An Infinite Number of Proofs of the Reciprocal Theorem

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Abstract

The reciprocal theorem is the following: for all but a finite number of n there exists n distinct reciprocals that sum to 1. We give an infinite number of proofs of this theorem. Twice.

1 Introduction

The following was problem number 2 (out of 5) on the *The University of Maryland High School Mathematics Competition* in 2010.

(a) The equations $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$ *and* $\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42} = 1$ *express 1 as the sum of three (respectively four) distinct positive integers. Find five distinct positive integers* $a < b < c < d < e$ *such that* $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = 1$.

(b) Prove that for any integer $m \geq 3$ *there exists* m *positive integers* $d_1 < d_2 < \cdots < d_m$ *such that* $\frac{1}{d_1} + \cdots + \frac{1}{d_m} = 1$.

The third author graded the 188 students who attempted this problem. 188 got Part a correct. We list all of those answer in the appendix. 160 got Part b correct. There were four different correct solutions. We will present their proofs in Section **??**.

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Since the students came up with four proofs the question arises: how many proofs are there? We use the following terminology:

Def 1.1 A *nice n-sequence of natural numbers* is a sequence $d_1 < \cdots < d_n$ such that $\sum_{i=1}^{n} \frac{1}{d_i} = 1$.

We consider the following weaker version which we refer to as *The Reciprocal Theorem*:
For all but a finite number of n there exists a nice n-sequence

We will sometimes need the following well known result (look up Egyptian Fractions).

Lemma 1.2

1. *If $\alpha \in \mathsf{Q}^{>0}$ then there exists n and $2 \le a_1 < \ldots < a_n$ such that $\sum_{i=1}^{n} \frac{1}{a_i} = \alpha$*
2. *If $x \in \mathsf{N}$ then there exists a nice sequence where every element is divisible by x. (This follows from Item 1 since you can multiply $\sum_{i=1}^{n} \frac{1}{a_i} = x$ by $\frac{1}{x}$.)*

2 Four Correct Submitted Solutions

Theorem 2.1 *Let $P(n)$ be the statement: There exists a nice n-sequence. Then $(\forall n \ge 3)[P(n)]$.*

Proof: We sketch the four correct solutions submitted. All were by induction on n.

Let Base3 and Base4 be the following equations:

$$\begin{aligned} \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{6} &= 1 \\ \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{7} + \tfrac{1}{42} &= 1 \end{aligned}$$

They will be used as base cases.

SOLUTION ONE: Base3 is the base case. For $n \ge 4$ use

$$\frac{1}{d} = \frac{1}{d+1} + \frac{1}{d(d+1)}$$

to go from $P(n-1)$ to $P(n)$.

133 students submitted this solution which was by far the most common.

SOLUTION TWO: Base3 and Base4 are the base cases. Use

$$\frac{1}{d} = \frac{1}{2d} + \frac{1}{3d} + \frac{1}{6d}$$

to go from $P(n-1)$ to $P(n+1)$.

21 students submitted this solution.

SOLUTION THREE: Base3 is the base case. Load the induction hypothesis with the additional assumption that d_n is even.

Use

$$\frac{1}{d} = \frac{1}{(3d/2)} + \frac{1}{3d}$$

to go from $P(n-1)$ to $P(n)$.

4 students submitted this solution.

SOLUTION FOUR: Base3 is the base case.

Use

$$\frac{1}{d_1} + \cdots + \frac{1}{d_{n-1}} = 1 \Rightarrow \frac{1}{2} + \frac{1}{2d_1} + \cdots + \frac{1}{2d_{n-1}} = 1$$

to go from $P(n-1)$ to $P(n)$.

4 students submitted this solution.

∎

3 An Infinite Number of Proofs Based on SOLUTION ONE

We rewrite SOLUTION ONE with an eye towards modifying it. We call this SOLUTION ONE-1. It will look a bit odd since some parts of it that generalize are not strictly needed here.

SOLUTION ONE-1: We prove $(\forall n \geq 3)[P(n)]$. Base3 is the base case. Let $f_1(x) = \frac{x(x-1)}{1}$. Load the induction hypothesis with the additional assumptions that

- $d_{n-1} \equiv 0 \pmod 1$ (this is always true).

- $d_{n-2} < d_{n-1}$ (this is always true)
- $d_n = f(d_{n-1})$.
- $d_{n-1} \ge 2$.

