The Tug of War Game

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Abstract

The tug of war game is a type of Richman game played on a line graph. Initially there is a token at some node. Linda tries to get the token to the leftmost node, while Rick tries to get the token to the rightmost node. Each player starts out with an amount of money (not necessarily equal) . One player has the "initiative." On each turn each player bids some of their money. The player with the higher bid moves the token one node in their direction, and gives the amount bid to the other player. If the bids are tied then the player with the initiative wins the bid.

We consider a number of variants of the game, which depend on (1) whether money is continuous or discrete, (2) whether the strategies are adaptive or nonadaptive. In the continuous version we determine, given the initial conditions, which player wins; draws are essentially not possible. We also obtain bounds on how many moves it takes to win. In the discrete case draws are possible. We determine, given the initial conditions, which player wins, or if it is a draw. We focus on the situation where one player always has the initiative; however, most of our results hold for any tie-breaking mechanism.

1 Introduction

Tug of War is a two-player game played on a line graph with N nodes, numbered $0, 1, 2, \ldots, N-1$. We call the two players Linda (for Left) and Rick (for Right). There is a single token, which is always on some node; the initial (or start) node is denoted p. Linda tries to move the token left to node 0, and Rick tries to move the token right to node $N-1$. Each player has an initial amount of money, L dollars for Linda and R dollars for Rick. At each turn, the players bid to move the token along one edge (towards that players goal). The player who makes the higher bid moves the token and pays the amount bid to the other player. One player has the initiative. In case of a tie, the player with the initiative wins the bid. The game ends when the token is either on node 0, in which case Linda wins, or on node $N-1$, in which case Rick wins. If neither player has enough money to win, the game is a *draw*.

We consider a number of variants of the game, which depend on (1) whether money is continuous or discrete, (2) whether the strategy is adaptive or nonadaptive. In the *continuous version* the initial amounts of money L and R are real numbers, and the bids are real numbers. In the *discrete version* the initial amounts of money L and R are natural numbers, and the bids must be natural numbers. In a *nonadaptive* strategy the amount of money a player bids depends only on the position of the token.

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A player must set the amount to be bid at each node at the beginning of the game. The amount can be different at different nodes. A player who does not have enough money to make the required bid must bid 0. In an *adaptive* strategy the amount of money a player bids depends on the position of the token *and* on how much money each player has. We focus on the situation where one player (Linda) always has the initiative; however, most of our results hold for any tie-breaking mechanism. We discuss this in the Appendix.

Throughout the paper we assume Linda has the initiative unless otherwise stated.

Summary of Results

We show that in the continuous version Linda wins if $\frac{L}{p} > \frac{R}{N-p-1}$ and Rick wins if $\frac{R}{N-p-1} > \frac{L}{p}$. If $\frac{L}{p} = \frac{R}{N-p-1}$ then the outcome depends on several factors and can, in fact, be a draw. We describe this precisely in Section 3.4.1. We initially present these results assuming that Linda has the initiative; however, we then show how to remove this condition.

We give formulas for how many turns it takes to win when there are N nodes, start position p, Linda has L dollars, and Rick has R dollars. In this summary we just describe the case where N is odd and the token is in the middle. In all cases the number-of-moves is independent of who has the initiative.

If $L > R$, then using a nonadaptive strategy Linda wins within about

$$\frac{N^2(L+R)}{2(L-R)} - \frac{3N}{2}$$

turns. If Linda uses an adaptive strategy then she can win within about

$$\frac{N^2}{4} \ln\left(\frac{L+R}{L-R}\right) + \frac{N}{2}$$

moves. In both the adaptive and nonadaptive case if $R > L$ then Rick wins, and the formula is similar but reversed. When $L = R$ and some player wins, the number of turns needed to win is arbitrarily large, unless the win is immediate.

We show that in the discrete version assuming Linda always has the initiative she wins if

$$\left\lfloor \frac{L}{p} \right\rfloor \geq \left\lceil \frac{R+1}{N-p-1} \right\rceil - 1 \ .$$

We also give exact conditions for when Rick wins, but the equation is slightly more complicated. Ignoring floors and ceilings and assuming $L \geq p^2/2$, Rick wins if approximately

$$\frac{R+1}{N-p-1} \geq \frac{L}{p} + \frac{N+1}{2} \ .$$

Otherwise the game is a draw. As we can see, in the discrete case there is a range of about $N/2$ dollars where the game is a draw.

If the initiative alternates each turn and $L, R \geq p^2/4$, then Linda wins if approximately

$$\frac{L}{p} \geq \frac{R+1}{N-p-1} + \frac{N+1}{4} \ ,$$

Rick wins if approximately

$$\frac{R+1}{N-p-1} \geq \frac{L}{p} + \frac{N+1}{4} \ ,$$

and otherwise it is a draw. Once again there is a range of about $N/2$ dollars where the game is a draw, but it has shifted by $N/4$.

Comparison to previous work

Tug of war is a special case of Richman games, named after the mathematician David Richman, Richman games are played on a general directed graphs rather than just undirected paths. Each player has a distinct node to which he or she is trying to move the token in order to win the game. The player who wins the bid (at a turn) can choose to move the token along any incident outgoing edge to a neighboring node. Richman games were originally studied by Lazarus et al. [1, 2].

Let $M = L + R$ be the total money. Then we can rewrite the condition for Linda winning the continuous version as $L > \frac{pM}{N-1}$, and the condition for Rick winning as $L < \frac{pM}{N-1}$ (or, equivalently, $R > \frac{(N-p-1)M}{N-1}$). This is essentially the way Lazarus, et al. write the condition except they do not include the M by normalizing money to total 1. This style is perhaps simpler than the way we write it, but our way has several advantages: (1) It explicitly shows the symmetry between L and R, which is especially useful when determining how many turns it takes to win using an adaptive strategy. (2) Because of the floors and ceilings, this style may be necessary in order to write the formulas for who wins in the discrete version.

Lazarus, et al. [1, 2] did the following. Let $G = (V, E)$ be a directed graph. They defined, for each node v, a sequence $R(v, 0), R(v, 1), \ldots$ that converges to the fraction of the money one of the players needs to win. If it converges in a finite number of steps t, then that player can win in t steps.

Our work differs from that of Lazarus, et al. [1, 2] in that (1) they only considered adaptive strategies and the continuous case, (2) they did not deal with the number-of-moves explicitly, and (3) they did not consider the discrete case.

Tug-of-war is especially interesting because it is the simplest non-trivial Richman game.

2 Preliminaries

Recall that N is the number of nodes, p is the start node, and L and R are the amount of money that Linda and Rick start with, respectively. These four variables, along with the rule of who has the initiative, define the game. We will always use t for the number of turns so far in the game, and q for the *current position* of the token (after t turns).

Observation 2.1 *Each edge in the graph between the start node p and the current node q must have been traversed. Other than those $|q-p|$ traversals, each edge is traversed exactly the same number of times in each direction. Let T be the total number of such pairs of traversals. Then the total number of turns taken so far in the game $t = 2T + |q-p|$.*

3 Continuous case

We begin with the continuous version. It gives insights into the discrete version and is easier to analyze.

3.1 When Linda Wins, Given that She Always Has the Initiative

Throughout this section we assume Linda always has the initiative. We analyze both the adaptive and nonadaptive cases. We determine when Linda wins and give an upper bound on how many moves it takes.

3.1.1 Nonadaptive Strategies

We assume that Linda must use a nonadaptive strategy. We make no restriction on Rick's strategy.

