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Canonical Partition Theorems for Parameter Sets

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A canonical (i.e., unrestricted) version of the partition theorem for k-parameter sets of Graham and Rothschild (*Trans. Amer. Math. Soc.* **159** (1971), 257–291) is proven. Some applications, e.g., canonical versions, of the Rado–Folkman–Sanders theorem and of the partition theorem for finite Boolean algebras are given. Also the Erdös–Rado canonization theorem (*J. London Math. Soc.* **25** (1950), 249–255) turns out to be an immediate corollary.

A. INTRODUCTION

"Classical" partition (Ramsey) theory investigates the behavior of structures with respect to colorings of substructures with only a small number of colors. The main question is whether it is possible to obtain monochromatic (i.e., constantly colored) substructures. For a survey on Ramsey theory see, e.g., [5]. Recent research considers more general colorings, viz., colorings with an arbitrary number of colors. Of course, generally one cannot expect to find monochromatic substructures. But possibly one always can be restricted to certain types of colorings, for example, structures on which the coloring is either constant or one-to-one. An example of such a theorem is the so-called "canonical version" of van der Waerden's theorem on arithmetic progressions, which is due to Erdös and Graham.

THEOREM. *For every positive integer k there exists a positive integer n such that for every coloring $\Delta: \{0,..., n-1\} \to \omega$ of the first n nonnegative integers with arbitrary many colors* (i.e., *with an infinite number of colors*) *there exists an arithmetic progression $a, a+d,..., a+(k-1)\cdot d$ of k terms such that $\Delta \restriction \{a,..., a+(k-1)\cdot d\}$ is either a constant coloring or a one-to-one coloring.*

However, things do not always behave so nicely; sometimes it is certainly not true that one can always be restricted to a constant or a one-to-one

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coloring. A prototypical result in this direction is the so-called "Erdös–Rado canonization theorem." This may be viewed as the generalization of Ramsey's theorem to arbitrary colorings.

THEOREM [3]. *Let k, m be positive integers. Then there exists a positive integer n such that for every coloring $\Delta : [n]^k \to \omega$ of the k-element subsets of $n = \{0,..., n-1\}$ with infinitely many colors there exists an m-element subset X of n and a—possibly empty—subset $\mathscr{K} \subseteq \{0,..., k-1\}$ of k such that for any two k-element subsets $\{\alpha_0,..., \alpha_{k-1}\}$ and $\{\beta_0,..., \beta_{k-1}\}$ of X, where $\alpha_0 < \cdots < \alpha_{k-1}$ and $\beta_0 < \cdots < \beta_{k-1}$, it follows that $\Delta(\{\alpha_0,..., \alpha_{k-1}\}) = \Delta(\{\beta_0,..., \beta_{k-1}\})$ iff $\alpha_i = \beta_i$ for all $i \in \mathscr{K}$.*

In other words, with respect to colorings of k-element subsets there exist 2^k different types of canonical colorings, viz., each $\mathscr{K} \subseteq \{0,..., k-1\}$ gives rise to such a type and obviously none of these types may be omitted without violating the assertion of the theorem.

In this paper we propose a definition of canonical colorings in arbitrary structures. Then we state and prove a canonical version of the partition theorem for k-parameter sets of Graham and Rothschild [4]. Since the partition theorem for k-parameter sets admits as immediate corollaries Ramsey's theorem as well as the Rado–Folkman–Sanders theorem on finite sums (or unions), the canonical partition theorem for k-parameter sets yields as corollaries the Erdös–Rado canonization theorem and a canonical Rado–Folkman–Sanders theorem. This also improves a result of Taylor [10].

One remark concerning our notation: Because the "type of a coloring" does not depend on the colors that were actually used but rather on the equivalence relation given by the fibres of the coloring we prefer to use the notion of "canonical sets of equivalence relations." Consequently we shall talk about "equivalence relations" instead of "colorings with an arbitrary large number of colors." However, the notion of colorings will be reserved for situations where partition results are applied.

B. CANONICAL SETS OF EQUIVALENCE RELATIONS

In this section we use the language of categories in order to define the notion of canonical sets of equivalence relations. Recall that a category $\mathbb{C}$ is given by a set of objects A, B, C,..., and for each two objects A, B of $\mathbb{C}$ a set $\mathbb{C}\binom{A}{B}$ of morphisms $f: B \to A$ is defined. Finally morphisms $f \in \mathbb{C}\binom{A}{B}$ and $g \in \mathbb{C}\binom{B}{C}$ may be composed yielding $f \cdot g \in \mathbb{C}\binom{A}{C}$ and this composition is associative. Since this is nearly all we need from category theory in connection with partition (Ramsey) theory sometimes such categories $\mathbb{C}$ are called "classes with binomial coefficients $C\binom{\cdot}{\cdot}$."

In all applications $\mathbb{C}\binom{A}{B}$ will be a set of rigidified monomorphisms, thus $\mathbb{C}\binom{A}{B}$ represents the set of B-subobjects of A.

Notation. For a set X we denote by $\Pi(X)$ the set of equivalence relations on X. In particular, $\Pi(\mathbb{C}\binom{A}{C})$ denotes the set of equivalence relations on the set of C-subobjects of A. If $\pi \in \Pi(X)$ and $y, z \in X$, then $y \approx z \pmod{\pi}$ indicates that y and z are equivalent modulo π.

Notation. Let $\pi \in \Pi(\mathbb{C}\binom{A}{C})$ and $f \in \mathbb{C}\binom{A}{B}$. Then $\pi_f \in \Pi(\mathbb{C}\binom{B}{C})$ denotes the equivalence relation which is induced from f, viz.,

$$g \approx h \pmod{\pi_f} \qquad \text{iff} \qquad f \cdot g \approx f \cdot h \pmod{\pi}.$$

DEFINITION. A set $\mathscr{A} \subseteq \Pi(\mathbb{C}\binom{B}{C})$ is a *canonical set of equivalence relations* (or shorthand: $\mathscr{A}$ is canonical) iff $\mathscr{A}$ is a set of minimal cardinality such that there exists an object A in $\mathbb{C}$ satisfying:

for every equivalence relation $\pi \in \Pi(\mathbb{C}\binom{A}{C})$ there exists an embedding $f \in \mathbb{C}\binom{A}{B}$ such that $\pi_f \in \mathscr{A}$. (can)

Remark. A priori it is not clear whether all minimal sets $\mathscr{A} \subseteq \Pi(\mathbb{C}\binom{B}{C})$ which satisfy (can) have the same cardinality or not. We do not know an example of a category $\mathbb{C}$ with minimal sets of different cardinalities. Possibly there exists some (weak) conditions satisfied in all relevant categories which imply that all minimal sets $\mathscr{A} \subseteq \Pi(\mathbb{C}\binom{A}{C})$ have the same cardinality.

It turns out that in general canonical sets of equivalence relations are not uniquely determined. In particular, the canonical version of Schur's theorem (i.e., Theorem D.4) yields an example for this. More examples may be found, e.g., in the category Fin Tree(m) of finite trees in which each element has at most m immediate successors (see [1]). Thus in general it does not make sense to say that a specific equivalence relation $\pi \in \Pi(\mathbb{C}\binom{B}{C})$ is a canonical equivalence relation. However, there always exist certain equivalence relations $\pi \in \Pi(\mathbb{C}\binom{B}{C})$ which necessarily belong to each canonical set $\mathscr{A}$. For example, the constant equivalence relation (where any two C-subobjects are equivalent) is of that type and if $|\mathbb{C}\binom{B}{C}| > 1$, also the one-to-one equivalence relation (where each C-subobject is only equivalent to itself) belongs to every canonical set $\mathscr{A}$.

DEFINITION. An equivalence relation $\pi \in \Pi(\mathbb{C}\binom{B}{C})$ is a *necessary equivalence relation* iff for every object A there exists an equivalence relation $\pi^* \in \Pi(\mathbb{C}\binom{A}{C})$ such that $\pi^*_f = \pi$ for every $f \in \mathbb{C}\binom{A}{B}$.

