Open Problems Column
Edited by William Gasarch
This Issue's Column!

This issue's Open Problem Column is by Natasha Dobrinen and William Gasarch. It is *When Ramsey Theory Fails, Settle For More Colors*.

Request for Columns!

I invite any reader who has knowledge of some area to contact me and arrange to write a column about open problems in that area. That area can be (1) broad or narrow or anywhere in between, and (2) really important or really unimportant or anywhere inbetween.

When Ramsey Theory Fails
Settle for More Colors
(Big Ramsey Degrees!)

Natasha Dobrinen and William Gasarch

1 Introduction

In this column we state a class of open problems in Ramsey Theory. The general theme is to take Ramsey-type statements that are false and weaken them by allowing the homogenous set to use more than one color. This concept is not new, and the theorems we state and/or prove are not new; however, the open questions that request easier proofs of the known theorems (or weaker versions) may be new. We use the phrase *an elementary proof*. This is not meant to be a technical or rigorous term. What we really mean is a proof that can be taught in an undergraduate combinatorics course. A good example of what we mean is the proof of Theorem 9.3.

The papers in this area tend to be rather abstract. This column aims to keep things simple. Hence we avoid stating things in general terms and instead concentrate on concrete examples.

We use the following standard notation.

Notation 1.1

1. N is the set of natural numbers. We do not include 0. ω is the set N in its natural order.

2. $\omega + n$ is the ordered set

$$1 < 2 < \cdots < \omega < \omega + 1 < \cdots < \omega + n - 1.$$

3. Z is the set of integers. ζ is the set Z in its natural order.

4. Q is the set of rationals. η is the set Q in its natural order.

5. R is the set of reals. λ is the set R in its natural order.

6. If $n \in \mathsf{N}$ then $[n]$ is the set $\{1, \ldots, n\}$.

The following notions are well-known concepts in Ramsey Theory.

Def 1.2 Let A be a set (finite or infinite). Let $a, d \in \mathsf{N}$.

1. $\binom{A}{a}$ is the set of all a-sized subsets of A.

2. Let COL be a finite coloring of $\binom{A}{a}$. A *homogenous set relative to* COL is a set $H \subseteq A$ such that COL restricted to $\binom{H}{a}$ is constant. We often say *homog* when the coloring is implicit.

3. Let COL be a finite coloring of $\binom{A}{a}$. A *d-homogenous set relative to* COL is a set H such that COL restricted to $\binom{H}{a}$ takes $\le d$ values. We often say *d-homog* when the coloring is implicit. Note that *homog* is *1-homog*.

4. Let A be an ordered set (e.g., η). Let COL be a finite coloring of $\binom{A}{a}$. An *order-homogenous set relative to* COL is a set $H \subseteq A$ of the same order type as A such that COL restricted to $\binom{H}{a}$ is constant. We often say *order-homog* when the coloring is clear.

5. Let A be an ordered set (e.g., η). Let COL be a finite coloring of $\binom{A}{a}$. A *d-order-homogenous set relative to* COL is a set H of the same order type as A such that COL restricted to $\binom{H}{a}$ takes $\le d$ values. We often say *d-order-homog* when the coloring is clear. Note that *order-homog* is *1-order-homog*.

The following is the well-known Ramsey's Theorem on N and the never-stated-but-well-known order-Ramsey Theory on ω. Part 2 (about ω) follows trivially from Part 1 (about N); however, we include Part 2 since this paper is about coloring orderings, not sets, so we want to lay out what is known, even if it's trivial.

Theorem 1.3 *Let $a, c \in \mathsf{N}$.*

1. *For all finite colorings of $\binom{\mathsf{N}}{a}$ there exists an infinite homog set.*
2. *For all finite colorings of $\binom{\omega}{a}$ there exists an order homog set (Note that the order-homog set will be infinite since it has order type ω).*

2 Ramsey on $\omega + n$

Let $n \geq 1$. The analog of Theorem 1.3.2 for $\omega + n$ fails.

Theorem 2.1 *There exists* COL: $\binom{\omega+n}{2} \rightarrow [2]$ *such that there is no order-homog set.*

Proof sketch: We define COL.

- If $x < y \in \mathsf{N}$ then $\mathrm{COL}(x, y) = \mathrm{RED}$.
- If $0 \leq x < y \leq n-1$ then $\mathrm{COL}(\omega + x, \omega + y) = \mathrm{RED}$.
- If $x \in \mathsf{N}$ and $0 \leq y \leq n-1$ then $\mathrm{COL}(x, \omega + y) = \mathrm{BLUE}$.