Use

$$\frac{1}{d_{n-2}} + \frac{1}{d_{n-1}} + \frac{1}{f_1(d_{n-1})} =$$

$$\frac{1}{d_{n-2}} + \frac{1}{d_{n-1}} + \frac{1}{d_{n-1}(d_{n-1}-1)+1} + \frac{1}{f_1(d_{n-1}(d_{n-1}-1)+1)}$$

to go from $P(n-1)$ to $P(n)$. Use $d_{n-1} \ge 2$ to show

$$d_{n-1} < d_{n-1}(d_{n-1}-1)+1$$

We now produce SOLUTION ONE-2:

SOLUTION ONE-2: We prove $(\forall n \ge 4)[P(n)]$. Base4 is the base case. Let $f_2(x) = \frac{x(x-2)}{2}$. Load the induction hypothesis with the additional assumptions that

- $d_{n-1} \equiv 0 \pmod 2$,
- $d_{n-2} < d_{n-1} - 1$.
- $d_n = f(d_{n-1})$.
- $d_{n-1} \ge 3$.

Use

$$\frac{1}{d_{n-2}} + \frac{1}{d_{n-1}} + \frac{1}{f_2(d_{n-1})} =$$

$$\frac{1}{d_{n-2}} + \frac{1}{d_{n-1}-1} + \frac{1}{(d_{n-1}-1)(d_{n-1}-2)+2} + \frac{1}{f_2((d_{n-1}-1)(d_{n-1}-2)+2)}$$

to go from $P(n-1)$ to $P(n)$. Use $d_{n-1} \ge 3$ to prove

$$d_{n-1} - 1 < (d_{n-1}-1)(d_{n-1}-2)+2.$$

Def 3.1 Let $a \in \mathsf{N}$. Let $f_a(x) = \frac{x(x-a)}{a}$.

Lemma 3.2 *Let $a \geq 1$ and $b, d, x \in \mathsf{N}$.*

1.
$$\frac{1}{d} + \frac{1}{f_a(d)} = \frac{1}{d-a}$$

2.
$$\frac{1}{x} + \frac{1}{x(x-1)+a} + \frac{1}{f_a(x(x-1)+a)} = \frac{1}{x-1}$$

3.
$$\frac{1}{d-a+1}+\frac{1}{(d-a+1)(d-a)+a}+\frac{1}{f_a((d-a+1)(d-a)+a)} = \frac{1}{d-a}$$

(This follows from item 2 with $x = d-a+1$.)

4.
$$\frac{1}{b} = \frac{1}{b+1} + \frac{1}{b(b+1)+a} + \frac{1}{f_a(b(b+1)+a)}$$

(This follows from item 2 with $x = b+1$. We use b to be consistent with a later use.)

5.
$$\frac{1}{d}+\frac{1}{f_a(d)} = \frac{1}{d-a+1}+\frac{1}{(d-a+1)(d-a)+a}+\frac{1}{f_a((d-a+1)(d-a)+a)}$$

(This follows from items 1 and 2)

Proof:
1)
$$\frac{1}{d} + \frac{a}{d(d-a)} = \frac{d-a}{d(d-a)} + \frac{a}{d(d-a)} = \frac{d}{d(d-a)} = \frac{1}{d-a}$$

2) We use the following:

$$\frac{1}{f_a(x(x-1)+a)} = \frac{a}{(x(x-1)+a)(x(x-1)} = \frac{1}{x(x-1)} - \frac{1}{x(x-1)+a}$$

Note that

$$\frac{1}{x} + \frac{1}{x(x-1)+a} + \frac{1}{f_a(x(x-1)+a)} =$$

$$\frac{1}{x} + \frac{1}{x(x-1)+a} + \frac{1}{x(x-1)} - \frac{1}{x(x-1)+a} =$$

$$\frac{1}{x} + \frac{1}{x(x-1)} = \frac{x-1}{x(x-1)} + \frac{1}{x(x-1)} = \frac{x}{x(x-1)} = \frac{1}{x-1}.$$

▮

Def 3.3 Let $a \geq 1$, $n \geq 3$. A *nice* (n,a)*-sequence* is a nice n-sequence $(d_1, \ldots, d_n)$ such that:

1. $d_{n-1} \equiv 0 \pmod a$
2. $d_{n-2} < d_{n-1} - a + 1$
3. $d_n = f_a(d_{n-1})$
4. $d_{n-1} \geq a + 1$.