Remark. Each canonical set of equivalence relations $\mathscr{A} \in \Pi(\mathbb{C}\binom{B}{C})$ contains every necessary equivalence relation $\pi \in \Pi(\mathbb{C}\binom{B}{C})$. But in general a canonical set may also contain equivalence relations which are not necessary. However, for certain categories it is true that the set of all necessary equivalence relations is canonical, e.g., the Erdös–Rado canonization theorem provides an example for this.

Notation. Let $\mathscr{A} \subseteq \Pi(\mathbb{C}(\binom{B}{C}))$ be a canonical set of equivalence relations. Adapting the well-known Ramsey-arrow we shall write $A \to_{\text{can}}^{\mathbb{C},\mathscr{A}} (B)^C$ in order to indicate that for every equivalence relation $\pi \in \Pi(\mathbb{C}(\binom{A}{C}))$ there exists an $f \in \mathbb{C}(\binom{A}{B})$ such that $\pi_f \in \mathscr{A}$. We shall suppress $\mathscr{A}$ and $\mathbb{C}$ if no confusion can arise.

C. Canonical Sets for the Hales–Jewett Class [A]

The partition theorem of Hales and Jewett [6] as well as its generalization, the partition theorem for k-parameter sets of Graham and Rothschild [4], plays a central role in partition theory for finite structures. In this section we study canonical versions of these theorems.

Basically the Hales–Jewett theorem considers partitions of vertices of the n-dimensional cube A^n, where A is a finite set. The result is that for sufficiently large n there always exists some monochromatic k-dimensional subcube. The notion of a k-dimensional subcube is defined purely combinatorically, that is, without any algebraic means.

Definition C.1. Let A be a finite set and let $k \leqslant n$ be nonnegative integers. Then $[A](\binom{n}{k})$ is the set of mappings $f: n \to A \cup \{\lambda_0, ..., \lambda_{k-1}\}$—where without restriction $A \cap \{\lambda_0, \lambda_1, ...\} = \varnothing$—which satisfy

(1) for every $j < k$ there exists some $i < n$ with $f(i) = \lambda_j$,

(2) $\min f^{-1}(\lambda_i) < \min f^{-1}(\lambda_j)$ for all $i < j < k$.

Usually the elements $f \in [A](\binom{n}{k})$ are called "k-parameter words of length n." Each $f \in [A](\binom{n}{k})$ represents a unique k-dimensional subcube in A^n, viz., the set $\{f \cdot (\lambda_0, ..., \lambda_{k-1}) \mid \lambda_0, ..., \lambda_{k-1} \in A\}$, where the parameters $\lambda_0, ..., \lambda_{k-1}$ which occur in f are replaced by elements of A.

Originally these kinds of embeddings have been considered in the Hales–Jewett theorem. Later on Graham and Rothschild [4] proved a more general partition theorem for partitions of k-dimensional subcubes (i.e., k-parameter words of length n).

In order to state this theorem explicitly we have to define the notion of a k-dimensional subcube of an m-dimensional subcube in an n-dimensional cube. Recall that for linear spaces these notions may be defined by multiplying the corresponding matrices. For parameter words the following composition is introduced:

DEFINITION C.2. Let $f \in [A]\binom{n}{m}$ and $g \in [A]\binom{m}{k}$. Then $f \cdot g \in [A]\binom{n}{k}$ is defined by

$$f \cdot g(i) = f(i) \quad \text{if} \quad f(i) \in A,$$
$$= g(j) \quad \text{if} \quad f(i) = \lambda_j.$$

Motivation. The product $f \cdot g$ defines the notion of "a k-dimensional subcube of an m-dimensional subcube of an n-dimensional cube."

The partition theorem for Hales–Jewett cubes says:

THEOREM C.3 [6, 4]. *Let A be a finite set and let δ, k, m be nonnegative integers. Then there exists a positive integer n satisfying:*

> for every coloring $\Delta: [A]\binom{n}{k} \to \delta$ of the k-dimensional subcubes in A^n with δ colors there exists an m-dimensional subcube $f \in [A]\binom{n}{m}$ with all its k-dimensional subcubes in one color, i.e., the coloring $\Delta_f: [A]\binom{m}{k} \to \delta$ with $\Delta_f(g) = \Delta(f \cdot g)$ is a constant coloring. (HJ)

Notation. For convenience we shall abbreviate (HJ) by $n \to^A (m)^k_\delta$.

For a short proof of this result see, e.g., [2]. Next a canonical version of this theorem will be presented. We need a bit more preparation.

Notation. Let $\sigma \in \Pi(X)$ and $\tau \in \Pi(Y)$ be equivalence relations. We shall write $\sigma \leqslant \tau$ iff $a \approx b \pmod{\sigma}$ for $a, b \in X \cap Y$ implies $a \approx b \pmod{\tau}$. Observe that $\leqslant$ is a quasiordering. However, $\sigma \leqslant \tau$ and $\tau \leqslant \sigma$ imply that $\sigma \restriction X \cap Y = \tau \restriction X \cap Y$. We shall use this quasiordering only when $X \subseteq Y$ or $Y \subseteq X$.

DEFINITION C.4. A sequence $\boldsymbol{\pi} = (\pi_0, ..., \pi_k)$ of equivalence relations $\pi_i \in \Pi(A \cup \{\lambda_0, ..., \lambda_i\})$, $i = 0, ..., k-1$, and $\pi_k \in \Pi(A \cup \{\lambda_0, ..., \lambda_{k-1}\})$ is *k-canonical* iff

(1) $\pi_0 \leqslant \pi_1 \leqslant \cdots \leqslant \pi_k$,

(2) if $\lambda_i \approx c \pmod{\pi_i}$ for some $c \in A \cup \{\lambda_0, ..., \lambda_{i-1}\}$, $i < k$, then $\pi_{i+1} \leqslant \pi_i$.

Remark. It turns out that k-canonical sequences determine certain necessary equivalence relations in $\Pi([A]\binom{m}{k})$, comparable to the fact that in the Erdös–Rado canonization theorem subsets $\mathscr{K} \subseteq \{0, ..., k-1\}$ determine necessary equivalence relations in $\Pi([m]^k)$. It remains to explain how k-canonical sequences determine equivalence relations in $\Pi([A]\binom{m}{k})$.

DEFINITION C.5. Let $\boldsymbol{\pi} = (\pi_0, ..., \pi_k)$ be k-canonical and let $f \in [A]\binom{m}{k}$.

The numbers $\omega_\pi(f, i)$, $i = 0,..., k-1$, are defined by $\omega_\pi(f, i) = \min\{l > \omega_\pi(f, i-1) \mid f(l) \approx \lambda_i \pmod{\pi_i}\}$, where for technical convenience $\omega_\pi(f, -1) = -1$ and $\omega_\pi(f, k) = m$.

I.e., $\omega_\pi(f, 0)$ denotes the first occurrence of an element which is equivalent to λ_0 modulo π_0. Then $\omega_\pi(f, 1)$ denotes the first occurrence after $\omega_\pi(f, 0)$ of an element which is equivalent to λ_1 modulo π_1, etc. Suppose that in f entries which are equivalent to λ_i modulo π_i shall be replaced by λ_i. Then the numbers $\omega_\pi(f, 0)$, $\omega_\pi(f, 1)$,..., give earliest possible entries for doing this.