Assume Linda sets her bids so that at node i she bids B_i. Note that the B_i must be be strictly decreasing. Otherwise, if for some i, $B_i < B_{i+1}$, Rick can force the token to move back-and-forth between nodes i and $i+1$ while retaining his money by winning the bid at node i for slightly more than B_i dollars, and letting Linda win the bid at node $i+1$ for B_{i+1} dollars. In this way Rick can increase his money arbitrarily and actually win the game.

Even if for some two nodes i and $i+1$, $B_i = B_{i+1}$, Rick can force the token to move back-and-forth between those two nodes forever by using an *exponential decay of bids*: At the kth such iteration, Rick wins the bid at node i by bidding $B + \epsilon/2^k$ for some arbitrarily small value $\epsilon > 0$ (and loses the bid at node $i+1$). In this way, he loses less than ϵ dollars after any finite number of turns. Hence he can keep doing this forever, forcing a draw (although he will continually lose money). Note that this strategy of Rick's does not work in the discrete case.

Intuitively, for Linda to guarantee a win the bid values should have three properties:

- They should be small enough so that Linda has enough money to win, i.e., $L \geq \sum_{i=1}^{p} B_i$.
- They should be large enough so that Rick does not have enough money to win, i.e., $R \leq \sum_{i=p}^{N-2} B_i$.
- They should be strictly decreasing (as noted above).

This is formalized in the following lemma.

Lemma 3.1 *Let B_i be a strictly decreasing function such that $L \geq \sum_{i=1}^{p} B_i$ and $R \leq \sum_{i=p}^{N-2} B_i$. If Linda always has the initiative then bidding B_i at node i is a winning strategy.*

Proof: Assume that the game has gone for t turns, and that the token is currently at node q. For each pair of traversals back-and-forth across an edge, say $(i, i+1)$, Linda gains and Rick loses (more than) $B_i - B_{i+1}$ dollars. Let b be the minimum of $B_i - B_{i+1}$ (for $1 \leq i < N-2$). Let T be as in Observation 2.1. By that observation Rick loses at least Tb for the T turns in which the token crosses back-and-forth across an edge. Furthermore, if $q \leq p$, Rick gains $\sum_{i=q+1}^{p} B_i$ for the remaining moves from p to q, so Rick's total money is $R - Tb + \sum_{i=q+1}^{p} B_i$. If $q > p$, Rick loses (more than) $\sum_{i=p}^{q-1} B_i$ for the remaining moves from p to q, so Rick's total money is (less than) $R - Tb - \sum_{i=p}^{q-1} B_i$. Since Rick cannot have a negative amount of money, (in either case) there is a finite upper bound on T, and hence on the total number of turns t. Rick cannot win since $R \leq \sum_{i=p}^{N-2} B_i$ (and Rick loses tied bids). Furthermore, Linda has enough money to cover her bids since $L \geq \sum_{i=1}^{p} B_i$. Thus, Linda must win. ∎

Theorem 3.2 *Assume that $\frac{L}{p} > \frac{R}{N-p-1}$, and Linda always has the initiative. Then she has a non-adaptive winning strategy.*

Proof: Consider the constant function $B_i = \frac{R}{N-p-1}$. Then $\sum_{i=1}^{p} B_i < L$ and $\sum_{i=p}^{N-2} B_i = R$. Now perturb B_i to make it strictly decreasing by increasing its values slightly. Then, as long as the increase is small enough, $\sum_{i=1}^{p} B_i < L$ and $\sum_{i=p}^{N-2} B_i \geq R$. By Lemma 3.1, Linda wins. ∎

Theorem 3.2 does not say how many turns it takes Linda to win. At one extreme, a player who can win without ever losing a bid (i.e., in p turns for Linda or $N-p-1$ turns for Rick) is said to have a *crushed* the opponent.

Lemma 3.3 *Assume that $L \geq (2^p - 1)R$ and Linda always has the initiative. Then she can crush Rick and wins in p turns with a nonadaptive strategy.*

Proof: Linda bids $2^k R$ at the kth turn. On the kth turn Rick will have exactly $2^k R$ dollars, so Linda always bids just enough to win the auction. Furthermore, after p turns she will have bid a total of $\sum_{k=1}^{p} 2^{k-1} R = (2^p - 1)R$, which is at most what she started with. ∎

At the other extreme, if $\frac{L}{p}$ is close to (but greater than) $\frac{R}{N-p-1}$, Linda will need many turns to win. We analyze the number of turns under this assumption.

In order to minimize the number of turns, the differences $B_i - B_{i+1}$ should be as large as possible so that, as the token moves back and forth between nodes i and $i+1$, Linda makes as much money as possible. These differences are limited by the fact that the sum of the bids from nodes 1 to p must be at most L (so that Linda has enough money to make all of her bids), and the sum of the bids from nodes p to $N-2$ must be at least R (so that Rick cannot win). As a first order approximation, all of the differences $B_i - B_{i+1}$ should be the same, since Rick can force the game to be played between any adjacent pair of nodes. Thus Linda's bids should be a linear function $B_i = a + (p-i)b$ (for constants a and b).

In order to satisfy the two conditions on L and R we need

$$\begin{aligned} L &\geq \sum_{i=1}^{p} B_i = \sum_{i=1}^{p} a + (p-i)b = pa + \frac{(p-1)p}{2} b \\ \text{and} \qquad R &\leq \sum_{i=p}^{N-2} B_i = \sum_{i=p}^{N-2} a + (p-i)b = (N-p-1)a - \frac{(N-p-2)(N-p-1)}{2} b \,. \end{aligned}$$

So

$$\frac{L}{p} \geq a + \frac{p-1}{2} b \qquad \text{and} \qquad \frac{R}{N-p-1} \leq a - \frac{N-p-2}{2} b \,.$$

Solving the two simultaneous inequalities gives $b \leq \frac{2}{N-3}\left(\frac{L}{p} - \frac{R}{N-p-1}\right)$. Letting b take on its maximum value and then solving for a gives

$$a = \frac{1}{N-3}\left(\frac{(N-p-2)L}{p} + \frac{(p-1)R}{N-p-1}\right) \tag{1}$$

$$\text{and} \qquad b = \frac{2}{N-3}\left(\frac{L}{p} - \frac{R}{N-p-1}\right) \,. \tag{2}$$

Theorem 3.4 *Assume that* $\frac{L}{p} > \frac{R}{N-p-1}$*, and Linda always has the initiative. Then she has a nonadaptive strategy to win within*

$$\frac{(N-3)(L+R) \;-\; \left[\frac{(N-p-2)L}{p} + \frac{(p-1)R}{N-p-1}\right]}{\frac{L}{p} - \frac{R}{N-p-1}} \;-\; 3p \;+\; 6$$

turns.

Proof: If $N = 3$ then Linda will win in one turn. Otherwise, using the value $B_i = a + (p-i)b$ for Linda's bid at node i, where a and b are from Equations 1 and 2, Linda must win eventually. By Observation 2.1, the game will take $2T + p$ turns, and when the game is over Rick will have at most $R + L - Tb$ dollars.