We leave it to the reader to show there is no order-homog set. ∎

What if instead of demanding an order-homog set we settled for a d-order homog set for some d. If we are 2-coloring, nothing of interest is true: (1) by Theorem 2.1 we cannot get a 1-order-homog set, and (2) $\omega + n$ itself is always an uninteresting 2-order-homog set. But what if we 3-color? c-color for $c \geq 3$? We give a result where the resulting set is d-homog, where d is *independent of the number of colors originally used!*

Theorem 2.2 *Let $n \geq 1$.*

1. *For all finite colorings of $\binom{\omega+n}{2}$ there exists an $(1 + \binom{n}{1} + \binom{n}{2})$-order-homog set.*
2. *There is a finite coloring of $\binom{\omega+n}{2}$ such that there is no $(\binom{n}{1} + \binom{n}{2})$-order-homog set.*

Proof:
a) Let COL be a finite coloring of $\binom{\omega+n}{2}$. By Theorem 1.3.2 there is a homog set of order type ω. Call this set H_{-1}.

For each $0 \le j \le n-1$ do the following: Find the least color c such that

$$H_j = \{x \in H_{j-1} : \mathrm{COL}(x, \omega + j) = c\} \text{ is infinite.}$$

Let H be $H_{n-1} \cup \{\omega, \omega+1, \ldots, \omega+n-1\}$. We now count how many different colors are used on $\binom{H}{2}$.

- COL restricted to $\binom{H_{n-1}}{2}$ takes 1 value.
- For all $0 \le j \le n-1$, COL restricted to $\omega \times \{\omega + j\}$ takes 1 value (by the way we defined H_j). Hence all together these take $\le n$ values, one for each j. We denote this $\binom{n}{1}$ as a hint about how to generalize.
- COL restricted to $\binom{\{\omega,\ldots,\omega+n-1\}}{2}$ takes $\le \binom{n}{2}$ values.

The total number of colors used is at most $1 + \binom{n}{1} + \binom{n}{2}$.

b) Let COL be the following coloring of $\binom{\omega+n}{2}$.

- If $x, y \in \mathsf{N}$ then $\mathrm{COL}(x,y) = 1$.
- If $x \in \mathsf{N}$ and $0 \le y \le n-1$ then $\mathrm{COL}(x, \omega+y) = y+2$
- Color $\binom{\{\omega,\omega+1\ldots,\omega+n-1\}}{2}$ with colors $\{n+2, \ldots, n+1+\binom{n}{2}\}$, each pair getting a different color.

We leave it to the reader to show that there is no $(\binom{n}{1} + \binom{n}{2})$-order-homog set. ∎

Note 2.3 One can prove a similar theorem for colorings of $\binom{\omega+n}{a}$. We leave it to the reader to work this out.

3 An Important Notation

Lets look at Theorem 2.2 in the case of $n = 2$. We get:

1. For all finite colorings of $\binom{\omega+2}{2}$ there exists a 4-order-homog set.

2. There is a finite coloring of $\binom{\omega+2}{2}$ such that there is no 3-order-homog set.

This is an example of the general theme of the research we are considering: no finite coloring of $\binom{\omega+2}{2}$ has a 3-order-homog set, but one does have a 4-order-homog set. We will later write this as $T(2, \omega + 2) = 4$. The first parameter is the arity, the second parameter is the linear ordering. Kechris, Pestov, and Todorcevic [KPT05] defined the following important notation where we vary the linear ordering and the arity.

Def 3.1

1. Let L be an infinite linear order and $a \in \mathsf{N}$. Then $T(a, L)$ is the least number such that the following holds: *For all finite colorings of* $\binom{L}{a}$ *there is a* $T(a, L)$*-order-homog set.* Note that $T(a, L)$ is independent of the number of colors used. (We will later extend this definition to structures other than linear orderings.)

2. L has *finite big Ramsey degrees* if, for all a, $T(a, L)$ exists. (We will not be using this notation; however we include it since it is used in the literature.)

Note 3.2

1. Kechris, Pestov, and Todorcevic actually defined T much more generally.

2. For most of the sections in this paper we deal with $T(a, X)$ where X is a linear order. In Section 9 we will generalize the notion of T and deal with $T(1, \mathsf{N} \times \mathsf{N})$.