EXAMPLE. Let $A = \{0, 1, 2\}$, $k = 3$, and let the 3-canonical sequence $(\pi_0, \pi_1, \pi_2, \pi_3)$ be given by

$$\pi_0 = \{\{0\}, \{1\}, \{2\}, \{\lambda_0\}\}, \qquad \pi_1 = \{\{0, \lambda_0\}, \{1, \lambda_1\}, \{2\}\},$$
$$\pi_2 = \{\{0, \lambda_0\}, \{1, \lambda_1\}, \{2, \lambda_2\}\}, \quad \pi_3 = \{\{0, \lambda_0\}, \{1, \lambda_1\}, \{2, \lambda_2\}\}.$$

Finally, let

$$f = (0, 1, 2, \underset{\uparrow}{\lambda_0}, 0, \underset{\uparrow}{1}, \underset{\uparrow}{2}, \lambda_1, 0, 1, 2, \lambda_0, \lambda_2, 0, 1) \in [A]\binom{15}{3}.$$

Then $\omega_\pi(f, 0) = 3$, $\omega_\pi(f, 1) = 5$, $\omega_\pi(f, 2) = 6$.

DEFINITION C.6. Let $\boldsymbol{\pi} = (\pi_0,..., \pi_k)$ be k-canonical. The partition $\boldsymbol{\pi}^m \in \Pi([A]\binom{m}{k})$ is defined as follows: $f \approx g \pmod{\boldsymbol{\pi}^m}$ iff

$$f(l) \approx g(l) \qquad \pmod{\pi_{i+1}},$$

for all $l < m$ with $\omega_\pi(f, i) < l \leqslant \omega_\pi(f, i+1)$ and $i = -1, 0,..., k-1$.

Examples

(1) Let $k = 0$. Each $\pi \in \Pi(A)$ is 0-canonical. Then $\pi^m \in \Pi([A]\binom{m}{0})$ is defined by applying π componentwise, i.e., $(a_0,..., a_{m-1}) \approx (b_0,..., b_{m-1}) \pmod{\pi^m}$ iff $a_i \approx b_i \pmod{\pi}$ for $i < m$.

(2) Let $k = 1$. A pair $\boldsymbol{\pi} = (\pi_0, \pi_1)$, where $\pi_0 \in \Pi(A \cup \{\lambda_0\})$ and $\pi_1 \in \Pi(A \cup \{\lambda_0\})$, is 1-canonical iff $\pi_0 \leqslant \pi_1$ and if $a \approx \lambda_0 \pmod{\pi_0}$ for some $a \in A$, then $\pi_0 = \pi_1$. Again $\boldsymbol{\pi}^m$ is defined by applying (π_0, π_1) componentwise, i.e., if $\pi_0 = \pi_1$, then $f \approx g \pmod{\boldsymbol{\pi}^m}$ iff $f(l) \approx g(l) \pmod{\pi_0}$ for every $l < m$ and if $\pi_0 \neq \pi_1$, then $f \approx g \pmod{\boldsymbol{\pi}^m}$ iff $\min f^{-1}(\lambda_0) = \min g^{-1}(\lambda_0)$ and $f(l) \approx g(l) \pmod{\pi_0}$ for every $j < m$ and $f(l) \approx g(l) \pmod{\pi_1}$ for every $\min f^{-1}(\lambda_0) < l < m$.

(3) Again let $k=1$, but now $A=\{0\}$. There exist three necessary equivalence relations: Let $f, g\in[A]\binom{m}{1}$,

$$f\approx g \quad \text{iff} \quad f=g,$$
$$\text{iff} \quad \min f^{-1}(\lambda_0)=\min g^{-1}(\lambda_0),$$
$$\text{without any restriction.}$$

The canonical version of the partition theorem for k-parameter words (viz., Theorem C.3) may be stated as follows:

THEOREM C.7. *Let A be a finite alphabet. Then $\{\boldsymbol{\pi}^m \mid \boldsymbol{\pi}$ k-canonical$\}$ is the set of necessary equivalence relations. Moreover, $\{\boldsymbol{\pi}^m \mid \boldsymbol{\pi}$ k-canonical$\}$ is a canonical set of equivalence relations.*

Proof of Theorem C.7. Fix the finite set A. We first show that $\{\boldsymbol{\pi}^m \mid \boldsymbol{\pi}$ k-canonical$\}$ satisfies (can), viz.,

LEMMA. *Let $k<m$ be nonnegative integers. Then there exists n such that for every equivalence relation $\pi\in\Pi([A]\binom{n}{k})$ there exists an $f\in[A]\binom{n}{m}$ such that*

$$\pi_f\in\{\boldsymbol{\pi}^m \mid \boldsymbol{\pi}\ k\text{-canonical}\}.$$

Proof of Lemma. Let n' be such that

$$n' \xrightarrow{A} (m+1)^m_{|\Pi([A]\binom{m}{k})|}.$$

Such a number n' exists according to the partition theorem for Hales–Jewett cubes. Then let n be such that

$$n \xrightarrow{A} (n')^{k+1}_{|\Pi[A]\binom{k+1}{k}|}.$$

It remains to show that n has the desired properties. Let $\pi\in\Pi([A]\binom{n}{k})$ be any equivalence relation. Consider first the coloring

$$\Delta' : [A]\binom{n}{k+1}\to\Pi\left([A]\binom{k+1}{k}\right) \qquad \text{with} \quad \Delta'(g)=\pi_g.$$

Then there exists an $f'\in[A]\binom{n}{n'}$ such that $\Delta'(f'\cdot g)=\Delta'(f'\cdot h)$ for all g, $h\in[A]\binom{n'}{k+1}$, in other words, $\pi_{f'\cdot g}=\pi_{f'\cdot h}$ for all $g, h\in[A]\binom{n'}{k+1}$.

Next consider the coloring

$$\Delta : [A]\binom{n'}{m}\to\Pi\left([A]\binom{m}{k}\right) \qquad \text{with} \quad \Delta(g)=\pi_{f'\cdot g}.$$

Then there exists an $f'' \in [A]\binom{n'}{m+1}$ such that $\Delta(f'' \cdot g) = \Delta(f'' \cdot h)$ for all $g, h \in [A]\binom{m+1}{m}$, i.e., $\pi_{f' \cdot f'' \cdot g} = \pi_{f' \cdot f'' \cdot h}$ for all $g, h \in [A]\binom{m+1}{m}$.

For convenience let us forget about f' and f'' and assume that $\pi \in \Pi([A]\binom{m+1}{k})$ satisfies

$$\pi_g = \sigma \qquad \text{for all} \quad g \in [A]\binom{m+1}{k+1}, \tag{1}$$

and

$$\pi_g = \tau \qquad \text{for all} \quad g \in [A]\binom{m+1}{m}, \tag{2}$$

where $\sigma \in \Pi([A]\binom{k+1}{k})$ and $\tau \in \Pi([A]\binom{m}{k})$.

Define the sequence $\boldsymbol{\pi} = (\pi_0, ..., \pi_k)$ as follows:

$$\begin{aligned} a \approx b \quad (\text{mod } \pi_i) \qquad &\text{iff} \\ &(\lambda_0, ..., \lambda_{i-1}, a, \lambda_i, ..., \lambda_{k-1}) \\ &\approx (\lambda_0, ..., \lambda_{i-1}, b, \lambda_i, ..., \lambda_{k-1}) \quad (\text{mod } \sigma). \end{aligned}$$

Recall that actually $\pi_i \in \Pi(A \cup \{\lambda_0, ..., \lambda_i\})$ as no λ_j can occur at the ith position of any parameter word for $j > i$.

The lemma is proved by a series of propositions showing that $\boldsymbol{\pi}$ is k-canonical and that $\boldsymbol{\pi}^m = \tau$. For convenience let $f \in [A]\binom{m+1}{k+2}$ be a fixed parameter word and $\pi_f \in \Pi([A]\binom{k+2}{k})$ the induced equivalence relation.

PROPOSITION 1. $\pi_i \leqslant \pi_{i+1}$ *for every* $i < k$.