We need to know a lower bound on how much money Rick has at the end to estimate the number of moves. Unless Linda wins in one turn, she must win the last two bids. So Rick must have at least $[a+(p-1)b] + [a+(p-2)b] = 2a + (2p-3)b$ dollars at the end of the game (since that is what Linda bids at nodes 1 and 2). Therefore,

$$\begin{aligned}
& R + L - Tb \;\geq\; 2a + (2p-3)b \\
\text{or} \qquad & T \;\leq\; \frac{L+R-a}{b} \;-\; 2p \;+\; 3 \\
& \quad = \frac{(N-3)(L+R) \;-\; \left[\frac{(N-p-2)L}{p} + \frac{(p-1)R}{N-p-1}\right]}{2\left(\frac{L}{p} - \frac{R}{N-p-1}\right)} \;-\; 2p \;+\; 3 \\
\text{or} \qquad & t \;\leq\; \frac{(N-3)(L+R) \;-\; \left[\frac{(N-p-2)L}{p} + \frac{(p-1)R}{N-p-1}\right]}{\frac{L}{p} - \frac{R}{N-p-1}} \;-\; 3p \;+\; 6\ .
\end{aligned}$$

∎

Corollary 3.5 *Assume that N is odd, the token starts in the middle (so $p = \frac{N-1}{2}$), $L > R$, and Linda always has the initiative. Then she has a nonadaptive strategy to win within*

$$\frac{(N-3)^2(L+R)}{2(L-R)} \;-\; \frac{3(N-1)}{2} \;+\; 6$$

turns.

Proof: Substitute $\frac{N-1}{2}$ for p in the previous theorem and simplify. ∎

If $L - R$ is large enough Linda's strategy from the previous theorem will set bids towards Rick's side to be negative. This is not a problem because Rick will not have enough money to get the token that far. But Linda can win in fewer moves by playing on a graph of virtual size $s < N$, setting a smaller and b larger. In this case we need

$$L \;\geq\; pa + \frac{(p-1)p}{2}b\ , \qquad R \;\leq\; (s-p-1)a - \frac{(s-p-2)(s-p-1)}{2}b\ , \qquad \text{and} \qquad a - b(s-p-1) \;=\; 0\ .$$

This does not seem to have a nice solution, so we simplify by decreasing L and increasing R slightly:

$$L \;\geq\; pa + \frac{p^2}{2}b\ , \qquad R \;\leq\; (s-p-1)a - \frac{(s-p-1)^2}{2}b\ , \qquad \text{and} \qquad a \;=\; b(s-p-1)\ .$$

Substituting $a = b(s-p-1)$ into the two inequalities gives:

$$L \;\geq\; pb(s-p-1) + \frac{p^2}{2}b \;=\; pb(t-\frac{p}{2})$$

$$\text{and} \qquad R \;\leq\; (s-p-1)b(s-p-1) - \frac{(s-p-1)^2}{2}b \;=\; \frac{(s-p-1)^2}{2}b\;.$$

Solving these two inequalities for b gives:

$$\frac{R}{(s-p-1)^2} \;\leq\; b \;\leq\; \frac{2L}{p(2(s-1)-p)}$$

Ignoring b and solving for s gives:

$$s \;\geq\; 1+p\left(1+\frac{R \pm \sqrt{2LR+R^2}}{2L}\right)\;,$$

so s can be set

$$s \;=\; \left\lceil 1+p\left(1+\frac{R + \sqrt{2LR+R^2}}{2L}\right)\right\rceil\;.$$

If $s < N$ then it is possible for Linda to play the game just in the interval $[0, s-1]$. In this case the number of moves can be obtained by taking the number of moves in Theorem 3.4 where N is replaced by s.

3.1.2 Adaptive strategies

We know that if $\frac{R}{N-p-1} < \frac{L}{p}$ Linda wins with a nonadaptive strategy, and we have an upper bound on the number of turns it takes. In the proof of Lemma 3.1 we presented a strategy for Linda that made $L - R$ increase every turn additively. In the adaptive case we will have a more complicated quantity called *the advantage* increase multiplicative instead of additively.

Here is a quick outline of the method:

1. Define a notion of Linda's advantage.

2. Define Linda's bids parameterized by a constant (α_q) at each node q.

3. Show that Linda's advantage changes multiplicatively, as the token moves back-and-forth across an edge.

4. Set the constants (α_q) so that the multiplicative changes are balanced (or almost balanced) across all of the edges.

Definition 3.6 *Assume the token is at node q, and Linda and Rick have L' and R' dollars, respectively. Then Linda's* advantage *is the difference*

$$\frac{L'}{q} - \frac{R'}{N-q-1}\;.$$

By picking Linda's bids carefully, we can ensure that this advantage increases multiplicatively as the token moves back-and-forth across an edge. Analogously, in the nonadaptive case, $L' - R'$ is Linda's advantage, and it increases additively as the token moves back-and-forth across an edge.

Linda's bids are a function of the number of nodes, N, the current location of the token, q, and the current amount of money that Linda and Rick have, L' and R', respectively. Linda bids

$$\alpha_q \frac{L'}{q} + (1-\alpha_q)\frac{R'}{N-q-1}$$

at node q for constants α_q ($0 \le \alpha_q \le 1$), which will be determined later. As we will see in the next lemma and its corollary, with this choice Linda's advantage increases multiplicatively as the token moves back-and-forth across an edge. Since $\frac{L'}{p} > \frac{R'}{N-p-1}$ and $0 \le \alpha_q \le 1$, Linda will always have enough money to bid $\alpha_q \frac{L'}{q} + (1-\alpha_q)\frac{R'}{N-q-1}$.

Rick will play the game back-and-forth across the edge with the smallest multiplicative gain. So if the multiplicative increases are not balanced the largest one can be decreased and all of the others can be slightly increased. For small N we are able to set the α_q to completely balance the increases. For general N the increases are only approximately balanced.

Lemma 3.7 *Assume the token is at node q, and Linda and Rick have L' and R' dollars, respectively, where $\frac{L'}{p} > \frac{R'}{N-p-1}$, Linda has the initiative, and she bids $\alpha_q \frac{L'}{q} + (1-\alpha_q)\frac{R'}{N-q-1}$ (for some constant α_q). If Linda wins the bid so that the token moves left, then Linda's advantage changes by a factor of exactly*

$$\frac{q}{q-1} - \alpha_q \frac{N-1}{(q-1)(N-q)} \tag{3}$$

If Rick wins the bid so that the token moves right, then Linda's advantage changes by a factor of more than

$$\frac{q}{q+1} + \alpha_q \frac{N-1}{(q+1)(N-q-2)} \tag{4}$$

Proof: Assume Linda wins the bid. When the token arrives at node $q-1$ the amount of money Linda and Rick have, respectively, is

$$L'' = L' - \alpha_q \frac{L'}{q} - (1-\alpha_q)\frac{R'}{N-q-1} = \left(1 - \frac{\alpha_q}{q}\right)L' - (1-\alpha_q)\frac{R'}{N-q-1}$$

and

$$R'' = R' + \alpha_q \frac{L'}{q} + (1-\alpha_q)\frac{R'}{N-q-1} = \left(1 + \frac{1-\alpha_q}{N-q-1}\right)R' + \alpha_q \frac{L'}{q}\ .$$