Lemma 3.4 *Let $a \geq 1$, $n \geq 3$. If there exists a nice n-sequence $(b_1, \ldots, b_n)$ such that $b_n \equiv 0 \pmod a$ and $b_{n-1} \geq a+1$ then there exists a nice $(n+3, a)$-sequence.*

Proof: Assume there exists a nice n-sequence $(b_1, \ldots, b_n)$ such that $b_n \equiv 0 \pmod a$. Let k be such that $b_n = ak$. Using this and Lemma **??**.3: we have

$$\begin{aligned} \tfrac{1}{b_n} = \tfrac{1}{b_n+1} + \tfrac{1}{b_n(b_n+1)+a} &= \tfrac{1}{b_n+1} + \tfrac{1}{ak(b_n+1)+a} \\ &= \tfrac{1}{b_n+1} + \tfrac{1}{a(k(b_n+1)+1)} + \tfrac{1}{f_a(a(k(b_n+1)+1))} \end{aligned}$$

Hence

$$\frac{1}{b_1} + \cdots + \frac{1}{b_{n-1}} + \frac{1}{b_n+1} + \frac{1}{a(k(b_n+1)+1)} + \frac{1}{f_a(a(k(b_n+1)+1))} = 1$$

Take $d_1 = b_1$, …, $d_{n-1} = b_{n-1}$, $d_n = b_n + 1$, $d_{n+1} = a(k(b_n+1)+1)$, $d_{n+2} = f_a(a(k(b_n+1)+1)$.

Conditions 1,3,4 are clearly true. Condition 2 holds by easy algebra ▮

Theorem 3.5 *Let $a \geq 1$, $n \geq 3$. For all but a finite number of n there exists a nice (n,a)-sequence*

Proof: We prove this by induction on n. We do not know what the base case is; however, one can use our proof of the base case to find it.
Base Case: By Lemma **??**.2 there exists $m \in \mathsf{N}$ and a nice m-sequence where the last term (in fact all terms, though we do no need that) are composite. By Lemma **??** there exists a nice $(m+3,a)$-sequence.

Induction Step: Assume $(d_1,\ldots,d_{n-1},d_n)$ is a nice (n,a)-sequence. Let $d_{n-1}=d$. Note that $d \equiv 0 \pmod a$ and $d_n = f(d)$. By Lemma **??**.5:

$$\frac{1}{d_{n-2}}+\frac{1}{d_{n-1}}+\frac{1}{f_a(d_{n-1})}=$$

$$\frac{1}{d_{n-2}}+\frac{1}{d_{n-1}-a+1}+\frac{1}{(d_{n-1}-a+1)(d_{n-1}-a)+a}+\frac{1}{f_a((d_{n-1}-a+1)(d_{n-1}-a)+a)}$$

We claim that

$$(d_1,\ldots,d_{n-2},d_{n-1}-a+1,(d_{n-1}-a+1)(d_{n-1}-a)+a,f_a((d_{n-1}-a+1)(d_{n-1}-a)+a))$$

is a nice $(n+1,a)$-sequence.

We first just prove that it's a nice n-sequence. Clearly the sum of the reciprocals adds to 1. Clearly $d_1<\cdots<d_{n-2}$ inductively. We have $d_{n-2}<d_{n-1}-a+1$ inductively since that is condition 2 for nice (n,a)-sequences. By algebra

$$d_{n-1}-a+1<(d_{n-1}-a+1)(d_{n-1}-a)+a<f_a((d_{n-1}-a+1)(d_{n-1}-a)+a))$$

We now prove the conditions for being a nice (n,a)-sequence. Since (the old) $d_{n-1}\equiv 0 \pmod a$, $d_{n-1}-a\equiv 0 \pmod a$ and hence

$$(d_{n-1}-a+1)(d_{n-1}-a)+a\equiv 0 \pmod a.$$

We need $(d_{n-1}-a+1)<((d_{n-1}-a+1)(d_{n-1}-a)+a)-a+1$. This is true by algebra.

We need $d_{n+1}=f(d_n)$. This is clearly true. ∎

The proof of Theorem **??** gives no bound on n_0. The following alternative proof does.

Theorem 3.6 *Let $a \in \mathsf{N}$. For all $n \ge a^{O(a)}$ there exists a nice (n,a)-sequence .*

Proof:

We need to find a $(a^{(a+o(1)a}, n)$-sequence for our base case. After that we use the induction step as in the proof of Theorem **??**.

Claim 1: For all primes p there exists a nice sequence of length $\le p^{O(p)}$ such that p divides the last term.

Proof of Claim 1:

We define operations on nice sequences. These operations will do most of the work for us.

1. Assume $(c_1, \ldots, c_n)$ and $(d_1, \ldots, d_m)$ are nice. We define

$$M(c_1, \ldots, c_n, d_1, \ldots, d_m) = (c_1, \ldots, c_{n-1}, c_n d_1, c_n d_2, \ldots, c_n d_m).$$

   It is easy to see that the output of M is a nice $(n+m-1)$-sequence.