Proof. Let $a \approx b$ (mod π_i), i.e.,

$$\begin{aligned} &(\lambda_0, ..., \lambda_{i-1}, a, \lambda_i, ..., \lambda_{k-1}) \\ &\quad \approx (\lambda_0, ..., \lambda_{i-1}, b, \lambda_i, ..., \lambda_{k-1}) \qquad (\text{mod } \sigma). \end{aligned}$$

First consider $(\lambda_0, ..., \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \lambda_i, \lambda_{i+2}, ..., \lambda_k) \in [A]\binom{k+2}{k+1}$. By (1) it follows in particular that

$$\begin{aligned} &(\lambda_0, ..., \lambda_{i-1}, a, \lambda_i, a, \lambda_{i+1}, ..., \lambda_{k-1}) \\ &\quad \approx (\lambda_0, ..., \lambda_{i-1}, b, \lambda_i, b, \lambda_{i+1}, ..., \lambda_{k-1}) \qquad (\text{mod } \pi_f). \end{aligned} \tag{3}$$

Next consider $(\lambda_0, ..., \lambda_{i+1}, a, \lambda_{i+2}, ..., \lambda_k) \in [A]\binom{k+2}{k+1}$, again by (1) it follows:

$$\begin{aligned} &(\lambda_0, ..., \lambda_{i-1}, a, \lambda_i, a, \lambda_{i+1}, ..., \lambda_{k-1}) \\ &\quad \approx (\lambda_0, ..., \lambda_{i-1}, b, \lambda_i, a, \lambda_{i+1}, ..., \lambda_{k-1}) \qquad (\text{mod } \pi_f), \end{aligned} \tag{4}$$

and thus by transitivity it follows from (3) and (4) that

$$(\lambda_0,\dots,\lambda_{i-1},b,\lambda_i,b,\lambda_{i+1},\dots,\lambda_{k-1}) \approx (\lambda_0,\dots,\lambda_{i-1},b,\lambda_i,a,\lambda_{i+1},\dots,\lambda_{k-1}) \quad (\text{mod } \pi_f). \tag{5}$$

This shows that for $g=(\lambda_0,\dots,\lambda_{i-1},b,\lambda_i,\lambda_{i+1},\dots,\lambda_k)\in[A]\binom{k+2}{k+1}$

$$(\lambda_0,\dots,\lambda_{i-1},\lambda_i,a,\lambda_{i+1},\dots,\lambda_{k-1}) \approx (\lambda_0,\dots,\lambda_{i-1},\lambda_i,b,\lambda_{i+1},\dots,\lambda_{k-1}) \quad (\text{mod } \pi_{f\cdot g})$$

holds and again by (1) $\pi_{f\cdot g}=\sigma$ which implies that $a\approx b$ (mod π_{i+1}).

Proposition 2. *Assume that* $c\approx\lambda_i$ (mod π_i) *for some* $c\in A\cup\{\lambda_0,\dots,\lambda_{i-1}\}$. *Then* $\pi_{i+1}\leqslant\pi_i$.

Proof. From the assumption it follows that

$$(\lambda_0,\dots,\lambda_{i-1},c,\lambda_i,\dots,\lambda_{k-1}) \approx (\lambda_0,\dots,\lambda_{i-1},\lambda_i,\lambda_i,\dots,\lambda_{k-1}) \quad (\text{mod } \sigma). \tag{6}$$

Let a, $b\in A\cup\{\lambda_0,\dots,\lambda_i\}$ with $a\approx b$ (mod π_{i+1}), i.e.,

$$(\lambda_0,\dots,\lambda_i,a,\lambda_{i+1},\dots,\lambda_{k-1}) \approx (\lambda_0,\dots,\lambda_i,b,\lambda_{i+1},\dots,\lambda_{k-1}) \quad (\text{mod } \sigma). \tag{7}$$

Consider first $(\lambda_0,\dots,\lambda_{i-1},\lambda_i,a,\lambda_{i+1},\dots,\lambda_k)\in[A]\binom{k+2}{k+1}$. By (6) and (1) it follows that

$$(\lambda_0,\dots,\lambda_{i-1},c,a,\lambda_i,\dots,\lambda_{k-1}) \approx (\lambda_0,\dots,\lambda_{i-1},\lambda_i,a,\lambda_i,\dots,\lambda_{k-1}) \quad (\text{mod } \pi_f), \tag{8}$$

where as before $f\in[A]\binom{m+1}{k+2}$ is some fixed parameter word.

Next consider $(\lambda_0,\dots,\lambda_{i-1},\lambda_i,\lambda_{i+1},\lambda_i,\dots,\lambda_k)\in[A]\binom{k+2}{k+1}$, by (7) and (1) it follows that

$$(\lambda_0,\dots,\lambda_{i-1},\lambda_i,a,\lambda_i,\lambda_{i+1},\dots,\lambda_{k-1}) \approx (\lambda_0,\dots,\lambda_{i-1},\lambda_i,b,\lambda_i,\lambda_{i+1},\dots,\lambda_{k-1}) \quad (\text{mod } \pi_f). \tag{9}$$

Finally consider $(\lambda_0,\dots,\lambda_{i-1},\lambda_i,b,\lambda_{i+1},\dots,\lambda_k)\in[A]\binom{k+2}{k+1}$, again by (6) and (1) it follows that

$$(\lambda_0,\dots,\lambda_{i-1},\lambda_i,b,\lambda_i,\dots,\lambda_{k-1}) \approx (\lambda_0,\dots,\lambda_{i-1},c,b,\lambda_i,\dots,\lambda_{k-1}) \quad (\text{mod } \pi_f). \tag{10}$$

Thus by transitivity it follows from (8–10) that

$$(\lambda_0,\ldots,\lambda_{i-1},c,a,\lambda_i,\ldots,\lambda_{k-1}) \approx (\lambda_0,\ldots,\lambda_{i-1},c,b,\lambda_i,\ldots,\lambda_{k-1}) \quad (\mathrm{mod}\ \pi_f). \tag{11}$$

This shows that for $g=(\lambda_0,\ldots,\lambda_{i-1},c,\lambda_i,\ldots,\lambda_k)\in[A]\binom{k+2}{k+1}$ holds

$$(\lambda_0,\ldots,\lambda_{i-1},a,\lambda_i,\ldots,\lambda_{k-1}) \approx (\lambda_0,\ldots,\lambda_{i-1},b,\lambda_i,\ldots,\lambda_{k-1}) \quad (\mathrm{mod}\ \pi_{f\cdot g}),$$

and since by (1) $\pi_{f\cdot g}=\sigma$ it follows that $a\approx b$ (mod π_i) as desired. ∎

COROLLARY. *The sequence* $\boldsymbol{\pi}=(\pi_0,\ldots,\pi_k)$ *is k-canonical.*

Next consider for $i<k$ the operators T_i which act on parameter words $h\in[A]\binom{m+1}{k}$ in such a way that $T_ih(l)=\lambda_i$ if $l=\omega_{\boldsymbol{\pi}}(h,i)$ and $T_ih(l)=h(l)$ in all other cases. In general T_ih need not be an element of $[A]\binom{m+1}{k}$, because possibly the parameters of T_ih are improperly ordered. However, in certain cases $T_ih\in[A]\binom{m+1}{k}$ and, moreover, then also its equivalence class modulo π does not change.

PROPOSITION 3. *Let* $h\in[A]\binom{m+1}{k}$ *and let* $i<k$. *Then* $T_iT_{i-1}\cdots T_0h\in[A]\binom{m+1}{k}$ *and additionally* $T_iT_{i-1}\cdots T_0h\approx T_{i-1}\cdots T_0h(\mathrm{mod}\ \pi)$. *In particular, for* $i=0$ *this means* $T_0h\approx h$ (mod π).

Proof. Proceed by induction on i. First observe that $T_{i-1}\cdots T_0h(\omega_{\boldsymbol{\pi}}(h,j))=\lambda_j$ for all $j<i$, so that in fact $T_iT_{i-1}\cdots T_0h\in[A]\binom{m+1}{k}$. We may assume that $T_{i-1}\cdots T_0h(\omega_{\boldsymbol{\pi}}(h,i))\neq\lambda_i$, otherwise there is nothing to show. By definition $T_{i-1}\cdots T_0h(\omega_{\boldsymbol{\pi}}(h,i))\approx\lambda_i$ (mod π_i).