Thus

$$
\begin{aligned}
&\frac{L''}{q-1} - \frac{R''}{N-(q-1)-1} \\
&= \quad \frac{(1-\frac{\alpha_q}{q})L' - \frac{1-\alpha_q}{N-q-1}R'}{q-1} \;-\; \frac{(1+\frac{1-\alpha_q}{N-q-1})R' + \frac{\alpha_q}{q}L'}{N-q} \\
&= \quad \left(\frac{1-\frac{\alpha_q}{q}}{q-1} - \frac{\alpha_q}{q(N-q)}\right)L' \;-\; \left(\frac{1+\frac{1-\alpha_q}{N-q-1}}{N-q} + \frac{1-\alpha_q}{(q-1)(N-q-1)}\right)R' \\
&= \quad \left(\frac{1}{q-1} - \frac{\alpha_q}{q}\left(\frac{1}{q-1}+\frac{1}{N-q}\right)\right)L' \;-\; \left(\frac{1}{N-q} + \frac{1-\alpha_q}{N-q-1}\left(\frac{1}{N-q}+\frac{1}{q-1}\right)\right)R' \\
&= \quad \left(\frac{1}{q-1} - \frac{\alpha_q}{q}\left(\frac{N-1}{(q-1)(N-q)}\right)\right)L' \;-\; \left(\frac{1}{N-q} + \frac{1-\alpha_q}{N-q-1}\left(\frac{N-1}{(N-q)(q-1)}\right)\right)R' \\
&= \quad \left(\frac{1}{q-1} - \frac{\alpha_q(N-1)}{q(q-1)(N-q)}\right)L' \\
&\qquad -\left(\frac{1}{N-q} + \frac{N-1}{(q-1)(N-q-1)(N-q)} - \frac{\alpha_q(N-1)}{(q-1)(N-q-1)(N-q)}\right)R' \\
&= \quad \left(\frac{1}{q-1} - \frac{\alpha_q(N-1)}{q(q-1)(N-q)}\right)L' \;-\; \left(\frac{q}{(q-1)(N-q-1)} - \frac{\alpha_q(N-1)}{(q-1)(N-q-1)(N-q)}\right)R' \\
&= \quad \left(\frac{q}{q-1} - \frac{\alpha_q(N-1)}{(q-1)(N-q)}\right)\frac{L'}{q} \;-\; \left(\frac{q}{q-1} - \frac{\alpha_q(N-1)}{(q-1)(N-q)}\right)\frac{R'}{N-q-1} \\
&= \quad \left(\frac{q}{q-1} - \frac{\alpha_q(N-1)}{(q-1)(N-q)}\right)\left(\frac{L'}{q} \;-\; \frac{R'}{N-q-1}\right)\;.
\end{aligned}
$$

Assume Rick wins the bid. When the token arrives at node $q+1$ the amount of money Linda and Rick have, respectively, is

$$
\begin{aligned}
L'' &> L' + \alpha_q\frac{L'}{q} + (1-\alpha_q)\frac{R'}{N-q-1} &= (1+\frac{\alpha_q}{q})L' + (1-\alpha_q)\frac{R'}{N-q-1} \\
\text{and} \qquad R'' &< R' - \alpha_q\frac{L'}{q} - (1-\alpha_q)\frac{R'}{N-q-1} &= (1-\frac{1-\alpha_q}{N-q-1})R' - \alpha_q\frac{L'}{q}\;.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{L''}{q+1} - \frac{R''}{N-(q+1)-1} \\
&\quad > \quad \frac{(1+\frac{\alpha_q}{q})L' + \frac{1-\alpha_q}{N-q-1}R'}{q+1} \;-\; \frac{(1-\frac{1-\alpha_q}{N-q-1})R' - \frac{\alpha_q}{q}L'}{N-q-2} \\
&\quad = \quad \left(\frac{1+\frac{\alpha_q}{q}}{q+1} + \frac{\alpha_q}{q(N-q-2)}\right)L' \;-\; \left(\frac{1-\frac{1-\alpha_q}{N-q-1}}{N-q-2} - \frac{1-\alpha_q}{(q+1)(N-q-1)}\right)R' \\
&\quad = \quad \left(\frac{1}{q+1} + \frac{\alpha_q}{q}\left(\frac{1}{q+1} + \frac{1}{N-q-2}\right)\right)L' \;-\; \left(\frac{1}{N-q-2} - \frac{1-\alpha_q}{N-q-1}\left(\frac{1}{N-q-2} + \frac{1}{q+1}\right)\right)R' \\
&\quad = \quad \left(\frac{1}{q+1} + \frac{\alpha_q}{q}\left(\frac{N-1}{(q+1)(N-q-2)}\right)\right)L' \;-\; \left(\frac{1}{N-q-2} - \frac{1-\alpha_q}{N-q-1}\left(\frac{N-1}{(N-q-2)(q+1)}\right)\right)R' \\
&\quad = \quad \left(\frac{1}{q+1} + \frac{\alpha_q(N-1)}{q(q+1)(N-q-2)}\right)L' \\
&\qquad\qquad - \left(\frac{1}{N-q-2} - \frac{N-1}{(N-q-1)(N-q-2)(q+1)} + \frac{\alpha_q(N-1)}{(q+1)(N-q-1)(N-q-2)}\right)R' \\
&\quad = \quad \left(\frac{1}{q+1} + \frac{\alpha_q(N-1)}{q(q+1)(N-q-2)}\right)L' \;-\; \left(\frac{q}{(q+1)(N-q-1)} + \frac{\alpha_q(N-1)}{(q+1)(N-q-1)(N-q-2)}\right)R' \\
&\quad = \quad \left(\frac{q}{q+1} + \frac{\alpha_q(N-1)}{(q+1)(N-q-2)}\right)\frac{L'}{q} \;-\; \left(\frac{q}{q+1} + \frac{\alpha_q(N-1)}{(q+1)(N-q-2)}\right)\frac{R'}{N-q-1} \\
&\quad = \quad \left(\frac{q}{q+1} + \frac{\alpha_q(N-1)}{(q+1)(N-q-2)}\right)\left(\frac{L'}{q} - \frac{R'}{N-q-1}\right) .
\end{aligned}
$$

∎

Lemma 3.8 *Assume Linda always has the initiative and she bids* $\alpha_q \frac{L'}{q} + (1-\alpha_q)\frac{R'}{N-q-1}$ *at each node* q *(for constants* α_q*), where* L' *and* R' *are Linda's and Rick's current money at node* q*, respectively. Then, for any* q*, whenever the token crosses back-and-forth between nodes* q *and* $q+1$ *(which does not have to be on successive turns), Linda's advantage changes by a factor greater than*

$$\left(1 \;+\; \alpha_q \frac{N-1}{q(N-q-2)}\right)\left(1 \;-\; \alpha_{q+1}\frac{N-1}{(q+1)(N-q-1)}\right) . \tag{5}$$

Proof: By Formula (4), when the token moves right from node q to $q+1$, Linda's advantage changes by a factor greater than

$$\frac{q}{q+1} \;+\; \alpha_q \frac{N-1}{(q+1)(N-q-2)} ,$$

and by Formula (3) (substituting $q+1$ for q), when the token moves left from node $q+1$ to q, Linda's advantage changes by a factor of

$$\frac{q+1}{q} \;-\; \alpha_{q+1}\frac{N-1}{q(N-q-1)} .$$

Multiplying the two factors together and simplifying gives the desired result. ∎

This lemma provides enough information to completely understand and analyze the four-node game. The analysis appears in Appendix B.

If, for general N, Linda fixes her bids to have the form

$$\alpha_q \frac{L'}{q} + (1-\alpha_q)\frac{R'}{N-q-1} ,$$

then in order for Rick to delay the game as long as possible he does best to play the game between the two nodes where Linda gains the smallest factor.