3. In this paper we will mostly look at $a = 2$; however, even the $a = 1$ case can be interesting. It is easy to see that $T(1, \omega + n) = n + 1$, $T(1, \zeta) = 2$, and $T(1, \eta) = 1$.

We restate Theorem 2.2 with this notation.

Theorem 3.3 *Let $n \geq 1$. $T(2, \omega + n) = 1 + \binom{n}{1} + \binom{n}{2}$.*

We restate Note 2.3 with this notation:

Note 3.4 We leave it as an exercise to work out what $T(a, \omega + n)$ is.

4 Ramsey on Countable Ordinals

The analog of Theorem 1.3.2 for finite sums of ω fails.

Theorem 4.1 *Let $n \in \mathsf{N}$. There exists* COL: $\binom{\omega+\cdots+\omega}{2} \rightarrow [2]$ *(there are n ω's) such that there is no order-homog set.*

Proof sketch: We define COL.

Def 4.2 Let $\beta_1 < \beta_2$ be ordinals. Then $FIN(\beta_1, \beta_2)$ means that there exists $n \in \mathsf{N}$ such that $\beta_1 + n = \beta_2$.

$$\mathrm{COL}(\beta_1, \beta_2) = \begin{cases} \text{RED} & \text{if } FIN(\beta_1, \beta_2) \\ \text{BLUE} & \text{if } \neg FIN(\beta_1, \beta_2) \end{cases} \tag{1}$$

We leave it to the reader to show there is no order-homog set. ∎

Theorem 4.1 is motivation for looking at (1) $T(a, \beta)$ where β is an ordinal (this section), and (2) $T(a, \beta)$ where β is a countable scattered ordering (next section).

The study of $T(a, \beta)$, where β is an *ordinal* has a rich history. A paper by Mašulović and Šobot [Mv19] summarizes that history and has the following theorem.

Theorem 4.3 *1. Let β be a countable ordinal. The following are equivalent:*

- *For all $a \in \mathsf{N}$, $T(a, \beta) < \infty$.*
- *$\beta < \omega^\omega$.*

2. If $\beta \geq \omega^\omega$ then (1) $T(1, \beta) < \infty$, and (2) for all $a \geq 2$, $T(a, \beta) = \infty$.

3. *Let $m \in \mathsf{N}$. Then $T(1, \omega \cdot m) = 1$, and for all $a \geq 2$, $T(a, \omega \cdot m) \leq m^a$.*

We will give an elementary proof of $T(2, \omega+\omega) = 4$ later in Theorem 9.4.

Theorem 4.3.3 follows from a more fine-tuned result of Mašulović and Šobot [Mv19] (Theorem 4.8 of that paper). In that paper, they use the notation $T(a, \beta)$ to denote the number of colorings of *embeddings* of a set of size a into β, rather than colorings of subsets of β of size a, finding exactly m^a colors for embeddings of a into $\omega \cdot m$.

The following are also known:

Theorem 4.4

1. *(Fraïssé [Fra00] (Page 189)) For all ordinals β, $T(1, \omega^\beta) = 1$.*
2. *(Fraïssé [Fra00] (Page 189)) For all ordinals β, $T(1, \beta) < \infty$.)*
3. *(Galvin, unpublished) The sequence $T(a, \omega^2)$ coincides with OEIS sequence A000311.*

Open Problem 4.5

1. For some countable ordinals $\beta < \omega^\omega$ and some $a \in \mathsf{N}$ give an elementary proof that $T(a, \beta) < \infty$. (Again note that we already have an elementary proof that $T(2, \omega + \omega) = 4$, which we present in Theorem 9.4.)
2. For $a \in \mathsf{N}$, and $\beta < \omega^\omega$, determine $T(a, \beta)$.

5 Countable Scattered Linear Orderings

The study of $T(a, \beta)$, where β ia a *countable scattered linear order* has a rich history. A paper by Mašulović [Maš19] summarizes that history and has the following theorem. We need some definitions before we can present it.

Def 5.1 Let L be a linear order.

1. L is *scattered* if there is no dense subset with more than one element.
2. We classify how complicated L can be.
    - Γ_0 contains the empty ordering $\emptyset$ and the 1-point ordering.