Consider $g\in[A]\binom{m+1}{k+1}$ which is defined by

$$g(l)=\begin{cases}\lambda_i & \text{if } l=\omega_{\boldsymbol{\pi}}(h,i),\\ T_{i-1}\cdots T_0h(l) & \text{if } T_{i-1}\cdots T_0h(l)\in A\cup\{\lambda_0,\ldots,\lambda_{i-1}\},\quad l\neq\omega_{\boldsymbol{\pi}}(h,i),\\ \lambda_{j+1} & \text{if } T_{i-1}\cdots T_0h(l)=\lambda_j \quad \text{for some } j\geqslant i,\end{cases}$$

in other words, a new parameter λ_i is introduced at the $\omega_{\boldsymbol{\pi}}(h,i)$th position of $T_{i-1}\cdots T_0h$ and the remaining parameters are appropriately renumbered.

Obviously

$$g\cdot(\lambda_0,\ldots,\lambda_{i-1},T_{i-1}\cdots T_0h(\omega_{\boldsymbol{\pi}}(h,i)),\lambda_i,\ldots,\lambda_{k-1}) = T_{i-1}\cdots T_0h,$$

and

$$g\cdot(\lambda_0,\ldots,\lambda_{i-1},\lambda_i,\lambda_i,\ldots,\lambda_{k-1})=T_iT_{i-1}\cdots T_0h.$$

But since

$$(\lambda_0,\ldots,\lambda_{i-1},T_{i-1}\cdots T_0 h(\omega_\pi(h,i)),\lambda_i,\ldots,\lambda_{k-1})$$
$$\approx(\lambda_0,\ldots,\lambda_{i-1},\lambda_i,\lambda_i,\ldots,\lambda_{k-1}) \qquad (\mathrm{mod}\ \sigma)$$

and $\pi_g=\sigma$ it follows that

$$T_i T_{i-1},\ldots,T_0 h\approx T_{i-1}\cdots T_0 h \qquad (\mathrm{mod}\ \pi). \quad ∎$$

PROPOSITION 4. $\boldsymbol{\pi}^{m+1}\leqslant\pi$.

Proof. Let $g,\ h\in[A]\binom{m+1}{k}$ with $g\approx h\,(\mathrm{mod}\ \boldsymbol{\pi}^{m+1})$. By definition $\omega_\pi(g,i)=\omega_\pi(h,i)$ for every $i<k$. By Proposition 3 we may also assume that

$$g(\omega_\pi(g,i))=h(\omega_\pi(h,i))=\lambda_i,$$

for every $i<k$.

Then proceed by induction on $|\{l<m+1\mid g(l)\neq h(l)\}|$ in order to show that $g\approx h\,(\mathrm{mod}\ \pi)$. Pick $l<m+1$ with $g(l)\neq h(l)$, say $\omega_\pi(g,i)<l<\omega_\pi(g,i+1)$, where for convenience $\omega_\pi(g,k)=m+1$.

Consider the parameter word $f\in[A]\binom{m+1}{k+1}$ which is defined by

$$\begin{aligned} f(\hat{l})&=\lambda_{i+1} && \text{if}\quad \hat{l}=l,\\ &=g(\hat{l}) && \text{if}\quad g(\hat{l})\in A\cup\{\lambda_0,\ldots,\lambda_{i-1}\},\qquad \hat{l}\neq l,\\ &=\lambda_{j+1} && \text{if}\quad g(l)=\lambda_j\quad\text{for some } j>i,\end{aligned}$$

i.e., the lth position of g is replaced by a new parameter and the remaining parameters are appropriately renumbered. Because $g(l)\approx h(l)$ $(\mathrm{mod}\ \pi_{i+1})$ it follows by (1) that

$$g=f\cdot(\lambda_0,\ldots,\lambda_{i-1},g(l),\lambda_i,\ldots,\lambda_{k-1})$$
$$\approx f\cdot(\lambda_0,\ldots,\lambda_{i-1},h(l),\lambda_i,\ldots,\lambda_{k-1})=g' \qquad (\mathrm{mod}\ \pi).$$

By induction it follows that $g'\approx h\,(\mathrm{mod}\ \pi)$ and thus by transitivity also $g\approx h\,(\mathrm{mod}\ \pi)$. ∎

PROPOSITION 5. *Let* $\boldsymbol{\pi}=(\pi_0,\ldots,\pi_k)$ *be a k-canonical sequence and let* $f\in[A]\binom{n}{m}$ *be an arbitrary parameter word. Then it follows that* $\boldsymbol{\pi}^m=(\boldsymbol{\pi}^n)_f$.

Proof. Let us first show that $\boldsymbol{\pi}^m\leqslant(\boldsymbol{\pi}^n)_f$. Let $g,\ h\in[A]\binom{m}{k}$ be with $g\approx h\,(\mathrm{mod}\ \boldsymbol{\pi}^m)$. Let $\hat{l}<n$, *say*

$$\omega_\pi(f\cdot g,i)<\hat{l}\leqslant\omega_\pi(f\cdot g,i+1), \tag{12}$$

We claim that $f\cdot g(\hat{l})\approx f\cdot h(\hat{l})$ $(\mathrm{mod}\ \pi_{i+1})$.

Case 1. Suppose $f(\hat{l}) \in A$. Then $f \cdot g(\hat{l}) = f(\hat{l}) = f \cdot h(\hat{l})$ and thus, in particular, $f \cdot g(\hat{l}) \approx f \cdot h(\hat{l}) \pmod{\pi_{i+1}}$.

Case 2. Suppose $f(\hat{l}) = \lambda_l$, say

$$\omega_{\pi}(g, j) < l \leqslant \omega_{\pi}(g, j+1).$$

Then $\min f^{-1}(\lambda_{\omega_{\pi}(g,j)}) < \min f^{-1}(\lambda_l)$ by condition (1) of Definition C.1. By (12) it follows that

$$\min f^{-1}(\lambda_l) \leqslant \hat{l} \leqslant \omega_{\pi}(f \cdot g, i+1) \leqslant \min f^{-1}(\lambda_{\omega_{\pi}(g,i+1)}).$$

Hence $j \leqslant i$ and, in particular, $\pi_{j+1} \leqslant \pi_{i+1}$. Thus the equivalence $g(l) \approx h(l) \pmod{\pi_{j+1}}$ implies

$$f \cdot g(\hat{l}) = g(l) \approx h(l) = f \cdot h(\hat{l}) \qquad \pmod{\pi_{i+1}}.$$

This shows $\boldsymbol{\pi}^m \leqslant (\boldsymbol{\pi}^n)_f$.

Next we show that $(\boldsymbol{\pi}^n)_f \leqslant \boldsymbol{\pi}^m$. Let g, $h \in [A]\binom{m}{k}$ be with $g \approx h$ $(\bmod (\boldsymbol{\pi}^n)_f)$, i.e., $f \cdot g \approx f \cdot h \pmod{\boldsymbol{\pi}^n}$. Let $l < m$, say

$$\omega_{\pi}(g, i) < l \leqslant \omega_{\pi}(g, i+1). \tag{13}$$

We claim that $g(l) \approx h(l) \pmod{\pi_{i+1}}$.