Lemma 3.9 *Assume that $\frac{L}{p} > \frac{R}{N-p-1}$, Linda always has the initiative, and her bids are of the form*

$$\alpha_q \frac{L'}{q} + (1-\alpha_q)\frac{R'}{N-q-1} .$$

Assume that her advantage never decreases (with these bids). Let $f > 1$ be the minimum factor for the advantage that Linda gains crossing back-and-forth across an edge. Then Linda wins within

$$2\left\lceil \log_f\left(\frac{L+R}{L-\frac{pR}{N-p-1}}\right)\right\rceil + p$$

turns.

Proof: Rick cannot win: By Lemma 3.7 Linda always has a positive advantage, and if the token gets to node $n-2$, Linda's bid will be at least R' (and Linda wins ties).

By the conditions of the lemma $\frac{L'}{p} - \frac{R'}{N-p-1} \geq \left(\frac{L}{p} - \frac{R}{N-p-1}\right) f^T$. Thus,

$$L' - \frac{pR'}{N-p-1} \geq \left(L - \frac{pR}{N-p-1}\right) f^T.$$

Also, since $L' \leq L+R$,

$$L' - \frac{pR'}{N-p-1} \leq L+R,$$

Putting these two facts together gives

$$\left(\frac{L}{p} - \frac{R}{N-p-1}\right) f^T \ \leq\ L+R .$$

Taking log of both sides gives

$$T \ \geq\ \log_f\left(\frac{L+R}{L-\frac{pR}{N-p-1}}\right) .$$

The result follows from Observation 2.1. ∎

In an optimal strategy, Linda's advantage as the token crosses back-and-forth across an edge must increase by the same factor for every edge (at least if there are a large number of turns). Otherwise, Rick will move the token to the edge with the least such value, and play the game across that edge.

Thus, for fixed N, we would like to calculate the α_q that makes all of the values from Formula (5) the same. Let f be this common factor. As before, $\alpha_1 = 1$ and $\alpha_{N-2} = 0$, since Linda will bet everything at node 1, and Rick will bet everything at node $N-2$, respectively. For any given value of N, we can calculate an asymptotically optimal strategy for Linda, which will make all of the values, α_q, from Formula (5) the same. We have done this for $N \leq 7$. In calculating these values, it is helpful but not necessary, to note that Rick must also use the same optimal strategy. So $\alpha_q = 1 - \alpha_{N-1-q}$, and, in particular for N odd, $\alpha_{(N-1)/2} = 1/2$.

N	α values	f
4	$\alpha_1 = 1,\ \alpha_2 = 0$	4
5	$\alpha_1 = 1,\ \alpha_2 = 1/2,\ \alpha_3 = 0$	2
6	$\alpha_1 = 1,\ \alpha_2 = 1 - \frac{6\sqrt{5}-10}{5},\ \alpha_3 = \frac{6\sqrt{5}-10}{5},\ \alpha_4 = 0$	$6 - 2\sqrt{5}$
7	$\alpha_1 = 1,\ \alpha_2 = 7/9,\ \alpha_3 = 1/2,\ \alpha_4 = 2/9,\ \alpha_5 = 0$	$4/3$

We have not been able to calculate the optimal values for α_q for general N. Using the values from the nonadaptive strategy, but letting them change adaptively, provides an upper bound and seems to give a good approximation to the optimal values for an adaptive strategy. In the nonadaptive strategy Linda bids $a + (p-q)b$ at node q, but we are assuming every move restarts the calculation so $q = p$. This means that she bids

$$a \quad = \quad \frac{1}{N-3}\left(\frac{(N-q-2)L}{q} \quad + \quad \frac{(q-1)R}{N-q-1}\right)$$

at node q. This only works if a has the right form; but it does! So $\alpha_q = \frac{N-q-2}{N-3}$. Let f_q be the factor that Linda's advantage increases going back-and-forth between nodes q and $q+1$. The following table gives the α_q and f_q values for $N \leq 7$ using this approximation.

N	α values	f values	minimum f
4	$\alpha_1 = 1,\ \alpha_2 = 0$	$f_1 = 4$	4
5	$\alpha_1 = 1,\ \alpha_2 = 1/2,\ \alpha_3 = 0$	$f_1 = 2,\ f_2 = 2$	2
6	$\alpha_1 = 1,\ \alpha_2 = 2/3,\ \alpha_3 = 1/3,\ \alpha_4 = 0$	$f_1 = 14/9,\ f_2 = 121/81,\ f_3 = 14/9$	121/81
7	$\alpha_1 = 1,\ \alpha_2 = 3/4,\ \alpha_3 = 1/2,\ \alpha_4 = 1/4,\ \alpha_5 = 0$	$f_1 = f_4 = 11/8,\ f_2 = f_3 = 21/16$	21/16

Notice how close the equivalent values in the two tables are.

Theorem 3.10 *Assume that $\frac{L}{p} > \frac{R}{N-p-1}$ and Linda always has the initiative. Then Linda has an adaptive strategy to win within*

$$2\left\lceil \log_f\left(\frac{L+R}{L - \frac{pR}{N-p-1}}\right)\right\rceil + p$$

turns, where

$$f \quad = \quad 1 + \frac{8(N-2)}{(N+1)(N-3)^2} \qquad \textit{for } N \textit{ odd}$$

$$\textit{and} \qquad f \quad = \quad 1 + \frac{8(N-1)(N-2)}{N^2(N-3)^2} \qquad \textit{for } N \textit{ even.}$$

Proof: From the conditions of Lemma 3.7, let $\alpha_q = \frac{N-q-2}{N-3}$, so that at each node q Linda bids

$$\frac{(N-q-2)L'}{(N-3)q} + \frac{(q-1)R'}{(N-3)(N-q-1)}$$

dollars, where L' and R' are the current amount of money Linda and Rick have, respectively. It turns out that for the given α_q values, Linda's advantage increases on every move (even when she wins the bid and gives money to Rick). By Formula (5) as the token crosses back-and-forth between nodes q and $q+1$ Linda's advantage increases by a factor of

$$\left(1+\frac{N-1}{q(N-3)}\right)\left(1-\frac{(N-q-3)(N-1)}{(q+1)(N-q-1)(N-3)}\right) \quad = \quad 1+\frac{2(N-1)(N-2)}{(q+1)(N-q-1)(N-3)^2}\ .$$

The result follows from Lemma 3.9. ∎

As in the nonadaptive case Linda can do better playing on a graph of virtual size $s < N$ when L is large enough. We will not expand on this.

We can get a simpler result when the token starts in the middle:

Corollary 3.11 *Assume that N is odd, the token starts in the middle (so $p = \frac{N-1}{2}$), $L > R$, Linda always has the initiative. As N gets asymptotically large Linda has an adaptive strategy to win within*

$$\frac{N^2}{4}\ln\left(\frac{L+R}{L-R}\right)+\frac{N}{2}$$

turns.

3.2 Linda Wins without the Initiative

We can use the results of the previous section to determine when Linda wins without having the initiative: Assume that $\frac{L}{p} > \frac{R}{N-p-1}$. Linda can play the game as if she has $L-\epsilon$ dollars for a very small constant $\epsilon > 0$, but adds ϵ/p to each of her bids. This gives her the effect of always having the initiative. She will always have enough money to make the bid since if she wins the bid on a net of p turns, she wins the game.

