Review of[1] of
Ramsey Theory for Discrete Structures
by Hans Jürgen Prömel
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Review by
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1 Introduction

Here are examples of theorems in Ramsey Theory:

1. For all 2-colorings of the *edges* of K_6 there is a monochromatic K_3. That is, there are three vertices such that all of the edges between them are the same color.

2. (Ramsey's Theorem) For all $c \in \mathbb{N}^{\neq 0}$ for all m there exists a number $n \le 2^{2m-1}$ such that for all c-colorings of the *edges* of K_n there is a monochromatic K_m. That is, there are m vertices such that all of the edges between them are the same color. (It is known that $n \ge 2^{m/2}$.)

3. For all 2-colorings of $\{1, \ldots, 9\}$ there exists a monochromatic arithmetic sequence of length 3.

4. (van der Waerden's Theorem, VDW) For all $c, k \in \mathbb{N}^{\neq 0}$ there exists $W = W(k, c)$ such that for all c-colorings of of $\{1, \ldots, W\}$ there exists a monochromatic arithmetic sequence of length k.

Ramsey's Theorem and VDW's theorem are similar *philosophically*: if you color a large enough object you get a nice monochromatic sub-object. Are they related mathematically? See the summary of Part II to find out!

The first book on Ramsey Theory (excluding specialized monographs) was by Graham, Rothchild and Spencer [1]; and the second was by books by Landman and Robertson [2]. The book under review could be entitled

A Second Course in Ramsey Theory.

The book is self contained; however, its rough going unless you are already somewhat familiar with the subject.

2 Summary of Contents

The book is in five Parts, each one of which is subdivided into chapters. We summarize by parts

Part I: Roots of Ramsey Theory

The first chapter, *Ramsey's Theorem* proofs Ramsey's theorem for hypergraphs and the canonical Ramsey theorem (which needs the proof of Ramsey's Theorem for hypergraphs even if all you want to prove is Can. Ramsey for Graphs). We state both theorems

Definition Let $c, k, m \in \mathbb{N}^{\neq 0}$. Assume you are given a c-colorings of the *edges* of the complete k-hypergraph of m vertices. A subset H of the vertices is *homogenous* if every edge that consists of k elements of H (that is, an elements of $\binom{H}{k}$) is the same color.

1. (The Hypergraph Ramsey Theorem) Let $c, k, m \in \mathbb{N}^{\neq 0}$. There exists n such that for all c-colorings of the *edges* of the complete k-hypergraph there is a homogenous set of size m
2. (The Canonical Ramsey Theorem for Graphs.) For all $m \in \mathbb{N}^{\neq 0}$ there exists n such that any coloring (with any number of colors, though bounded by $\binom{n}{2}$ since you are coloring the edges) of the edges there exists a set H of size m such that either (1) H is a homogenous, (2) for $x, y \in H$ the color of (x, y) depends only on $\min\{x, y\}$ (called a *min-homogenous set*, (3) for $x, y \in H$ the color of (x, y) depends only on $\max\{x, y\}$ (called a *max-homogenous set*, or (4) all elements of $\binom{H}{2}$ are colored differently (called a *rainbow set*). To prove this one needs the 4-hypergraph Ramsey Theorem.

The second chapter, *From Hilbert's Cube Lemma to Rado's Thesis* takes you through many theorems that involve coloring $\mathbb{N}$ (and in one case $\mathbb{Q}$). Of particular interest: the book gives an exact condition on a matrix A such that, which we call COND, such that the following are equivalent:

1. Matrix A satisfies condition COND
2. For all finite colorings of $\mathbb{N}^{\neq 0}$ there exists numbers $x_1, \ldots, x_m$ such that $A\vec{x} = 0$.
3. For all finite colorings of $\mathbb{Z}^{\neq 0}$ there exists numbers $x_1, \ldots, x_m$ such that $A\vec{x} = 0$.
4. For all finite colorings of $\mathbb{Q}^{\neq 0}$ there exists numbers $x_1, \ldots, x_m$ such that $A\vec{x} = 0$.

Part II: A Starting Point for Ramsey Theory: Parameter Sets

This is the hardest chapter in the book. We discuss what is proven.