- For an ordinal $\beta > 0$ let

$$\Gamma_\beta = \left\{ \sum_{i \in \mathsf{Z}} S_i : S_i \in \bigcup_{\alpha < \beta} \Gamma_\alpha \text{ for all } i \in \mathsf{Z} \right\}.$$

Hausdorff proved the following:

Theorem 5.2 *The set of countable scattered linear orderings is exactly*

$$\bigcup_{\beta \text{ a countable ordinal}} \Gamma_\beta.$$

Def 5.3 Let L be a countable scattered linear order. The *Hausdorff rank of* L is the least β such that $L \in \Gamma_\beta$.

We can now present Mašulović's theorem.

Theorem 5.4 *Let L be a countable scattered linear order. The following are equivalent:*

- *For all* $a \in \mathsf{N}$, $T(a, L) < \infty$.
- *The Hausdorff rank of L is finite.*

Open Problem 5.5

1. Find an elementary proof that $T(2, \zeta + \zeta)$ exists. Slightly more complicated countable scattered linear orderings, and $a \geq 3$, can also be considered.
2. For $a, b \in \mathsf{N}$, for scattered linear orders $L \in \Gamma_b$, find $T(a, L)$.
3. Let $a, b \in \mathsf{N}$. Then for every $L \in \Gamma_b$, $T(a, L)$ exists. Find a function f such that, for all $L \in \Gamma_b$, $T(a, L) \leq f(a, b)$ or show that no such function exists. Find a function g such that, for all $L \in \Gamma_b$, $T(a, L) = g(a, b)$ or show that no such function exists.

6 Ramsey on ζ

The analog of Theorem 1.3.2 for ζ fails.

Theorem 6.1 *There exists* COL: $\binom{\zeta}{2} \rightarrow [2]$ *such that there is no order-homog set.*

Proof sketch: We define COL.

$$\mathrm{COL}(i,j) = \begin{cases} \mathrm{RED} & \text{if } i,j \geq 0 \\ \mathrm{BLUE} & i < 0 \text{ or } j < 0 \end{cases} \tag{2}$$

We leave it to the reader to show there is no order-homog set. ∎

Mašulović and Šobot [Mv19] proved that $T(a, \omega{+}\omega) \leq 2^a$. In this context $\omega + \omega$ and ζ are similar. Hence:

Theorem 6.2 *For all* $a \in \mathsf{N}$, $T(a,\zeta) \leq 2^a$.

We give an elementary proof that $T(2,\zeta) = 4$ later in Theorem 9.4.

Open Problem 6.3

1. Find an elementary proof that, for all $a \geq 3$, $T(a,\zeta)$ exists.
2. Find the values of $T(a,\zeta)$.

7 Ramsey on η

The analog of Theorem 1.3.2 for η fails by the following theorem of Sierpinski.

Theorem 7.1 *There exists* COL: $\binom{\eta}{2} \rightarrow [2]$ *such that there is no order-homog set.*

Proof sketch: Let $q_1, q_2, \ldots,$ be some enumeration of η. We define COL. Assume $i < j$.

$$\mathrm{COL}(q_i,q_j) = \begin{cases} \mathrm{RED} & \text{if } q_i < q_j \text{ (the enum-order and the } \eta\text{-order agree)} \\ \mathrm{BLUE} & \text{if } q_i > q_j \text{ (the enum-order and the } \eta\text{-order disagree)} \end{cases} \tag{3}$$

We leave it to the reader to show there is no order-homog set. ∎

Theorem 7.2

1. *$T(2, \eta) = 2$. (This was first proven by Galvin, unpublished.)*

2. *For all a, $T(a, \eta)$ exists. (This was first proven by Laver [Lav84].)*

3. *$T(a, \eta)$ is the coefficient of x^{2a+1} in the Taylor series for the tangent function, hence*

$$T(a, \eta) = \frac{B_{2a+1}(-1)^{a+1}(1 - 4^{a+1})}{(2(a+1))!}.$$

(This was proven by Devlin [Dev79], see also Vuksanovic [Vuk02] and Halpern & Lauchli [HL66].)

The proof of Theorem 7.2 uses the Halpern-Lauchi Theorem [HL66], which is a Ramseyian theorem on products of trees. Their proof uses logic and is difficult. Harrington gave an innovative proof of the Halpern-Lauchli Theorem using the method of forcing. The first published version of this proof appeared in Todorcevic and Farah [TF95]. A purely combinatorial proof appears in Todorcevic [Tod10]. Dodos and Kanellopolous [DK16] give an easier purely combinatorial proof of a weaker result (which still suffices to prove Theorem 7.2.2).