Using these ideas with Theorem 3.2 gives:

Theorem 3.12 *Assume that $\frac{L}{p} > \frac{R}{N-p-1}$. Then Linda has a nonadaptive winning strategy (independent of the initiative).*

Using these ideas with Lemma 3.3 gives:

Lemma 3.13 *Assume that $L > (2^p-1)R$. Then Linda crushes Rick with a nonadaptive strategy (independent of the initiative) and hence wins in p moves.*

Using these ideas with Theorem 3.4 gives:

Lemma 3.14 *Assume that $\frac{L}{p} > \frac{R}{N-p-1}$. Let $\epsilon > 0$ be a (small) constant such that $\frac{L-\epsilon}{p} > \frac{R}{N-p-1}$. Then Linda has a nonadaptive strategy to win within*

$$\frac{(N-3)(L-\epsilon+R) - \left[\frac{(N-p-2)(L-\epsilon)}{p}+\frac{(p-1)R}{N-p-1}\right]}{\frac{L-\epsilon}{p}-\frac{R}{N-p-1}} - 3p + 6$$

turns (independent of the initiative).

Letting ϵ be arbitrarily small gives:

Lemma 3.15 *Assume that $\frac{L}{p} > \frac{R}{N-p-1}$. Then Linda has a nonadaptive strategy to win within*

$$\frac{(N-3)(L+R) \;-\; \left[\frac{(N-p-2)L}{p} + \frac{(p-1)R}{N-p-1}\right]}{\frac{L}{p} - \frac{R}{N-p-1}} \;-\; 3p \;+\; 6$$

turns (independent of the initiative).

Corollary 3.16 *Assume that N is odd, the token starts in the middle (so $p = \frac{N-1}{2}$), and $L > R$. Then Linda has a nonadaptive strategy to win within*

$$\frac{(N-3)^2(L+R)}{2(L-R)} \;-\; \frac{3(N-1)}{2} \;+\; 6$$

turns (independent of the initiative).

Using the above ideas with Theorem 3.10 gives:

Corollary 3.17 *Assume that $\frac{L}{p} > \frac{R}{N-p-1}$. Let $\epsilon > 0$ be a (small) constant such that $\frac{L-\epsilon}{p} > \frac{R}{N-p-1}$. Then Linda has an adaptive strategy to win within*

$$2\left\lceil \log_f\left(\frac{L+R}{L-\epsilon-\frac{pR}{N-p-1}}\right)\right\rceil + p$$

turns (independent of the initiative), where

$$\begin{aligned} f &= 1+\frac{8(N-2)}{(N+1)(N-3)^2} && \textit{for } N \textit{ odd} \\ \textit{and} \quad f &= 1+\frac{8(N-1)(N-2)}{N^2(N-3)^2} && \textit{for } N \textit{ even.} \end{aligned}$$

Corollary 3.18 *Assume that $\frac{L}{p} > \frac{R}{N-p-1}$. Then Linda has an adaptive strategy to win within*

$$2\left\lceil \log_f\left(\frac{L+R}{L-\frac{pR}{N-p-1}}\right)\right\rceil + p$$

turns, where

$$\begin{aligned} f &= 1+\frac{8(N-2)}{(N+1)(N-3)^2} && \textit{for } N \textit{ odd} \\ \textit{and} \quad f &= 1+\frac{8(N-1)(N-2)}{N^2(N-3)^2} && \textit{for } N \textit{ even.} \end{aligned}$$

turns (independent of the initiative).

Corollary 3.19 *Assume that N is odd, the token starts in the middle (so $p = \frac{N-1}{2}$), and $L > R$. As N gets asymptotically large Linda has an adaptive strategy to win within*

$$\frac{N^2}{4}\ln\left(\frac{L+R}{L-R}\right) + \frac{N}{2}$$

turns (independent of the initiative).

3.3 Nobody has an advantage

Finally, a player that always has the initiative still wins when $\frac{R}{N-p-1} = \frac{L}{p}$:

Lemma 3.20 *Assume $\frac{R}{N-p-1} = \frac{L}{p}$, and that Linda may use an adaptive strategy. Then Lindawins.*

Proof: Linda bids L'/p at every node, where L' is Linda's current money, until Rick wins a bid. If Rick never wins a bid then Linda wins the game (in p turns). Otherwise, at some point Rick bids more than L'/p. This produces an advantage for Linda, and by Theorem 3.2 she wins. ∎

When $\frac{R}{N-p-1} = \frac{L}{p}$, there is no upper bound on how many turns it takes Linda to win, since Rick can win the bid by bidding an arbitrarily small amount more than L'/p.

Lemma 3.21 *Assume $\frac{R}{N-p-1} = \frac{L}{p}$, and both Linda and Rick are required to use a nonadaptive strategy. Then Linda wins.*

Proof: Linda bids $\frac{L}{p}$ at every node. Let B_i be Rick's bid at node i. For each pair of traversals back-and-forth across the edge between nodes i and $i+1$, Linda gains and Rick loses (more than) $B_i - \frac{L}{p}$ dollars. Let b be the minimum of $B_i - \frac{L}{p}$ (for $1 \leq i < N-2$). By Observation 2.1, Rick loses at least Tb for the T turns in which the token crosses back an forth across an edge. Furthermore, if $q \leq p$, Rick gains $\sum_{i=q+1}^{p} \frac{L}{p}$ for the remaining moves from p to q, so Rick's total money is $R - Tb + (p-q)\frac{L}{p}$. If $q > p$, Rick loses (more than) $\sum_{i=p}^{q-1} B_i$ for the remaining moves from p to q, so Rick's total money is (less than) $R - Tb - \sum_{i=p}^{q-1} B_i$. Since Rick cannot have a negative amount of money, (in either case) there is a finite upper bound on T, and hence on the total number of turns t. Rick cannot win since $R = \sum_{i=p}^{N-2} \frac{L}{p}$ (and Rick loses tied bids). Furthermore, Linda has enough money to cover her bids since $L = \sum_{i=1}^{p} \frac{L}{p}$. Thus, Linda must win. ∎

A player who can win in one turn is said to have an *immediate win*.

Lemma 3.22 *Assume that $\frac{R}{N-p-1} = \frac{L}{p}$, neither player has an immediate win, Linda must use a nonadaptive strategy, and Rick may use an adaptive strategy. Then the game is a draw.*

Proof: We assume that Linda always has the initiative and show that she does not have a non-adaptive winning strategy. The other cases are symmetric but easier.

We can assume that $p \leq N-3$: If the token starts at node $N-2$, Linda must bid exactly L/p (or else Rick will win immediately) and the token will move to node $N-3$.

Case 1: $B_p > L/p$. Then Rick lets Linda win the first bid. Now Rick has an advantage and he wins by Theorem 3.12.

Case 2: $B_p \leq L/p$. Find the least $q \geq p$ such that $B_q \leq B_{q+1}$. If no such q exists, Rick has enough money to simply win.

Case 2a: $B_q < B_{q+1}$. Then Rick wins by making the token move back-and-forth between nodes q and $q+1$ by bidding $(B_q + B_{q+1})/2$ at node q.

Case 2b: $B_q = B_{q+1}$. Rick makes the game last forever using an exponential decay of bids. ∎

Corollary 3.23 *Assume N is odd, and $p = \frac{N-1}{2}$, and $L = R$. A player who always has the initiative wins with an adaptive strategy, or if both players use a nonadaptive strategy. Neither player wins if the player who always has the initiative uses a nonadaptive strategy and the other player uses an adaptive strategy.*

3.4 Final Formulas and Interpretation

3.4.1 Token is in General Position

To obtain the result for when Rick wins take formulas for Linda wins and reverse the roles of $\frac{L}{p}$ and $\frac{R}{N-p-1}$. Any additive p terms become additive $N-p-1$.
Who wins.