Case 1. Suppose $\omega_{\pi}(f \cdot g, i+1) \geqslant \min f^{-1}(\lambda_l)$. In particular then

$$\begin{aligned} g(l) &= f \cdot g(\min f^{-1}(\lambda_l)) \approx f \cdot h(\min f^{-1}(\lambda_l)) \\ &= h(l) \qquad \pmod{\pi_{i+1}}. \end{aligned}$$

Case 2. Suppose $\omega_{\pi}(f \cdot g, i+1) < \min f^{-1}(\lambda_l)$, say

$$\omega_{\pi}(f \cdot g, i+j) < \min f^{-1}(\lambda_l) \leqslant \omega_{\pi}(f \cdot g, i+j+1),$$

where $j \geqslant 1$. By Definition C.1 and (13) it follows that $f \cdot g(\hat{l}) \in A \cup \{\lambda_0, ..., \lambda_i\}$ for each $\hat{l} < \min f^{-1}(\lambda_l)$. Thus there exist elements $a_1, ..., a_j \in A \cup \{\lambda_0, ..., \lambda_i\}$ such that $\lambda_{i+1} \approx a_1 \pmod{\pi_{i+1}}, ..., \lambda_{i+j} \approx a_j$ $\pmod{\pi_{i+j}}$ and by condition (2) of Definition C.4 this implies that $\pi_{i+j+1} \leqslant \pi_{i+j} \leqslant \cdots \leqslant \pi_{i+1}$. Since $g(l), h(l) \in A \cup \{\lambda_0, ..., \lambda_{i+1}\}$ by (13), the equivalence

$$\begin{aligned} g(l) &= f \cdot g(\min f^{-1}(\lambda_l)) \approx f \cdot h(\min f^{-1}(\lambda_l)) \\ &= h(l) \qquad \pmod{\pi_{i+j+1}} \end{aligned}$$

implies that

$$g(l) \approx h(l) \pmod{\pi_{i+1}}. \qquad \blacksquare$$

COROLLARY. $\boldsymbol{\pi}^m \leqslant \tau$.

Proof. Proposition 4 and Proposition 5.

Recall that we did not really use (2), i.e., the fact that for each $g \in [A]\binom{m+1}{n}$ it follows that $\pi_g = \tau$. However, (2) is needed in order to establish the reverse of the above corollary:

PROPOSITION 6. $\tau \leqslant \boldsymbol{\pi}^m$.

Proof. Assume that there exist $g, h \in [A]\binom{m}{k}$ with $g \approx h \pmod{\tau}$ but $g \not\approx h \pmod{\boldsymbol{\pi}^m}$. By Proposition 3 we may assume that $g(\omega_\pi(g, i)) = \lambda_i$ and $h(\omega_\pi(g, i)) = \lambda_i$ for $0 \leqslant i \leqslant k$. In particular then there exist $i \in \{-1, 0,..., k-1)\}$ and $l < m$ such that

$$\omega_\pi(h, i) < l \leqslant \omega_\pi(h, i+1), \qquad g(l) \not\approx h(l) \pmod{\pi_{i+1}}, \tag{14}$$

and say

$$h(l) \in A \cup \{\lambda_0,..., \lambda_i\}.$$

Then consider the parameter words

$$f_1 = (\lambda_0,..., \lambda_{l-1}, \lambda_l, \lambda_l,..., \lambda_{m-1}) \in [A]\binom{m+1}{m},$$

and

$$f_2 = (\lambda_0,..., \lambda_{l-1}, g(l), \lambda_l,..., \lambda_{m-1}) \in [A]\binom{m+1}{m} \qquad \text{if} \quad g(l) \in A,$$

resp.

$$f_2 = (\lambda_0,..., \lambda_{l-1}, \lambda_{\min g^{-1}(\lambda_j)}, \lambda_l,..., \lambda_{m-1}) \in [A]\binom{m+1}{m} \qquad \text{if} \quad g(l) = \lambda_j.$$

Since by (2) $\pi_{f_1} = \tau = \pi_{f_2}$ and by the assumption $g \approx h \pmod{\tau}$ it follows that

$$\begin{aligned}(h(0),..., h(l-1), h(l), h(l),..., h(m-1))&\\ \approx (g(0),..., g(l-1), g(l), g(l),..., g(m-1)) &\pmod{\pi},\end{aligned}$$

and

$$\begin{aligned}(g(0),..., g(l-1), g(l), g(l),..., g(m-1))&\\ \approx (h(0),..., h(l-1), g(l), h(l),..., h(m-1)) &\pmod{\pi}.\end{aligned}$$

By transitivity follows

$$\begin{aligned}(h(0),..., h(l-1), h(l), h(l),..., h(m-1))&\\ \approx (h(0),..., h(l-1), g(l), h(l),..., h(m-1)) &\pmod{\pi}.\end{aligned} \tag{15}$$

Recall that by (14) the parameters $\lambda_0,\dots,\lambda_i$ occur under $h(0),\dots,h(l-1)$, where possibly $i=-1$. Thus consider the parameter word $f\in[A]\binom{m+1}{k+1}$ with

$$\begin{aligned} f(\hat{l}) &= h(\hat{l}) \quad &&\text{if} \quad h(\hat{l})\in A\cup\{\lambda_0,\dots,\lambda_i\}, \quad \hat{l}\neq l,\\ &=\lambda_{i+1} &&\text{if} \quad \hat{l}=l,\\ &=\lambda_{j+1} &&\text{if} \quad \hat{l}\neq l \quad \text{and} \quad h(\hat{l})=\lambda_j \quad \text{for some } j>i. \end{aligned}$$

Then by (15)

$$\begin{aligned} &f\cdot(\lambda_0,\dots,\lambda_i,h(l),\lambda_{i+1},\dots,\lambda_{k-1})\\ &\qquad\approx f\cdot(\lambda_0,\dots,\lambda_i,g(l),\lambda_{i+1},\dots,\lambda_{k-1}) \qquad (\text{mod } \pi). \end{aligned}$$

Since by (1) $\pi_f=\sigma$ it follows that

$$\begin{aligned} &(\lambda_0,\dots,\lambda_i,h(l),\lambda_{i+1},\dots,\lambda_{k-1})\\ &\qquad\approx(\lambda_0,\dots,\lambda_i,g(l),\lambda_{i+1},\dots,\lambda_{k-1}) \qquad (\text{mod } \sigma), \end{aligned}$$

showing that $g(l)\approx h(l)$ (mod π_{i+1}), which contradicts (14). ∎

COROLLARY. *We have $\boldsymbol{\pi}^m=\tau$, which completes the proof of the lemma.* ∎

It remains to be shown that $\{\boldsymbol{\pi}^m\mid\boldsymbol{\pi}\ k\text{-canonical}\}$ is a set of minimal cardinality satisfying (can). By Proposition 5 each $\boldsymbol{\pi}^m$ is necessary, thus the minimality condition is trivially satisfied. Thus Theorem C.7 is proved. ∎

D. REMARKS AND COROLLARIES

Let us first show how the canonical version of the partition theorem for k-parameter sets implies the Erdös–Rado canonization theorem, viz.,

Proof of Erdös–Rado Canonization Theorem

Consider the alphabet $A=\{0\}$. For positive integers k and m let n be such that $n\to^{[A]}_{\text{can}}(m)^k$. Such an n exists according to Theorem C.7. We claim that n has the desired properties. Let $\pi\in\Pi([n]^k)$ be any equivalence relation. Consider the equivalence relation $\sigma\in\Pi([A]\binom{n}{k})$ which is defined by $f\approx g$ (mod σ) iff

$$\begin{aligned} &\{\min f^{-1}(\lambda_0),\dots,\min f^{-1}(\lambda_{k-1})\}\\ &\qquad\approx\{\min g^{-1}(\lambda_0),\dots,\min g^{-1}(\lambda_{k-1})\} \ (\text{mod } \pi). \end{aligned}$$

Then let $f\in[A]\binom{n}{m}$ be such that σ_f is necessary, say $\sigma_f=(\pi_0,\dots,\pi_k)^m$, where $(\pi_0,..., \pi_k)$ is a k-canonical sequence. Define $X = \{\min f^{-1}(\lambda_i) \mid i = 0,..., m-1\}$ and $\mathscr{K} = \{i < k \mid 0 \not\approx \lambda_i \pmod{\pi_i}\}$. We claim that any two k-element subsets $\{a_0,..., a_{k-1}\}$ and $\{b_0,..., b_{k-1}\}$ of X, where the a_i's and b_i's are arranged in increasing order, are equivalent modulo π iff $a_i = b_i$ for all $i \in \mathscr{K}$.