2. Assume $(c_1, \ldots, c_n)$ is n-nice. We define

$$E(c_1, \ldots, c_n) = (c_1, \ldots, c_{n-1}, c_n + 1, c_n(c_n + 1)).$$

   It is easy to see that the output of E is a nice $(n+1)$-sequence.

3. Assume $\vec{c}$ is nice and ends with t. Assume p does not divide t. Let

$$\vec{d} = E(\vec{c})$$

$$\vec{c}_2 = M(\vec{c}, \vec{d})$$

$$(\forall i \ge 3)[\vec{c}_i = M(\vec{c}_{i-1}, \vec{c})].$$

   Let $F(\vec{c}) = \vec{c}_{p-1}$. It is easy to see that $F(\vec{c})$ is a nice $(p(n-1)-n+3)$-sequence (we will later just bound this by pn). The last term of $F(\vec{c})$ is $t^{p-1}(t+1)$. Since p does not divide t, by Fermat's little theorem, the last term is $\equiv t+1 \pmod{p}$.

4. Assume $\vec{c}$ is nice and ends with a number that is $\equiv t \pmod{p}$. Assume p does not divide t. Then $F^{(i)}(\vec{c})$ is a nice $\le (pn)^i$-sequence whose last term is $\equiv t+i \pmod{p}$.

If $p=2$ or $p=3$ then we use the sequence $(2,3,6)$. Assume $p \ge 5$. Let $\vec{c}=(2,3,6)$. Let t be such that $6 \equiv t \pmod{p}$. Then $F^{(p-t)}(\vec{c})$ is a nice $(3p)^{p-t}$-sequence with last term $\equiv t+(p-t) \equiv 0 \pmod{p}$. Note that $(3p)^{p-t} \le p^{O(p)}$.

End of Proof of Claim 1

Let $a = p_1^{e_1} \cdots p_L^{e_L}$. By Claim 1 we can create, for each $1 \le i \le L$, a nice $(p_i)^{O(p_i)}$-sequence.

$\vec{c_i}$ whose last term is divisible by p_i.

If $\vec{c}$ is a nice n_1-sequence with last term $\equiv 0 \pmod{p}$ and $\vec{d}$ is a nice n_2-sequence with last term $\equiv 0 \pmod{q}$ then $M(\vec{p},\vec{q})$ is a nice (n_1+n_2-1)-sequence with last term $\equiv 0 \pmod{pq}$. Hence

$$M(\vec{c}_1,\vec{c}_1,\ldots,\vec{c}_1,\vec{c}_2,\ldots,\vec{c}_2,\ldots,\vec{c}_n)$$

(where we take each c_i e_i times) is a nice sequence of length

$$\sum_{i=1}^{L} e_i (p_i)^{O(p_i)}$$

Since $e_i \le \log a$, $L \le \log a$, and $p_i \le a$, this sum is

$$\le (\log a)^2 a^{O(a)} \le a^{O(a)}.$$

The last term is divisible by a. Since $a^{O(a)}+3 \le a^{O(a)}$ by Lemma **??**, we have a nice $a^{O(a)}$-sequence. ∎

The proof of Theorem **??** gives the bound $n_0 \le a^{O(a)}, a$. In the appendix we give empirical evidence that indicates $n_0 \le O(\log a)$.

4 Another Infinite Number of Proofs

We first give two proofs that a high school student taking the exam could have given but just happened not to.

Theorem 4.1 *For all $n \ge 3$, $P(n)$ holds.*

Proof: We use base3 for the base case. We load the induction hypothesis with the assumption that $d_n \equiv 0 \pmod 6$.

SOLUTION FIVE-a: Assume $(d_1, \ldots, d_n)$ is a nice sequence. Assume $d_n = 6d$. Then since $\frac{1}{6d} = \frac{1}{9d} + \frac{1}{18d}$ $(d_1, \ldots, d_{n-1}, 9d, 18d)$ is a nice sequence.

SOLUTION FIVE-b: Assume $(d_1, \ldots, d_n)$ is a nice sequence. Assume $d_n = 6d$. Then since $\frac{1}{6d} = \frac{1}{8d} + \frac{1}{24d}$ $(d_1, \ldots, d_{n-1}, 8d, 24d)$ is a nice sequence. ▮

The next theorem generates an infinite number of proofs using the idea of Theorem **??**.

