{\alpha} \in \tilde{A}_I$, $\tilde{\alpha} = a_1, a_2, ..., a_s$, we can map $B_{\tilde{\alpha}}$ onto Ω_{n-s+1} by dropping $a_1, a_2, ..., a_{s-1}$ and then shifting $m-1$ units to the left. That is, we map $\alpha \in B_{\tilde{\alpha}}$, $\alpha = a_1, a_2, ..., a_s, b_{s+1}, ..., b_n$, into $\beta = 1, b_{s+1} - (m-1), ..., b_n - (m-1)$. Clearly j occurs in α if and only if $j - (m-1)$ occurs in β.

Using (4) we therefore have

$$\begin{aligned}|\{\alpha \in B_{\tilde{\alpha}} : j \text{ occurs in } \alpha\}| &= |\{\beta \in \Omega_{n-s+1} : j - (m-1) \text{ occurs in } \beta\}| \\ &\leqslant \tfrac{3}{4}\,|\Omega_{n-s+1}| \\ &= \tfrac{3}{4} \cdot 2^{n-s} \\ &= \tfrac{3}{4}\,|B_{\tilde{\alpha}}|. \end{aligned} \tag{6}$$

Combining (5) and (6) we obtain

$$\begin{aligned} |A_I \cap A_j| &= \sum_{\tilde{\alpha} \in \tilde{A}_I} |B_{\tilde{\alpha}} \cap A_j| \\ &\leqslant \sum_{\tilde{\alpha} \in \tilde{A}_I} \tfrac{3}{4} |B_{\tilde{\alpha}}| \\ &= \tfrac{3}{4} |A_I|. \end{aligned} \tag{7}$$

Hence by induction (using (3) and (7)), $\Pr(A_I \cap A_j) \leqslant \frac{3}{4} \Pr(A_I) \leqslant (\frac{3}{4})^{|I|+1}$. Note also that $\Pr(A_{I \cup \{1\}}) \leqslant (\frac{3}{4})^{|I|}$. In particular, for all $S \in \mathcal{S}_n$, $\Pr(S \subseteq \alpha) \leqslant (\frac{3}{4})^{k-1}$.

For each S in $\mathcal{S}_n$, let E_S denote the event "$S \subseteq \alpha$." The probability that some S in $\mathcal{S}_n$ occurs in α satisfies

$$\begin{aligned} \Pr\left(\bigcup_{S \in \mathcal{S}_n} E_S \right) &\leqslant \sum_{S \in \mathcal{S}_n} \Pr(E_S) \\ &\leqslant |\mathcal{S}_n| \left(\frac{3}{4} \right)^{k-1} \\ &\leqslant \frac{(2n-1)^2}{2(k-1)} \left(\frac{3}{4} \right)^{k-1}. \end{aligned}$$

If $n < \frac{1}{2} + \sqrt{(k-1)/2}\, (\frac{4}{3})^{(k-1)/2}$, then $[(2n-1)^2/2(k-1)](\frac{3}{4})^{k-1} < 1$ and hence $\Pr(\bigcap_{S \in \mathcal{S}_n} \overline{E_S}) > 0$. That is, there exists $\alpha \in \Omega_n$ that does not contain a k-term arithmetic progression. Therefore $G(k, 2) > \sqrt{(k-1)/2}\, (\frac{4}{3})^{(k-1)/2}$. ∎

The proof of Theorem 1 can easily be modified to show that $G(k, r) > \sqrt{(k-1)/2}\ (1/p)^{(k-1)/2}$, where $p = p(r) = (1/r)(1 + 1/r)^{r-1}$, for all $k \geqslant 3$, $r \geqslant 2$. But this is much weaker than the following result.

THEOREM 2. *For all $k \geqslant 3$, $r \geqslant 2$,*

$$G(k, 2r-1) > \frac{r^{k-2}}{ek} (1 + o(1)).$$

Before proving Theorem 2, we state the form of the Lovász local lemma we use [1].