Let $\{a_0,..., a_{k-1}\}$ and $\{b_0,..., b_{k-1}\}$ be two k-element subsets of X—both arranged in increasing order. Consider $g \in [A]\binom{m}{k}$, which is defined as

$$\begin{aligned} g(i) &= 0 && \text{if } i < l, && \text{where } f(a_0) = \lambda_l, \\ &= \lambda_j && \text{if } l \leqslant i < l^*, && \text{where } f(a_j) = \lambda_l, \quad f(a_{j+1}) = \lambda_{l^*}, \end{aligned}$$

and for technical convenience $f(a_k)$ shall be m.

Let $h \in [A]\binom{m}{k}$ be defined analogously for the set $\{b_0,..., b_{k-1}\}$. Then by construction it follows that $\min(f \cdot g)^{-1}(\lambda_i) = a_i$ and $\min(f \cdot h)^{-1}(\lambda_i) = b_i$ for $i < k$. Hence $\{a_0,..., a_{k-1}\} \approx \{b_0,..., b_{k-1}\} \pmod{\pi}$ iff $g \approx h$ $(\mathrm{mod}(\pi_0,..., \pi_k)^m)$.

Observe that the definition of σ implies that $0 \approx \lambda_i \pmod{\pi_j}$ for every $i < j < k$. Hence the only information that one may get out of $(\pi_0,..., \pi_k)$ is the set $\mathscr{K}$. In particular then $g \approx h\,(\mathrm{mod}(\pi_0,..., \pi_k)^m)$ iff $\min\, g^{-1}(\lambda_i) = \min h^{-1}(\lambda_i)$ for all $i \in \mathscr{K}$. ∎

The next corollary is a canonical version of the finite union theorem (resp., the finite sum theorem). Sometimes this theorem is known as the "Rado–Folkman–Sanders theorem." The finite sum theorem is a special case of Rado's [8] much more general theorem on partition regular systems of equations.

THEOREM D.1. Finite sum theorem. *Let δ, m be positive integers. Then there exists a positive integer n such that for every coloring $\Delta: \{1,..., n\} \to \delta$ there exist m positive integers $x_0,..., x_{m-1} \in \{1,..., n\}$ with all finite sums $\sum \{x_i \mid i \in I\}$, where $I \subseteq \{0,..., m-1\}$ is nonempty, in the same color.*

Finite union theorem. *Let δ, m be positive integers. Then there exists a positive integer n such that for every coloring $\Delta: \mathscr{P}(n) \backslash \{\varnothing\} \to \delta$ of the nonempty subsets of $\{0,..., n-1\}$ there exist m mutually disjoint and nonempty sets $X_1,..., X_{m-1} \in \mathscr{P}(n) \backslash \{\varnothing\}$ with all finite unions $\bigcup \{X_i \mid i \in I\}$, where $I \subseteq \{0,..., m-1\}$ is nonempty, in the same color.*

The finite union theorem is an immediate corollary from the finite sum theorem and Ramsey's theorem. On the other hand the finite sum theorem may be immediately deduced from the finite union theorem using binary expansion, i.e., by associating to each positive integer $k < 2^n$ a nonempty subset of $\{0,..., n-1\}$.

Moreover, the finite union theorem turned out to be a special case of the partition theorem for k-parameter sets [4], viz. observe that for $A = \{0\}$ the nonempty subsets of $\{0,\dots,n-1\}$ are in a one-to-one correspondence with $[A]\binom{n}{1}$ by $X \Leftrightarrow f$ as follows: $f(i)=\lambda_0$ iff $i \in X$ and $f(i)=0$ iff $i \notin X$. Then any positive integer n with $n \to^{[A]} (m)^1_\delta$ satisfies the requirements of the theorem.

Applying the canonical version of the partition theorem for k-parameter sets yields a canonical version of the finite union theorem, viz.

THEOREM D.2. *Let m be a positive integer. Then there exists a positive integer n with the following property: for every equivalence relation $\pi \in \Pi(\mathscr{P}(n)\backslash\{\varnothing\})$ on the set of nonempty subsets of $\{0,\dots,n-1\}$ there exist m mutually disjoint and nonempty sets $X_0,\dots,X_{m-1} \in \mathscr{P}(n)\backslash\{\varnothing\}$ such that exactly one of the following three cases is valid for each two nonempty sets $I, J \in \mathscr{P}(m)\backslash\{\varnothing\}$:*

(i) $\bigcup\{X_i \mid i \in I\} \approx \bigcup\{X_i \mid i \in J\} \pmod{\pi}$ *without any restriction on I or J.*

(ii) $\bigcup\{X_i \mid i \in I\} \approx \bigcup\{X_i \mid i \in J\} \pmod{\pi}$ *iff* $\min I = \min J$.

(iii) $\bigcup\{X_i \mid i \in I\} \approx \bigcup\{X_i \mid i \in J\} \pmod{\pi}$ *iff* $I = J$.

One immediately observes that the equivalence relations given by (i), (ii), and (iii) are necessary equivalence relations. Thus the equivalence relations given by (i), (ii), and (iii) form a canonical set of equivalence relations.

This improves a result of Taylor [10] who showed that one can always restrict to five different kinds of equivalence relations. However, in the infinite case (i.e., a canonical version of Hindman's theorem [7]) there exist 5 necessary equivalence relations and these turned out to form a canonical set of equivalence relations, viz.

THEOREM D.3 [10]. *For every equivalence relation $\pi \in \Pi(\mathscr{P}_{\text{fin}}(\omega)\backslash\{\varnothing\}$ on the finite and nonempty subsets of the nonnegative integers there exist infinitely many mutually disjoint and nonempty sets $X_0, X_1,\dots, \in \mathscr{P}_{\text{fin}}(\omega)$ such that exactly one of the following five cases is valid for each two nonempty sets $I, J \in \mathscr{P}_{\text{fin}}(\omega)\backslash\{\varnothing\}$:*

(α) $\bigcup\{X_i \mid i \in I\} \approx \bigcup\{X_i \in J\} \pmod{\pi}$ *without any restriction on I or J.*

(β) $\bigcup\{X_i \mid i \in I\} \approx \bigcup\{X_i \mid i \in J\} \pmod{\pi}$ *iff* $\min I = \min J$.

(γ) $\bigcup\{X_i \mid i \in I\} \approx \bigcup\{X_i \mid i \in J\} \pmod{\pi}$ *iff* $\max I = \max J$.

(δ) $\bigcup\{X_i \mid i \in I\} \approx \bigcup\{X_i \mid i \in J\} \pmod{\pi}$ *iff* $\min I = \min J$ and $\max I = \max J$.

(ε) $\bigcup\{X_i \mid i \in I\} \approx \bigcup\{X_i \mid i \in J\}$ *iff* $I = J$.

That is, it turns out that equivalence relations (γ) and (δ) may be eliminated in the finite case.

Using binary expansions of positive integers an analogous canonical version may be established for the finite sum theorem. It could be worthwhile to note that this provides an example where canonical sets of equivalence relations are not uniquely determined. In particular, the set of necessary equivalence relations is not a canonical set, i.e., it does not satisfy condition (can). For example, let us look in detail at the special case $m = 2$ of the finite union theorem. This is an old result, sometimes known as "Schur's theorem."

THEOREM D.4 [9]. *For every positive integer δ there exists a positive integer n such that for every coloring $\Delta: \{1,..., n\} \to \delta$ there exist three numbers x, y, $z \in \{1,..., n\}$ with $x + y = z$ and $\Delta(x) = \Delta(y) = \Delta(z)$.*

For convenience we shall assume that $x \leqslant y \leqslant z$. Figure 1 shows the lattice of equivalence relations on $\{x, y, z\}$.