- $\frac{L}{p} > \frac{R}{N-p-1}$: For any method of assigning initiative, Linda wins with a nonadaptive (or adaptive) strategy.
- $\frac{R}{N-p-1} > \frac{L}{p}$: For any method of assigning initiative, Rick wins with a nonadaptive (or adaptive) strategy.
- $\frac{L}{p} = \frac{R}{N-p-1}$. Assume that Linda has the initiative and neither player has an immediate win. (1) If Linda use an adaptive strategy then she wins. (2) If Linda and Rick both use a unadaptive strategy then she wins. (3) If Linda uses a nonadaptive strategy and Rick uses an adaptive strategy then the game is a draw.

3.4.2 Token is in the Middle

Throughout this section N is odd and the token starts in the middle. This is the most natural case and the formulas become simpler.
Who wins.

- $L > R$: When the token starts in the middle, for any method of assigning initiative, Linda wins with a nonadaptive (or adaptive) strategy.
- $R > L$: For any method of assigning initiative, Rick wins with a nonadaptive (or adaptive) strategy.
- $L = R$, Linda has the initiative, and neither player has an immediate win: A player that always has the initiative (1) If Linda use an adaptive strategy then she wins. (2) If Linda and Rick both use a unadaptive strategy then she wins. (3) If Linda uses a nonadaptive strategy and Rick uses an adaptive strategy

Number of moves to win. We assume that N is large so the results are asymptotic in N. These apply for any method of assigning initiative.

- $L > R$:
  - Nonadaptive strategy: Linda wins within

$$\frac{N^2(L+R)}{2(L-R)} - \frac{3N}{2}$$

    turns (independent of Rick's strategy).

- Adaptive strategy: Linda wins within
$$\frac{N^2}{4}\ln\left(\frac{L+R}{L-R}\right)+\frac{N}{2}$$
turns (independent of Rick's strategy).

- $\frac{R}{N-p-1} > \frac{L}{p}$:
  - Nonadaptive strategy: Rick wins within
$$\frac{N^2(L+R)}{2(R-L)}-\frac{3N}{2}$$
turns (independent of Linda's strategy).
  - Adaptive strategy: Rick wins within
$$\frac{N^2}{4}\ln\left(\frac{R+L}{R-L}\right)+\frac{N}{2}$$
turns (independent of Linda's strategy).

As we can see, the number of turns for Linda to win with a nonadaptive strategy has a factor of $\frac{L+R}{L-R}$ whereas the adaptive strategy has a factor of ln of this value. Both strategies have a factor that is quadratic in the board size. Finally the nonadaptive strategy has a constant factor of $\frac{1}{2}$ whereas the adaptive strategy has a constant factor of only $\frac{1}{4}$.

4 The Discrete Case

We classify exactly when the discrete game is a win for Linda, a win for Rick, or a draw, assuming Linda has the initiative. All of the strategies, both to win and draw, are nonadaptive. We do not analyze how many turns it takes to win.

4.1 Linda Wins

First, consider what Linda needs to win (assuming she has the initiative). Assume Linda sets her bids so that at node i she bids B_i. Intuitively, for Linda to guarantee a win the bid values should have three properties.

- They should be small enough so that Linda has enough money to win, i.e., $L \geq \sum_{i=1}^{p} B_i$.
- They should be large enough so that Rick does not have enough money to win, i.e., $R < \sum_{i=p}^{N-2}(B_i+1)$.
- They should be non-increasing. Otherwise, if for some i, $B_i < B_{i+1}$, Rick can force the token to move back-and-forth between nodes i and $i+1$ while retaining his money (by winning the bid at node i for B_{i+1} dollars, and letting Linda win the bid at node $i+1$ for B_{i+1} dollars).

This is formalized in the following lemma.

Lemma 4.1 *Assume there exists a function B_i that is non-increasing such that $L \geq \sum_{i=1}^{p} B_i$ and $R < \sum_{i=p}^{N-2}(B_i + 1)$. If Linda always has the initiative then bidding B_i at node i is a winning strategy.*

Proof: We will show that some player must win, but Rick cannot. It then follows that Linda must win. For each pair of traversals back-and-forth across the edge between nodes i and $i+1$, Linda gains and Rick loses (at least) $B_i - B_{i+1} + 1$ dollars. Let b be the minimum of $B_i - B_{i+1} + 1$ (for $1 \leq i < N-2$).

By Observation 2.1, if $q \leq p$, Linda's money after t turns is at least $L + Tb - \sum_{i=q+1}^{p} B_i$. Otherwise it is at least $L + Tb + \sum_{i=p}^{q-1}(B_i + 1)$. In either case, she has enough money to cover her next bid. Similarly, if $q \leq p$, Rick's money after t turns is at most $R - Tb + \sum_{i=q+1}^{p} B_i$. Otherwise it is at most $R - Tb - \sum_{i=p}^{q-1}(B_i + 1)$. In either case, if T is large enough Rick's money will be negative, which is impossible, so the game must have a finite limit on the number of turns. Furthermore, if Rick wins so that $q = N-1$, his money is at most $R - Tb - \sum_{i=p}^{N-2}(B_i + 1)$, which is negative and therefore impossible. ∎

Lemma 4.2 *If Linda always has the initiative then she wins with a nonadaptive strategy if there exists an integer k such that*

$$\frac{R+1}{N-p-1} - 1 \leq k \leq \frac{L}{p} \ .$$

Proof: Let k be an integer satisfying the conditions of the lemma. Let $B_i = k$, so Linda bids k at every node. By Lemma 4.1 Linda wins. ∎

4.2 Rick Draws

Conversely, consider what Rick needs to draw (assuming Linda has the initiative). Assume Rick sets his bids so that at node i he bids B_i. Intuitively, for Rick to guarantee a draw the bid values should have three properties.

- They should be small enough so that Rick has enough money to cover all his bids, i.e., $R \geq \sum_{i=p}^{N-2} B_i$.
- They should be large enough so that Linda does not have enough money to win, i.e., $L < \sum_{i=1}^{p} B_i$.
- They should be non-decreasing. Otherwise, if for some i, $B_{i+1} < B_i$, Linda can force the token move back-and-forth between nodes i and $i+1$ and gain money at each iteration (by bidding B_{i+1} at both nodes).

This is formalized in the following lemma.