Open Problem 7.3

1. Find an elementary proof that $T(2, \eta)$ exists.

2. Find an elementary proof that, for all a, $T(a, \eta)$ exists.

8 Ramsey on λ

The analog of Theorem 1.3.2 for λ fails by the following theorem of Sierpinski.

Theorem 8.1 *There exists* $\mathrm{COL}\colon \binom{\lambda}{2} \to [2]$ *such that there is no continuum-sized order-homog set.*

Proof sketch: Let $<$ be the usual ordering on λ. Let $\prec$ be a well ordering of λ. We define COL. Assume $r < s$.

$$\mathrm{COL}(r,s) = \begin{cases} \mathrm{RED} & \text{if } r \prec s \text{ (the usual order and the well order agree)} \\ \mathrm{BLUE} & \text{if } s \prec r \text{ (the usual order and the well order disagree)} \end{cases} \tag{4}$$

We leave it to the reader to show there is no continuum-sized order-homog set. ▮

Dilip Raghaven and Stevo Todorcevic [RT18] proved that, assuming a certain type of large cardinal exists, then for each coloring of pairs of real numbers into finitely many colors, there is a subset which is a topological copy of the rationals on which there are no more than two colors. Suffice to say, this is a hard problem. We state open problems; however, we do not suggest finding an elementary proof since this seems unlikely.

Open Problem 8.2

1. Determine for which a, $T(a,\lambda)$ exists assuming various large cardinal assumptions.

2. Either remove the large cardinal assumptions or prove a reversal, e.g., If $T(a,\lambda)$ exists then an inaccessible cardinal exists.

9 Ramsey on $\mathsf{N} \times \mathsf{N}$

In the prior sections we colored an *ordered set* and wanted an *ordered homog set* of the same order type. We now look at a failed Ramsey theory where we color $\mathsf{N} \times \mathsf{N}$.

Def 9.1 Let $d \in \mathsf{N}$. Let COL be a finite coloring of $\mathsf{N} \times \mathsf{N}$.

1. A *bip-homogenous set relative to* COL is a pair of sets $X, Y \subseteq \mathsf{N}$ such that COL restricted to $X \times Y$ is constant. We often say *bip-homog* when the coloring is implicit. (The 'bip' stands for bipartite.)

2. A *d-bip-homogenous set relative to* COL is a pair of sets $X, Y \subseteq \mathsf{N}$ such that COL restricted to $X \times Y$ takes on only d values. We often say *d-bip-homog* when the coloring is implicit.

3. A d-bip-homogenous set X, Y relative to COL is *infinite* if both X and Y are infinite.

4. $T(\mathsf{N} \times \mathsf{N})$ is the least t such that, for all finite colorings of $\mathsf{N} \times \mathsf{N}$, there exists an infinite t-bip-homog set. Note that we do not need a numeric parameter for arity since we are only coloring $\mathsf{N} \times \mathsf{N}$.

5. One can generalize the above concepts to coloring A^k (where A is any set) or cross products of different sets.

The analog of Theorem 1.3.2 for $\mathsf{N} \times \mathsf{N}$ fails.

Theorem 9.2 *There exists* COL: $\mathsf{N} \times \mathsf{N} \to [2]$ *such that there is no infinite bip-homog set.*

Proof sketch: We define COL.

$$\mathrm{COL}(x, y) = \begin{cases} \mathrm{RED} & \text{if } x \le y \\ \mathrm{BLUE} & \text{if } x > y \end{cases} \tag{5}$$

We leave it to the reader to show there is no infinite set as in the premise. ∎

The following is folklore; however, we provide a proof.

Theorem 9.3 $T(\mathsf{N} \times \mathsf{N}) = 2$.

Proof: Let COL: $\mathsf{N} \times \mathsf{N} \to [c]$ for some $c \in \mathsf{N}$. Apply the 1-d Ramsey Theorem (that is, the infinite pigeon-hole-principle) to the first row so that it is now (with renumbering) all colored 1. Then apply the 1-d Ramsey Theorem to the first column (though not including the first element of the first row) so that it is now (with renumbering) all colored 2 (it's okay if it's all colored 1, though we assume not). In the diagram below, we either indicate what color the grid point is colored or leave it blank to indicate that the color could be anything.