Theorem 4.2

1. *If there exists a nice sequence of length n_0 with its last term composite then, for all $n \geq n_0$, $P(n)$.*

2. *There exists an infinite number of nice sequence of last term composite.*

Proof:
1) Let $(c_1, \ldots, c_{n_0})$ be the nice sequence with last term composite. Let e be a nontrivial factor of c_{n_0}. Note that:

- $\frac{1}{c_{n_0}} = \frac{1}{c_{n_0}(e+1)/e} + \frac{1}{c_{n_0}(e+1)}$ (note that since e divides c_{n_0} we are writing $\frac{1}{c_{n_0}}$ as the sum of two reciprocals), and

- $c_{n_0}(e+1)/e < c_{n_0}(e+1)$.

We prove that, for all $n \geq n_0$, there exists a nice sequence of length n with last term divisible by c_{n_0}.
Base Case: Use $(c_1, \ldots, c_{n_0})$.
Induction Step: Assume that there is a nice sequence of length n, $(d_1, \ldots, d_n)$ with $d_n \equiv 0 \pmod{c_{n_0}}$. Let $d_n = c_{n_0}x$. Then $\frac{1}{d_n} = \frac{1}{c_{n_0}x} = \frac{1}{c_{n_0}x(e+1)/e} + \frac{1}{c_{n_0}x(e+1)}$. Hence $(d_1, \ldots, d_{n-1}, c_{n_0}x(c_{n_0}+1)/e, c_{n_0}x(c_{n_0}+1))$ is a nice sequence of length $n+1$ with last term divisible by c_{n_0}.

2) This follows from by Lemma **??**.2. ▮

In the proof of Theorem **??** we write $\frac{1}{c_{n_0}}$, with the aid of a divisor e, as $\frac{1}{x}+\frac{1}{y}$, where y divides c_{n_0}. In the table below we show what this sum looks like for $4 \le c_{n_0} \le 12$ and possible e.

c_{n_0}	e	$e+1$	$y = \frac{c_{n_0}(e+1)}{e}$	$\frac{1}{c_{n_0}} = \frac{1}{c_{n_0}(e+1)/e} + \frac{1}{c_{n_0}(e+1)}$
4	2	3	6	$\frac{1}{4}=\frac{1}{6}+\frac{1}{12}$
6	2	3	9	$\frac{1}{6}=\frac{1}{9}+\frac{1}{18}$
6	3	4	8	$\frac{1}{6}=\frac{1}{8}+\frac{1}{24}$
8	2	3	12	$\frac{1}{8}=\frac{1}{12}+\frac{1}{24}$
8	4	5	10	$\frac{1}{8}=\frac{1}{10}+\frac{1}{40}$
9	3	4	12	$\frac{1}{9}=\frac{1}{12}+\frac{1}{36}$
10	2	3	15	$\frac{1}{10}=\frac{1}{15}+\frac{1}{30}$
10	5	6	12	$\frac{1}{10}=\frac{1}{12}+\frac{1}{60}$
12	2	3	18	$\frac{1}{12}=\frac{1}{18}+\frac{1}{36}$
12	3	4	16	$\frac{1}{12}=\frac{1}{16}+\frac{1}{48}$
12	4	5	15	$\frac{1}{12}=\frac{1}{15}+\frac{1}{60}$
12	6	7	14	$\frac{1}{12}=\frac{1}{14}+\frac{1}{84}$

A The Students Answers to Part a

The students submitted 32 correct solutions to Part a. We list all correct submitted solutions in lexicographic order, along with how many students submitted each one. We also note which of SOLUTION ONE, TWO, THREE, FOUR, FIVE-a, FIVE-b would lead to the answer they gave. For example, since we gave $(2,3,6)$ and $(2,3,7,42)$ as solutions, and SOLUTION ONE takes $(2,3,7,42)$ and produces $(2,3,7,43,1886)$, that solution to Part a is linked to SOLUTION ONE to Part b.

Solution	Numb	Comment
(2,3,7,43,1806)	91	Linked to SOLUTION ONE.
(2,3,7,48,336)	3	
(2,3,7,56,168)	1	
(2,3,7,63,126)	6	Linked to SOLUTION THREE.
(2,3,7,70,105)	1	
(2,3,8,25,600)	1	
(2,3,8,30,120)	1	
(2,3,8,32,96)	6	Linked to SOLUTION FIVE-b
(2,3,8,36,72)	5	
(2,3,8,42,56)	11	
(2,3,9,21,126)	2	
(2,3,9,24,72)	4	
(2,3,9,27,54)	3	Linked to SOLUTION FIVE-a
(2,3,10,20,60)	5	
(2,3,11,22,33)	1	
(2,3,12,15,60)	1	
(2,3,12,16,48)	1	
(2,3,12,14,84)	2	Linked to SOLUTION FOUR.
(2,3,12,18,36)	12	Linked to SOLUTION TWO.
(2,4,5,25,100)	3	
(2,4,5,30,60)	1	
(2,4,6,14,84)	3	
(2,4,6,16,48)	1	
(2,4,6,18,36)	2	
(2,4,6,20,30)	1	
(2,4,7,12,42)	4	
(2,4,7,14,28)	2	
(2,4,8,12,24)	6	
(2,4,8,10,40)	2	
(2,5,6,10,30)	1	
(2,5,6,12,20)	2	
(3,4,5,6,20)	3	