The Hales-Jewitt (HJ) theorem is a a generalization of VDW theorem. From the HJ theorem one can prove, not just VDW's theorem, but the Gallai-Witt theorem, which is a multidimensional version of VDW's theorem. All of this is proven.

The Graham-Rothchild (GR) Theorem is a generalization of the HJ theorem!! From the GR theorem one can prove both *The Hypergraph Ramsey Theorem* and *VDW's theorem*. Hence, this shows Ramsey's Theorem and VDW's Theorem are both part of the same phenomena and not just *philosophically similar*. The GR theorem is very powerful but very abstract. We give one more corollary of GR's theorem:

1. For all $c, m \in \mathbb{N}^{\neq 0}$ there exists n such that for all c-colorings of $\{1, \ldots, n\}$ there exists a $A \subseteq \{1, \ldots, n\}$ of size m such that every nonempty subset of A has the same color sum.

This chapter also proves Canonical versions (allow infinitely many colors) of the The HJ theorem, VDW's theorem, and The GR theorem.

Part III: Back to the Roots: Sets

This Part feels like a breath of fresh air compared to the hard abstractions in Part II. Topics are:

1. Some Exact and Some Asymptotic Ramsey Number. This is an easy chapter that is usually at the beginning of a course in Ramsey Theory.

2. The Paris-Harrington result about a Ramsey Theorem which is true but cannot be proved in Peano Arithmetic.

3. Product theorems. This is essentially Ramsey Theorem for bipartite, tripartite, k-partite graphs. Example: for all c, m there exists n such that for all c-coloring of $K_{n,n}$ there exists a monochromatic $K_{m,m}$.

4. A Quasi Ramsey Theorem. This is about the Erdős Discrepancy conjecture (not proven at the time but proven by Terry Tao in 2016). The idea is that every long sequence of $\{+1, -1\}$ there will be some arithmetic progression where the sum is large.

5. Partition Relations for Cardinals. Ramsey Theory on infinite cardinals. Example (actually a counterexample) There is a 2-coloring of pairs of reals such that there is no homogenous set of size the continuum.

Part IV: Graphs and Hypergraphs

Consider the following triviality: if G has a clique of size 6 in it then any 2-coloring of the edges of G has a monochromatic K_3. But what if G's largest clique is of size 5? 4? 3? 2? 1? IF its 1 or 2 then the theorem will be false. But the following is true:

1. Let c be a number of colors and F be a graph. There is a graph G with the same max clique size as F such that, for all c-colorings of the edges of G, there is a monochromatic F.

Most of this Part is about theorems like that. There is also material on (1) Random Graphs, (2) Rado's graph- a countable graph can be embedded in it, and (3) some more rather hard theorems related to the GR theorem.

Part V: Density Ramsey Theory

The original proof of van der Waerden's theorem gave Ackerman-like bounds on $W(k, c)$. Erdős made the following conjecture hoping it would lead to an alternative proof of VDW's theorem with smaller bounds:

If $A \subseteq \mathbb{N}$ of positive upper density then, for all k, A has arithmetic sequences of length k.

Roth proved the $k = 3$ case. Szemeredi proved the $k = 4$ case and later the general case. Szemeredi's proof *used* VDW's theorem and hence did not lead to smaller bounds. (Shelah and later Gowers got much better bounds though not through Erdős's conjecture.) Szemeredi's Theorem is called *a density theorem*. His proof was purely combinatorial though difficult. Later a density version of the HJ was proven; however, the proof was not purely combinatorial. Gowers proposed a polymath project (many people contributing) to try to find a purely combinatorial proof of the HJ density theorem. The project was a success. This book presents that proof.

3 Opinion

This is a good but tough book. I would recommend the reader already know Ramsey's theorem on hypergraphs and van der Waerden's theorem before beginning to read it.

However, if one knows some Ramsey Theory this book will teach you *more of it*. There is much in this book that is interesting and not that well known. And its good having it all in one place. But a warning- the subject his hard, hence the book is hard.

References

[1] R. Graham, B. Rothschild, and J. Spencer. *Ramsey Theory*. Wiley, New York, 1990.

[2] B. Landman and A. Robertson. *Ramsey Theory on the integers*. AMS, Providence, 2004.