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A lower bound for off-diagonal van der Waerden numbers ☆

Yusheng Li [a,*], Jinlong Shu [b]

[a] *Department of Mathematics, Tongji University, Shanghai 200092, China*
[b] *Department of Mathematics, East China Normal University, Shanghai 200062, China*

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ABSTRACT

Let $w(m,n)$ be the van der Waerden number in two colors. It is shown that $w(m,n)$ is at least $c(\frac{n}{\log n})^{m-1}$ for fixed m, where $c = c(m)$ is a positive constant.

1. Introduction

Let m and n be positive integers. Define the van der Waerden number $w(m,n)$ to be the smallest integer N such that if $[N] = \{1, 2, \dots, N\}$ are colored by red and blue, there is a red m-AP (arithmetic progression of m distinct terms) or a blue n-AP.

It is easy to see that $w(1,n) = n$, and for $n \geqslant 2$, $w(2,n) = 2n$ if n is odd and $2n-1$ otherwise. However, it is hard to prove the existence of $w(m,n)$ for fixed $m \geqslant 3$ or $m = n$; see van der Waerden [12,13] and Graham, Rothschild and Spencer [8]. As usual, an existence proof often gives an upper bound. A recent result of Gowers [6] is

$$w(n,n) \leqslant 2^{2^{2^{2^{2^{n+9}}}}}.$$

Although this upper bound is rather large, it greatly improved much larger bounds resulting from van der Waerden's proof (see [8] for a description) and the work of Shelah [9]. Szabó [11] proved

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* Corresponding author.
E-mail addresses: li_yusheng@tongji.edu.cn (Y. Li), jlshu@math.ecnu.edu.cn (J. Shu).

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that $w(n,n) \geqslant (1-o(1))\frac{2^{n-1}}{en}$, and Berlekamp [2] proved $w(p+1,p+1) \geqslant p2^p$ for prime p. The two lower bounds are very close, but the proofs are completely different. This may suggest that the lower bounds are much closer to the truth than the known upper bounds.

For the van der Waerden function $w(3,n)$, it is known that

$$n^{2-1/\log\log n} \leqslant w(3,n) \leqslant n^{cn^2}.$$

The upper bound is due to Bourgain [3], and the lower bound is a special case of that of Brown, Landman and Robertson [4] who proved that

$$w(m,n) \geqslant \frac{n^{m-1}}{n^{1/\log\log n}} \tag{1}$$

for fixed $m \geqslant 3$ and large n. It is suggested that $w(3,n)$ might be bounded from above by some polynomial on n (perhaps even a quadratic!); see [4,7]. It would be very interesting to see whether or not the van der Waerden function $w(m,n)$ behaves similarly to the graph Ramsey function $r(m,n)$. We will prove the following result.

Theorem 1. *Let $m \geqslant 3$ be fixed. Then*

$$w(m,n) \geqslant c\left(\frac{n}{\log n}\right)^{m-1}$$

for all large n, where $c = c(m) > 0$ is a constant.

2. The proof

Brown, Landman and Robertson [4] proved (1) by the Lovász local lemma [5], in which they used the symmetric form of the lemma by elegantly balancing the probabilities of monochromatic m-AP and n-AP. For random events $A_1, A_2, \ldots, A_n$, define a graph D, called *dependency graph*, on vertex set $\{1,2,\ldots,n\}$, in which every event A_i is mutually independent of all A_j with $\{i,j\} \notin E(D)$, i.e., each A_i is independent of any Boolean combination of the $\{A_j\colon \{i,j\} \in E(D)\}$. The following is the general form of the local lemma; see [1,10].

Theorem 2. *Let $A_1, A_2, \ldots, A_n$ be random events. Suppose that there exist real numbers $x_1, x_2, \ldots, x_n$ such that $0 < x_i < 1$ and*

$$\Pr(A_i) \leqslant x_i \prod_{\{i,j\}\in E(D)} (1-x_j).$$

Then $\Pr(\bigcap_{i=1}^n \overline{A}_i) > 0$.

The following form of the local lemma given by Spencer is slightly more convenient for some applications, in which $y_i = x_i / \Pr(A_i)$.

Corollary 1. *Let $A_1, A_2, \ldots, A_n$ be events. If there exist positive numbers $y_1, y_2, \ldots, y_n$ such that $y_i \Pr(A_i) < 1$ and*

$$\log y_i \geqslant -\sum_{ij\in E(D)} \log\bigl(1 - y_j \Pr(A_j)\bigr),$$

then $\Pr(\bigcap \overline{A_i}) > 0$.

Proof of Theorem 1. Color each integer of $[N]$ by red and blue randomly and independently, in which each integer is colored red with probability p and blue with probability $q = 1 - p$, where the probability p will be determined.