B For $1 \leq a \leq 149$ What is Smallest n_0?

In Theorem **??** and **??** we did, for each a, find an n_0 and a proof that for all $n \geq n_0$ there is a nice n-sequence. In Theorem **??** no bound on n_0 was given (though one could probably be derived), and in Theorem **??** we obtained $n_0 \leq 2^a a^{(1+o(1))a}$.

We wrote a program that would for $0 \leq a \leq 149$, find the first n_0 that works, and found how many nice (n_0, a)-sequences there are. In the table below we list the results. We include the sequence itself. We separate out the last three terms of each sequence since that is where the conditions apply.

Based on the data it looks like a bound of $n_0 \leq O(\log n)$ may be true. This quite far from our current theoretical bounds.

I HAVE COMMENTED OUT THE GRAPH FOR NOW SINCE I HAD A HARD TIME INTERFACING WITH IT AND YOU WILL BE CHANGING IT ANYWAY.

The above graph depicts the actual values of n_0 (minimum number of terms needed to have a base case in Theorem 3.6). The green curve is a graph of $5 + \log(a + 1)$; the orange curve represents $4 + \log(a)$ and gives a tighter bound. Note that the minimum number of terms for $a = 136$ has not yet been found, but is expected to be 8.

a	n_0	# distinct base cases	$d_1, \ldots, d_{n_0-3}$	d_{n_0-2}	d_{n_0-1}	d_{n_0}
0	3	1		2	3	6
1	4	2	2	3	8	24
				4	6	12
2	4	1	2	3	9	18
3	5	5	$2, 3$	8	28	168
			$2, 3$	12	16	48
			$2, 4$	5	24	120
			$2, 4$	6	16	48
			$2, 4$	8	12	24
4	5	2	$2, 3$	10	20	60
			$2, 4$	5	25	100
5	5	5	$2, 3$	7	48	336
			$2, 3$	8	30	120
			$2, 3$	9	24	72
			$2, 3$	12	18	36
			$2, 4$	6	18	36
6	5	1	$2, 3$	7	49	294
7	5	1	$2, 3$	8	32	96
8	5	1	$2, 3$	9	27	54
9	5	1	$2, 4$	5	30	60
10	6	7	$2, 3, 7$	44	935	78540
			$2, 3, 8$	33	99	792
			$2, 3, 9$	22	110	990
			$2, 3, 11$	14	242	5082
			$2, 3, 11$	15	121	1210
			$2, 3, 11$	22	44	132
			$2, 4, 5$	22	231	4620
11	5	1	$2, 3$	8	36	72
12	6	4	$2, 3, 7$	78	104	728
		4	$2, 3, 8$	26	325	7800
		4	$2, 3, 12$	13	169	2028
		4	$2, 4, 6$	13	169	2028
13	5	1	$2, 3, 7$	43	1820	234780

a	n_0	# distinct base cases	$d_1, \ldots, d_{n_0-3}$	d_{n_0-2}	d_{n_0-1}	d_{n_0}
14	6	22	$2,3,7$	45	645	27090
			$2,3,7$	70	120	840
			$2,3,8$	25	615	24600
			$2,3,8$	30	135	1080
15	6	6	$2,3,7$	48	352	7392
16	6	1	$2,3,7$	51	255	3570
17	6	15	$2,3,7$	45	648	22680
18	6	1	$2,3,9$	19	361	6498
19	6	19	$2,3,7$	45	650	20475
20	5	1	$2,3$	7	63	126
21	6	4	$2,3,7$	44	946	39732
22	6	1	$2,3,7$	46	506	10626
23	6	17	$2,3,7$	44	948	36498
24	6	2	$2,3,8$	25	625	15000
25	6	4	$2,3,7$	91	104	312
26	5	3	$2,3,7$	45	657	15330
27	6	11	$2,3,7$	44	952	31416
28	7	63	$2,3,7,43$	1827	157151	851444118
29	6	14	$2,3,7$	45	660	13860
30	7	55	$2,3,7,43$	1953	24025	18595350
31	6	1	$2,3,7$	48	368	3864
32	6	3	$2,3,8$	32	128	384
33	6	1	$2,3,7$	51	272	1904
34	5	6	$2,3,7$	45	665	11970
35	6	5	$2,3,8$	27	252	1512
36	7	34	$2,3,7,43$	1813	467791	5913813822
37	5	1	$2,3$	9	380	3420
38	6	4	$2,3,7$	91	117	234
39	6	5	$2,3,8$	25	640	9600
40	7	28	$2,3,7,45$	738	4346	456330