LOVÁSZ LOCAL LEMMA. *Let $A_1, ..., A_m$ be events with $\Pr(A_i) \leqslant p$ for all i. Suppose that each A_i is mutually independent of all but at most d of the other A_j's. If $ep(d+1) < 1$, then $\Pr(\bigcap \overline{A_i}) > 0$.*

Proof of Theorem 2. (In the case of $r=2$). To simplify the notation, we carry out the proof only in the case $r=2$. The proof for the general case is essentially the same.

Fix $k \geqslant 3$ and fix n. Let $\mathcal{M}$ be the set of all sequences $\alpha = a_1, a_2, ..., a_n$ such that $a_i \in \{2i-1, 2i\}$, $1 \leqslant i \leqslant n$. Thus α contains exactly one of the two elements in each of the *blocks* $[1, 2], [3, 4], ..., [2n-1, 2n]$.

Let the symbols S, T denote k-term arithmetic progressions contained in $[1, 2n]$ with common differences at least two. Give $\mathcal{M}$ the uniform probability distribution and again let E_S denote the event "$S \subseteq \alpha$". Then $|\mathcal{M}| = 2^n$ and $|\{\alpha \in \mathcal{M} : S \subseteq \alpha\}| = 2^{n-k}$, so $\Pr(E_S) = 2^{-k}$.

The event E_S is mutually independent of all the other events E_T for all T that have no blocks in common with S (that is, for no i, $1 \leqslant i \leqslant n$, is it true that $[2i-1, 2i] \cap S \neq \emptyset$ and $[2i-1, 2i] \cap T \neq \emptyset$). To see this, note that a random $\alpha \in \mathcal{M}$ can be constructed by randomly and independently choosing each element a_i from $[2i-1, 2i]$ with uniform probability. Thus even if we know the chosen element of α for each block besides those of S, the probability of E_S remains unchanged, and any assumption on the events E_T for T that have no blocks in common with S is determined by these chosen elements.

For each S, the number of T such that S and T do have a block in common is bounded above by $4nk$. (To see this note that the number of k-term arithmetic progressions in $[1, 2n]$ which contain any given element of $[1, 2n]$ is bounded above by $2n$ (in fact, by about $(\log 2)(2n)$). Since S meets k blocks, T will have a block in common with S only if T contains one of the $2k$ elements of these k blocks.)

Now we can apply the Lovász local lemma with $p = 2^{-k}$, $d = 4nk$. If $n < (2^{k-2}/ek)(1-\varepsilon)$, then $ep(d+1) < 1$, so $\Pr(\bigcap \overline{E_S}) > 0$. Therefore if $n < (2^{k-2}/ek)(1-\varepsilon)$, there is $\alpha \in \mathcal{M}$, $\alpha = a_1, a_2, ..., a_n$, which contains no k-term arithmetic progression. Since $a_{j+1} - a_j \leqslant 3$ for all j, this shows that $G(k, 3) > (2^{k-2}/ek)(1 + o(1))$. ∎

We conclude with several remarks.

Apparently nothing at all is known concerning an upper bound for $G(k, 2)$ (and hence for any $G(k, r)$) other than the inequality $G(k, 2) \leqslant W(k, 2)$.

Since $G(k, 2)$ may well be much smaller than $W(k, 2)$ and since an upper bound for $G(k, 2)$ would not automatically give an upper bound for $W(k, 2)$ (in sharp constrast to the fact that an upper bound for $G(k, 3)$ *would* automatically give an upper bound for $W(k, 2)$ via Nathanson's inequality $W(k, 2) \leqslant G(2k-1, 3)$), it may be far easier to find an explicit upper bound for $G(k, 2)$ than an upper bound for $W(k, 2)$.

Some values of $G(k, 2)$ are: $G(3, 2) = 5$, $G(4, 2) = 10$, $G(5, 2) = 19$, $G(6, 2) = 37$ (see Rabung's paper [3] for some related values).

It would also be interesting to find a function $f(k)$ such that $W(k, 2) \leqslant G(f(k), 2)$.

Note added in proof. Noga Alon has suggested a modification of our proof which improves the $(\frac{4}{3})^{(k-1)/2}$ term in Theorem 1 to $(c - \varepsilon(k))^{k/2}$, where $\varepsilon(k)$ tends to zero as k tends to infinity, and where c is an absolute constant which exceeds 3/2.

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