$\vdots$						$\cdots$
2						$\cdots$
2						$\cdots$
2						$\cdots$
2						$\cdots$
2						$\cdots$
2						$\cdots$
1	1	1	1	1	1	$\cdots$

Repeat this procedure on the second row and column to get

$\vdots$	$\vdots$					$\cdots$
2	4					$\cdots$
2	4					$\cdots$
2	4					$\cdots$
2	4					$\cdots$
2	4					$\cdots$
2	3	3	3	3	3	$\cdots$
1	1	1	1	1	1	$\cdots$

Keep doing this. In the end you have an $\mathsf{N} \times \mathsf{N}$ grid such that row i is almost always color (we'll say) r_i and column j is almost always color (we'll say) s_j. Since the number of colors is finite, there is some color r such that there are an infinite number of i with $r_i = r$. Get rid of all of the other rows. Make sure that each row (going up) has at least as many non-r's as the previous one. We now have something like this (after renumbering the colors):

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
2	5	3	2	3	6	7	6	1	$\cdots$
2	5	3	2	3	6	7	1	1	$\cdots$
2	5	3	2	3	6	1	1	1	$\cdots$
2	5	3	2	3	6	1	1	1	$\cdots$
2	5	3	2	3	6	1	1	1	$\cdots$
2	5	3	1	1	1	1	1	1	$\cdots$
1	7	1	1	1	1	1	1	1	$\cdots$

As before, let s_j be the color that is almost all of column j (after we have gotten rid of many rows and have a picture like the one above). Since the number of colors is finite, there is some color s such that there are an infinite number of j with $s = s_j$. Get rid of all of the other columns. We now have a 2-homog grid. ∎

Using the proof of Theorem 9.3 we can obtain elementary proofs of a few results.

Theorem 9.4

1. $T(2, \omega + \omega) = 4$.
2. $T(2, \zeta) = 4$.

Proof:
1a) Let COL be a finite coloring of $\binom{\omega+\omega}{2}$.

Let COL_1 be COL restricted to the first copy of ω. By Ramsey's Theorem there exists an order-homog set A_1 relative to COL_1. Let COL_2 be COL restricted to the second copy of ω. By Ramsey's Theorem there exists an order-homog set A_2 relative to COL_2.

Restrict COL to $A_1 \times A_2$. This is a coloring of $T(\mathsf{N} \times \mathsf{N})$. By Theorem 9.3 there exists $A_1' \subseteq A_1$ and $A_2' \subseteq A_2$ such that COL restricted to $A_1' \times A_2'$ is a 2-bip-homog set. $T(2, \omega + \omega) \le 4$

1b) We color $\binom{\omega+\omega}{2}$ as follows:

1. if x, y are in the first copy of ω, color $\{x, y\}$ 1.
2. if x, y are in the second copy of ω, color $\{x, y\}$ 2.
3. if x is in the first copy of ω and y is in the second copy of ω then $\{x, y\}$ is colored 3 if $x < y$ and 4 if $x \ge 4$.

We leave it to the reader to show that this coloring does not have a 3-homog set.

2) This proof is similar to that of Part 1, with the negative integers being the first ω and the positive integers being the second ω. ∎

The following is also folklore; however, we leave it to the reader.

Theorem 9.5 *Let N^n denote the n-fold Cartesian product of N. For all $n \geq 1$, $T(\mathsf{N}^n)$ exists.*

Open Problem 9.6

1. Find an elementary proof that $T(\mathsf{N}^n) < \infty$.
2. Find the numbers $T(\mathsf{N}^n)$, for $n \geq 3$.

10 Ramsey on The Random Graph and Hypergraph

Def 10.1 Let $G = (V, E)$ be an infinite graph. Let $d \in \mathsf{N}$. Let COL_V be a finite coloring of V and COL_E be a finite coloring of E.

1. A *G-homog set relative to* COL_V is a set $X \subseteq V$ such that (1) COL_V restricted to X is constant, and (2) the induced subgraph of G with vertex set X is isomorphic to G. We often say *homog* when the graph and the coloring are implicit.
2. The reader can define d-G-homog.
3. A *G-homog set relative to* COL_E is a set $X \subseteq V$ such that (1) COL_E restricted to the induced subgraph of G with vertex set X is constant, and (2) the induced subgraph of G with vertex set X is isomorphic to G. We often say *homog* when the graph and the coloring are implicit.
4. The reader can define d-G-homog.
5. $T(1, G)$ is the least d such that the following is true: For all finite colorings COL_V of the *vertices* of G there exists a d-homog set.
6. $T(2, G)$ is the least d such that the following is true: For all finite colorings COL_E of the *edges* of G there exists a d-homog set.
7. The reader can define these notions for a-ary hypergraphs.