a	n_0	# distinct base cases	$d_1, \ldots, d_{n_0-3}$	d_{n_0-2}	d_{n_0-1}	d_{n_0}
41	6	13	$2, 3, 7$	43	1848	79464
42	6	1	$2, 3, 7$	43	1849	77658
43	6	4	$2, 3, 7$	44	968	20328
44	6	3	$2, 3, 7$	45	675	9450
45	7	83	$2, 3, 7, 43$	1932	27738	16698276
46	7	27	$2, 3, 7, 44$	940	54332	62753460
47	6	3	$2, 3, 7$	48	384	2688
48	6	1	$2, 3, 7$	49	343	2058
49	6	2	$2, 3, 8$	25	650	7800
			$2, 4, 5$	25	150	300
50	7	127	$2, 3, 7, 43$	1904	35139	24175632
51	6	3	$2, 3, 8$	26	364	2184
52	7	18	$2, 3, 7, 43$	1855	68423	88265670
53	6	1	$2, 3, 8$	27	270	1080
54	6	2	$2, 3, 11$	15	165	330
			$2, 4, 5$	22	275	1100
55	6	3	$2, 3, 7$	48	392	2352
			$2, 3, 7$	56	224	672
			$2, 3, 8$	28	224	672
56	6	1	$2, 3, 9$	19	399	2394
57	7	52	$2, 3, 7, 43$	1827	157180	425800620
58	7	11	$2, 3, 7, 45$	826	2714	122130
59	6	7	$2, 3, 8$	25	660	6600
60	7	15	$2, 3, 7, 44$	1708	2074	68442
61	7	52	$2, 3, 7, 43$	1953	24056	9309672
62	6	4	$2, 3, 7$	45	693	6930
63	7	48	$2, 3, 7, 43$	2688	5568	478848
64	7	124	$2, 3, 7, 43$	1820	234845	848260140
65	6	1	$2, 3, 7$	44	990	13860
66	7	6	$2, 3, 7, 43$	1809	1089085	17701987590
67	7	100	$2, 3, 7, 43$	1904	35156	18140496
68	6	1	$2, 3, 7$	46	552	3864
69	6	4	$2, 3, 7$	45	700	6300
70	7	6	$2, 3, 7, 45$	639	44801	28224630
71	6	1	$2, 3, 8$	27	288	864
72	7	11	$2, 3, 7, 44$	1022	9709	1281588

a	n_0	# distinct base cases	$d_1, \ldots, d_{n_0-3}$	d_{n_0-2}	d_{n_0-1}	d_{n_0}
73	7	28	$2, 3, 7, 43$	1813	467828	2957140788
74	6	1	$2, 3, 8$	25	675	5400
75	7	72	$2, 3, 7, 43$	1824	183084	440866272
76	6	2	$2, 3, 7$	44	1001	12012
			$2, 3, 11$	14	308	924
77	6	3	$2, 3, 8$	26	390	1560
78	7	10	$2, 3, 7, 44$	948	36577	16898574
79	6	1	$2, 3, 10$	16	320	960
80	7	24	$2, 3, 7, 44$	972	18792	4340952
81	7	23	$2, 3, 7, 46$	492	26486	8528492
82	7	4	$2, 3, 7, 44$	996	12865	1981210
83	6	5	$2, 3, 7$	44	1008	11088
84	7	70	$2, 3, 7, 43$	3570	3740	160820
85	6	1	$2, 3, 7$	43	1892	39732
86	7	43	$2, 3, 7, 43$	1827	157209	283919454
87	7	116	$2, 3, 7, 43$	1848	79552	71835456
88	7	6	$2, 3, 7, 43$	1869	53667	32307534
89	6	2	$2, 3, 7$	45	720	5040
			$2, 3, 9$	20	270	540
90	7	100	$2, 3, 7, 43$	1807	3263553	117036820446
91	7	50	$2, 3, 7, 43$	1932	27784	8362984
92	7	39	$2, 3, 7, 43$	1953	24087	6214446
93	7	19	$2, 3, 7, 44$	987	14570	2243780
94	7	47	$2, 3, 7, 44$	1045	8075	678300
95	7	90	$2, 3, 7, 44$	928	214464	478898112
96	7	2	$2, 3, 7, 45$	679	8827	794430
			$2, 3, 8, 32$	97	9409	903264
97	6	1	$2, 3, 7$	49	392	1176
98	7	87	$2, 3, 7, 44$	927	285615	823713660
99	6	1	$2, 3, 8$	25	700	4200
100	7	5	$2, 3, 7, 43$	1818	273710	741480390
101	7	95	$2, 3, 7, 43$	1904	35190	12105360
102	7	2	$2, 3, 7, 44$	927	285619	791735868
			$2, 3, 8, 25$	618	20703	4140600
103	6	1	$2, 3, 8$	26	416	1248
104	6	2	$2, 3, 7$	45	745	4410
			$2, 4, 5$	21	525	2100
105	7	14	$2, 3, 7, 43$	1855	68476	44167020