The canonical version of Schur's theorem says:

THEOREM D.5. *Consider the sets $\{\pi_0, \pi_1, \pi_4\}$, $\{\pi_0, \pi_2, \pi_4\}$, $\{\pi_0, \pi_3, \pi_4\}$. Then each of these sets of equivalence relations forms a canonical set of equivalence for Schur's theorem.*

Analogously canonical versions of the finite sum theorem may be established for $m > 2$.

The next application is a canonical version of the partition theorem for finite Boolean algebras [4]. Let us denote by $\mathscr{P}(n)$ the Boolean algebra of subsets of an n-element set. Let $BA\binom{n}{k}$ be the set of $\mathscr{P}(k)$-subalgebras (i.e., $\mathscr{P}(k)$-sublattices) of $\mathscr{P}(n)$.

THEOREM D.6 [4]. *For nonnegative integers δ, m, k there exists a positive integer n such that for every coloring*

$$\Delta : BA \binom{n}{k} \to \delta$$

of the $\mathscr{P}(k)$-subalgebras of $\mathscr{P}(n)$ with δ colors there exists a $\mathscr{P}(m)$-

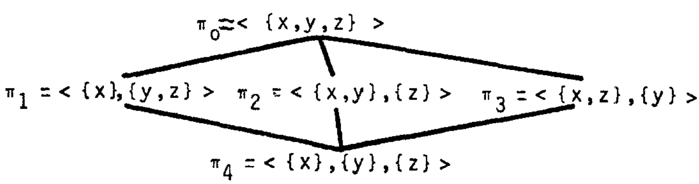

FIGURE 1

subalgebra $f \in BA(\binom{n}{m})$ with all its $\mathscr{P}(k)$-subalgebras in one color, i.e., $\Delta_f: BA(\binom{m}{k}) \to \delta$ with $\Delta_f(g) = \Delta(f \cdot g)$ is constant.

One easily observes that this theorem is just a reformulation of the partition theorem for k-parameter words for the two-element alphabet $A = \{0, 1\}$. Consider $\mathscr{P}(n)$; its elements may be viewed as being 0, 1-sequences of length n(i.e., characteristic functions). Thus the $\mathscr{P}(0)$-sublattices are in a one-to-one correspondence onto $[A](\binom{n}{0})$. Analogously the $\mathscr{P}(k)$-sublattices of $\mathscr{P}(n)$ are in a one-to-one correspondence onto $[A](\binom{n}{k})$. A $\mathscr{P}(k)$-sublattice of $\mathscr{P}(n)$ is given by a set $X_0 \subseteq \{0,..., n-1\}$ and k mutually disjoint nonempty sets $X_1,..., X_k \subseteq \{0,..., n-1\}$ with $X_i \cap X_0 = \varnothing$, $i = 1,..., k$. X_0 is the 0-element of the $\mathscr{P}(k)$-sublattice and $X_0 \cup X_1,..., X_0 \cup X_k$ are the atoms. The sets $X_0,..., X_k$ may be encoded using a single $f \in [A](\binom{n}{k})$, viz.,

$$
\begin{aligned}
f(i) &= 1 && \text{iff} \quad i \in X_0, \\
&= \lambda_j && \text{iff} \quad i \in X_{j+1}, \\
&= 0 && \text{else.}
\end{aligned}
$$

For example, with respect to colorings of $\mathscr{P}(0)$-sublattices, i.e., colorings of points, there exist only two canonical equivalence relations, viz., the constant equivalence relation, where each two elements are equivalent, and the one-to-one equivalence relation (identity), where each element is only equivalent to itself.

However, with respect to colorings of $\mathscr{P}(1)$-sublattices, i.e., colorings of two-element chains, there already exist 10 canonical equivalence relations. Let $\{A, A \cup B\}$, respectively $\{C, C \cup D\}$, where $A \cap B = C \cap D = \varnothing$ and B and D are both nonempty, be two arbitrary 2-element chains, then we have the following possibilities for canonical equivalence relations: $\{A, A \cup B\}$ is equivalent to $\{C, C \cup D\}$ iff

(1) $A = C$ and $B = D$ (the one-to-one equivalence relation which corresponds to the 1-canonical sequence $(\{\{0\}, \{1\}, \{\lambda_0\}\}, \{\{0\}, \{1\}, \{\lambda_0\}\})$),

(2) $\{l \in A \mid l < \min B\} = \{l \in C \mid l < \min D\}$ and $B = D$ (this corresponds to the 1-canonical sequence $(\{\{0\}, \{1\}, \{\lambda_0\}\}, \{\{0, 1\}, \{\lambda_0\}\})$),

(3) $\min B = \min D$, $\{l \in A \mid l < \min B\} = \{l \in C \mid l < \min D\}$, and $A \cup B = C \cup D$ (corresponding to $(\{\{0\}, \{1\}, \{\lambda_0\}\}, \{\{0\}, \{1, \lambda_0\}\})$),

(4) $A = C$ and $\min B = \min D$ (corresponding to $(\{\{0\}, \{1\}, \{\lambda_0\}\}, \{\{1\}, \{0, \lambda_0\}\})$),

(5) $\min B = \min D$ and $\{l \in A \mid l < \min B\} = \{l \in C \mid l < \min D\}$ (corresponding to $(\{\{0\}, \{1\}, \{\lambda_0\}\}, \{\{0, 1, \lambda_0\}\})$),

(6) $B = D$ (corresponding to $(\{\{0, 1\}, \{\lambda_0\}\}, \{\{0, 1\}, \{\lambda_0\}\})$),

(7) $\min B = \min D$ (corresponding to $(\{\{0, 1\}, \{\lambda_0\}\}, \{\{0, 1, \lambda_0\}\})$),

(8) $A \cup B = C \cup D$ (corresponding to $(\{\{0\}, \{1, \lambda_0\}\}, \{\{0\}, \{1, \lambda_0\}\})$),

(9) $A = C$ (corresponding to $(\{\{0, \lambda_0\}, \{1\}\}, \{\{0, \lambda_0\}, \{1\}\})$),

(10) no restriction, the constant equivalence relation (which corresponds to $(\{\{0, 1, \lambda_0\}\}, \{\{0, 1, \lambda_0\}\})$).

In general the numbers of canonical equivalence relations with respect to colorings of $\mathscr{P}(k)$-subalgebras grow rapidly, i.e., much faster than the Bell numbers B_{k+2} of numbers of partitions of a $(k+2)$-element set.

References

1. W. Deuber, H. J. Prömel, and B. Voigt, Canonical equivalence relations for chains in trees, to appear.
2. W. Deuber and B. Voigt, Partitionseigenschaften endlicher affiner und projektiver Räume, *Europ. J. Combinatorics* **3** (1982), 329–340.
3. P. Erdös and R. Rado, A combinatorial theorem, *J. London Math. Soc.* **25** (1950), 249–255.
4. R. L. Graham and B. L. Rothshild, Ramsey's theorem for n-parameter sets, *Trans. Amer. Math. Soc.* **159** (1971), 257–291.
5. R. L. Graham, B. L. Rothshild, and J. H. Spencer, "Ramsey Theory," Wiley, New York, 1980.
6. A. W. Hales and R. I. Jewett, Regularity and positional games, *Trans. Amer. Math. Soc.* **106** (1963), 222–229.
7. N. Hindman, Finite sums from sequences within cells of a partition of N, *J. Combin. Theory Ser. A* **17** (1974),
8. R. Rado, Studien zur Kombinatorik, *Math. Z.* **36** (1933), 242–280.
9. I. Schur, Über die Kongruenz $x^m + y^m \equiv z^m \pmod{p}$, *Jahresber. Deutsch. Math.-Verein.* **25** (1916), 114.
10. A. D. Taylor, A canonical partition relation for finite subsets of ω, *J. Combin. Theory Ser. A* **21** (1976), 137–146.