Def 10.2 R is the infinite random graph (often called *the Rado graph*). R_a is the infinite random a-ary hypergraph.

Theorem 10.3

1. *(Erdős, Hajnal, and Posa [EHP75])* $T(2, R) \geq 2$.
2. *(Pouzet and Sauer [PS96])* $T(2, R) \leq 2$, *hence* $T(2, R) = 2$.
3. *(Coulson, Dobrinen, and Patel [CDP20]) For all* $a \geq 3$, $T(a, R_a)$ *exists.*
4. *(Balko, Chodounský, Hubička, Konečný, and Vena [BCH+20])* $T(2, R_3) < \infty$.

(3) is a special case of a much more general theorem in [CDP20].

Open Problem 10.4

1. Find an elementary proof that $T(1, R) < \infty$.
2. Find an elementary proof that $T(2, R) < \infty$.
3. Find an elementary proof that $T(a, R_a) < \infty$.

11 Acknowledgement

We thank Stephen Fenner, Erik Metz, and Anthony Ostuni for proofreading and discussion.

References

[BCH+20] Martin Balko, David Chodounský, Jan Hubička, Matej Konečný, and Lluis Vena. Big Ramsey spectra of 3-uniform hypergraphs are finite, 2020. `https://arxiv.org/pdf/2008.00268.pdf`.

[CDP20] Rebecca Coulson, Natasha Dobrinen, and Rehana Patel. Ramsey theory of Fraïssé structures with a certain amalgamation property, 2020. In Preparation.

[Dev79] Denis Devlin. *Some partition theorems and ultrafilters on* ω. PhD thesis, Dartmouth College, 1979. `https://www.cs.umd.edu/~gasarch/BLOGPAPERS/DevlinPhd.pdf`.

[DK16] Pandelis Dodosa and Vassilis Kanellopoulos. *Ramsey theory for product spaces*. AMS, 2016. `http://users.uoa.gr/~pdodos/Publications/RT.pdf`.

[EHP75] Paul Erdős, Andras Hajnal, and Lajos Posa. Strong embeddings of graphs into coloured graphs. In *Infinite and Finite Sets*, pages 585–595, New York, 1975. North-Holland and American Elsevier. Coll. Math. Soc. János Bolyai 10.

[Fra00] Roland Fraïssé. *Theory of relation (Revised Edition)*. North-Holland Publishing, Amsterdamn, 2000.

[HL66] James D. Halpern and Hans Läuchli. A partition theorem. *Trans. Amer. Math. Soc.*, 124:360–367, 1966. `https://www.cs.umd.edu/~gasarch/BLOGPAPERS/HL-1966.pdf`.

[KPT05] Alexander Kechris, Vladmir Pestove, and Stevo Todorčević. Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups. *Geometric and Funcational Analysis*, 15:106–189, 2005.

[Lav84] Richard Laver. Products of infinitely many trees. *Journal of the London Mathematical Society*, 29:385–396, 1984.

[Maš19] Dragan Mašulović. Big Ramsey spectra of countable chains, 2019. `https://arxiv.org/pdf/1912.03022.pdf`.

[Mv19] Dragan Mašulović and Branislav Šobot. Countable ordinals and big Ramsey degrees, 2019. `https://arxiv.org/pdf/1904.03637.pdf`.

[PS96] Maurice Pouzet and Norbert Sauer. Edge partitions of the Rado graph. *Combinatorica*, 16, 1996.

[RT18] Dilip Raghavan and Stevo Todorčević. Proof of a conjectuer of Galvin, 2018. `https://arxiv.org/abs/1809.00922`.

[TF95] Stevo Todorčević and Ilijas Farah. *Some applications of the method of forcing*. Yenisei-Moscow, 1995.

[Tod10] Stevo Todorčević. *Introduction to Ramsey spaces.* Princeton University Press, 2010. https://www.jstor.org/stable/j.ctt7sxqn.

[Vuk02] Vojkan Vuksanovic. A proof of a parition theorem for $[q]^n$. *Proceedings of the American Mathematical Society*, 130(10):2857–2864, 2002. https://www.cs.umd.edu/~gasarch/BLOGPAPERS/partitionQ.pdf.