For each S of m-AP of $[N]$, let A_S signify the event that S is monochromatically red. For each T of n-AP of $[N]$, let B_T signify the event that T is monochromatically blue. Then

$$\Pr(A_S) = p^m, \qquad \Pr(B_T) = q^n.$$

We thus separate the involved events into two types, type A and type B. To use Corollary 1, we shall find $y_1, y_2, \ldots$ that consist of only two distinct numbers: one type y_i for A-events and the other type y_i for B-events. Let $x \in [N]$ be fixed. The number of m-APs in $[N]$ that contain x is at most $mN/(m-1)$, since there are m positions that x may occupy and the common difference of an m-AP cannot exceed $N/(m-1)$. Similarly, the number of n-APs in $[N]$ that contain x is at most $nN/(n-1)$. Obviously, two events are dependent if and only if they have some integer in common. Hence, each A event is mutually independent of all but at most $m^2N/(m-1)$ other A events and mutually independent of all but at most $mnN/(n-1)$ of the B events; each B event is mutually independent of all but at most $mnN/(m-1)$ of the A events and mutually independent of all but at most $n^2N/(n-1)$ other B events.

We will prove that the hypotheses of Corollary 1 are satisfied. To do this, we will show the existence of positive a and b such that $ap^m < 1$ and $bq^n < 1$, and the following inequalities hold:

$$\log a \geqslant -\frac{m^2N}{m-1}\log\bigl(1 - ap^m\bigr) - \frac{mnN}{n-1}\log\bigl(1 - bq^n\bigr), \tag{2}$$

$$\log b \geqslant -\frac{mnN}{m-1}\log\bigl(1 - ap^m\bigr) - \frac{n^2N}{n-1}\log\bigl(1 - bq^n\bigr). \tag{3}$$

We shall take $y_i = a$ for each A event, and $y_i = b$ for each B event. By Corollary 1, if such a and b are available, then there exists a red/blue coloring of $[N]$ in which there is neither red m-AP nor blue n-AP; in other words $w(m,n) > N$. To this end, let $c = c(m) > 0$ be an arbitrary constant with

$$c < \left(\frac{m-1}{m^2}\right)^m. \tag{4}$$

We shall choose N, p, b and a in order by

$$N = c\left(\frac{n}{\log n}\right)^{m-1},$$

$$p = \frac{m\log(nN)}{(m-1)n},$$

$$b = \frac{1}{nNq^n}, \qquad a = b^{m/n}.$$

It is easy to verify that $(1-p)^n \sim e^{-np}$ as $n \to \infty$ and

$$b = \frac{1}{nN(1-p)^n} \sim \frac{e^{np}}{nN} = \frac{(nN)^{m/(m-1)}}{nN} = (nN)^{1/(m-1)}.$$

Thus from the fact that $\log(nN) \sim m\log n$ we have

$$\log b \sim \frac{\log(nN)}{m-1} \sim \frac{m}{m-1}\log n.$$

As $bq^n \to 0$ and $\log(1+x) \sim x$ for $x \to 0$, the second term in the right-hand side of (3) is asymptotically equal to $nNbq^n = 1$. Note that $a \to 1$, $ap^m \to 0$ and $p \sim m^2 \log n/((m-1)n)$, and the first term in the right-hand side of (3) is asymptotically

$$\frac{m}{m-1} nNap^m \sim \frac{cm}{m-1} \frac{n^m}{(\log n)^{m-1}} \left(\frac{m^2 \log n}{(m-1)n}\right)^m = \frac{cm^{2m+1}}{(m-1)^{m+1}} \log n.$$

Therefore (3) is satisfied for large n if

$$\frac{m}{m-1} > \frac{cm^{2m+1}}{(m-1)^{m+1}},$$

which is valid from the equivalence to (4). We then verify inequality (2), which should hold as the proportion of the right-hand sides in (2) and (3) is m/n and we have chosen $a = b^{m/n}$. In details, the second term in the right-hand side of (2) is asymptotically

$$mNbq^n = \frac{m}{n} = o\left(\frac{\log n}{n}\right),$$

and the first term in that side is asymptotically

$$\frac{m^2}{m-1} Nap^m \sim \frac{cm^2}{m-1} \left(\frac{n}{\log n}\right)^{m-1} \left(\frac{m^2 \log n}{(m-1)n}\right)^m = c\left(\frac{m^2}{m-1}\right)^{m+1} \frac{\log n}{n}.$$

The left-hand side of (2) is

$$\log a = \frac{m}{n} \log b \sim \frac{m^2}{m-1} \frac{\log n}{n}.$$

So (2) is satisfied if

$$\frac{m^2}{m-1} > c\left(\frac{m^2}{m-1}\right)^{m+1},$$

which is equivalent to (4) also. Therefore we have obtained that

$$w(m,n) \geqslant c\left(\frac{n}{\log n}\right)^{m-1}$$

for all large n. $\square$

We conclude this note with a problem: prove or disprove that

$$w(m,n) \geqslant cn^{m-1}$$

for fixed m.

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