a	n_0	# distinct base cases	$d_1, \ldots, d_{n_0-3}$	d_{n_0-2}	d_{n_0-1}	d_{n_0}
106	7	4	$2, 3, 7, 43$	2247	9309	800574
107	6	1	$2, 3, 8$	27	324	648
108	7	2	$2, 3, 8, 27$	218	23653	5109048
			$2, 4, 5, 21$	436	11554	1213170
109	6	1	$2, 4, 5$	22	330	660
110	7	26	$2, 3, 7, 43$	1813	467865	1971583110
111	6	1	$2, 3, 7$	48	448	1344
112	7	2	$2, 3, 7, 43$	1813	467865	1971583110
			$2, 3, 7, 48$	339	38081	12795216
113	6	1	$2, 3, 9$	19	456	1368
114	7	50	$2, 3, 7, 45$	644	29095	7331940
115	7	44	$2, 3, 7, 44$	928	214484	396366432
116	7	69	$2, 3, 7, 44$	936	72189	44468424
117	7	6	$2, 3, 7, 48$	354	6726	376656
118	6	1	$2, 3, 7$	51	357	714
119	6	2	$2, 3, 8$	25	720	3600
			$2, 3, 10$	16	360	720
120	7	14	$2, 3, 7, 43$	1815	364331	1096636310
121	7	14	$2, 3, 7, 61$	135	115412	109064340
122	7	18	$2, 3, 7, 45$	738	4428	154980
123	7	36	$2, 3, 7, 44$	930	143344	165562320
124	7	11	$2, 3, 7, 45$	875	2375	42750
125	6	1	$2, 3, 7$	45	756	3780
126	7	3	$2, 3, 7, 45$	635	80137	50486310
127	7	16	$2, 3, 7, 43$	2688	5632	242176
128	6	1	$2, 3, 7$	43	1935	27090
129	7	90	$2, 3, 7, 43$	1820	234910	424247460
130	7	2	$2, 3, 7, 43$	1834	118424	106936872
			$2, 3, 7, 45$	655	16637	20962
131	6	1	$2, 3, 7$	44	1056	7392
132	7	48	$2, 3, 7, 43$	1824	183141	252002016
133	7	4	$2, 3, 7, 43$	1809	1089152	8851538304
134	7	67	$2, 3, 7, 43$	1890	40770	12271770
135	7	41	$2, 3, 7, 43$	1904	35224	9087792
136	8+					
137	7	42	$2, 3, 7, 44$	966	21390	3294060
138	7	2	$2, 3, 7, 43$	1807	3263581	76622354718
			$2, 3, 7, 44$	973	18487	2440284

a	n_0	# distinct base cases	$d_1, \ldots, d_{n_0-3}$	d_{n_0-2}	d_{n_0-1}	d_{n_0}
139	6	1	$2, 4, 5$	21	560	1680
140	7	18	$2, 3, 7, 44$	940	54426	20954010
141	7	6	$2, 3, 7, 45$	639	44872	14134680
142	7	27	$2, 3, 7, 43$	2002	18590	2398110
143	7	67	$2, 3, 7, 44$	1008	11232	864864
144	7	14	$2, 3, 7, 45$	1218	1450	13050
145	7	9	$2, 3, 7, 44$	1022	9782	645612
146	6	1	$2, 3, 7$	49	441	882
147	7	18	$2, 3, 7, 44$	925	854848	4936747200
148	7	1	$2, 3, 7, 49$	298	22052	3241644
149	6	1	$2, 3, 8$	25	750	3000