Lemma 4.3 *Assume there exists a function B_i that is non-decreasing such that $R \geq \sum_{i=p}^{N-2} B_i$ and $L < \sum_{i=1}^{p} B_i$. Then Rick bidding B_i at node i prevents Linda from winning.*

Proof: We will show that Rick always has enough money to make his bid, and Linda cannot win, therefore Rick must at least draw. For each pair of traversals back-and-forth across the edge between nodes i and $i+1$, Rick gains (at least) $B_{i+1} - B_i$ (for $1 \leq i < N-2$). Since B_i is non-decreasing this value is never negative. Let b be the minimum of $B_{i+1} - B_i$ (for $1 \leq i < N-2$). By Observation 2.1, if $q \leq p$, Rick's money after t turns is at least $R + Tb + \sum_{i=q+1}^{p} B_i$. Otherwise it is at least $R + Tb - \sum_{i=p}^{q-1} B_i$. In either case, Rick still has enough money to cover his next bid. Similarly, if Linda wins so that $q = 0$, Linda's money is at most $L - Tb - \sum_{i=1}^{p} B_i$, which is negative and therefore impossible. ∎

Lemma 4.4 *Rick has a nonadaptive strategy to prevent Linda from winning if there exists an integer k such that*

$$\frac{L+1}{p} \leq k \leq \frac{R}{N-p-1} \ .$$

Proof: Let k be an integer satisfying the conditions of the lemma. Rick bids k at every node. By Lemma 4.3 Rick prevents Linda from winning. ∎

4.3 Rick Wins

Consider what Rick needs to win (assuming Linda has the initiative). Intuitively, for Rick to guarantee a win the bid values B_i should have three properties:

- They should be small enough so that Rick has enough money to win, i.e., $R \geq \sum_{i=p}^{N-2} B_i$.
- They should be large enough so that Linda does not have enough money to win, i.e., $L < \sum_{i=1}^{p} B_i$.
- They should be strictly increasing. Otherwise, if for some i, $B_i \geq B_{i+1}$, Linda can force the token to move back-and-forth between nodes i and $i+1$ and maintain (or increase) her money at each iteration (by bidding $B_i - 1$ at node i, and B_{i+1} at node $i+1$).

If Linda has a "small" amount of money, it is too conservative to make B_i strictly increasing across the whole range 1 to $n-2$. It is sufficient for B_i to be strictly increasing across the range s to $n-2$, for some s between 1 and $p-1$, as long as $L < \sum_{i=s}^{p} B_i$ so that Linda cannot move the token past s. Then the game is actually played over the the range $s-1$ to $N-1$. Although this introduces the new variable, s, the final theorem will not involve it.

Lemma 4.5 *Let $s \in \mathsf{N}$ satisfy $0 < s \leq p$, and $B_i \in \mathsf{N}$ be a is monotonically strictly increasing function for $i \geq s$. If $R \geq \sum_{i=p}^{N-2} B_i$ and $L < \sum_{i=s}^{p} B_i$, then Rick bidding B_i at node i is a winning strategy.*

Proof: We will show that some player must win, but Linda cannot. It then follows that Rick must win. For each pair of traversals back-and-forth across the edge between nodes i and $i+1$, Rick gains and Linda loses (at least) $B_{i+1} - B_i$ dollars. Let b be the minimum of $B_{i+1} - B_i$ (for $1 \leq i < N-2$).

By Observation 2.1, if $q \leq p$, Rick's money after t turns is at least $R + Tb + \sum_{i=q+1}^{p} B_i$. Otherwise it is at least $R + Tb - \sum_{i=p}^{q-1} B_i$. In either case, he has enough money to cover his next bid. Similarly, if $q \leq p$, Linda's money after t turns is at most $L - Tb - \sum_{i=q+1}^{p} B_i$. Otherwise it is at most $L - Tb - \sum_{i=p}^{q-1} B_i$. In either case, if T is large enough Linda's money will be negative, which is impossible, so the game has a finite limit on the number of turns. Furthermore, Linda wins if $q = 0$ but then her money is negative, which is impossible. ∎

Lemma 4.6 *Rick wins with a nonadaptive strategy if there is an integer value k such that*

$$\begin{array}{llll} & k \geq p \quad \textit{and} & \dfrac{L+1}{p} + \dfrac{p-1}{2} \;\leq\; k \;\leq\; \dfrac{R}{N-p-1} - \dfrac{N-p}{2} + 1 \\ \textit{or} & k \leq p \quad \textit{and} & \dfrac{\sqrt{8L+9}-1}{2} \;\leq\; k \;\leq\; \dfrac{R}{N-p-1} - \dfrac{N-p}{2} + 1 \;. \end{array}$$

Proof: We will derive the value for k in the process of proving the lemma. For Rick to win, Lemma 4.5 gives the following lower bound on Rick's money: $R \geq \sum_{i=p}^{N-2} B_i$, where B_i is strictly increasing. In order to make the right side as small as possible we need $B_p = k$ and the values to increase by exactly one. Thus, Rick can guarantee a win, if there is an integer k such that

$$R \;\geq\; \sum_{i=p}^{N-2} k + (i-p) \;=\; \sum_{i=0}^{N-p-2} k + i \;=\; (N-p-1)k + \frac{(N-p-2)(N-p-1)}{2}$$

Solving for k gives

$$k \;\leq\; \frac{R}{N-p-1} - \frac{N-p}{2} + 1 \;.$$

By Lemma 4.5 we need a value s such that $L < \sum_{i=s}^{p} B_i$. In order to make the right side as large as possible, once again, we need the values to increase by exactly one. So, k must satisfy the condition that $L < \sum_{i=s}^{p} k - (p-i)$ or $L \leq (\sum_{i=s}^{p} k - (p-i)) - 1$.

If $k \geq p$, then $s = 1$, giving

$$L \;\leq\; \left(\sum_{i=1}^{p} k - (p-i)\right) - 1 \;=\; \sum_{i=1}^{p} (k-p) + \left(\sum_{i=1}^{p} i\right) - 1 \;=\; p(k-p) + \frac{p(p+1)}{2} - 1 \;.$$

Solving for k gives

$$k \;\geq\; \frac{L+1}{p} + \frac{p-1}{2} \;.$$

Thus Rick wins if there is an integer value k such that $k \geq p$ and

$$\frac{L+1}{p} + \frac{p-1}{2} \;\leq\; k \;\leq\; \frac{R}{N-p-1} - \frac{N-p}{2} + 1 \;.$$

If $k \leq p$, then $s = p - k + 1$, giving

$$L \leq (\sum_{i=p-k+1}^{p} k - (p-i)) - 1 \;=\; (\sum_{i=1}^{k} i) - 1 \;=\; \frac{k(k+1)}{2} - 1 \;.$$

Solving for k gives

$$k \;\geq\; \frac{\sqrt{8L+9}-1}{2} \;.$$

Thus Rick wins if there is an integer value k such that $k \leq p$ and

$$\frac{\sqrt{8L+9}-1}{2} \;\leq\; k \;\leq\; \frac{R}{N-p-1} - \frac{N-p}{2} + 1 \;.$$

∎

4.4 Linda Draws

Finally, consider what Linda needs to guarantee a draw. Her bid values should have three properties:

- They should be small enough so that Linda has enough money to cover her bets, i.e., $L \geq \sum_{i=1}^{p} B_i$.
- They should be large enough so that Rick does not have enough money to win, i.e., $R < \sum_{i=p}^{N-2}(B_i + 1)$.
- They should be increasing by at most one. Otherwise, if for some i, $B_{i+1} > B_i + 1$, Rick can force the token to move back-and-forth between nodes i and $i+1$ while increasing his money (by winning the bid at node i for $B_i + 1$ dollars, and letting Linda win the bid at node $i+1$ for B_{i+1} dollars).

Lemma 4.7 *Assume Linda always has the initiative. Let $s \in \mathsf{N}$ satisfy $0 < s \leq p$, and $B_i \in \mathsf{N}$ be a function such that $B_{i+1} - B_i \leq 1$ for $i \geq s$ (i.e., B_i increases by at most one). If $L \geq \sum_{i=s}^{p} B_i$, and $R < \sum_{i=p}^{N-2}(B_i + 1)$, then then Linda bidding B_i at node i prevents Rick from winning.*

Proof: We will show that Linda always has enough money to cover her bid, and Rick cannot win, therefore Linda must at least draw. For each pair of traversals back-and-forth across the edge between nodes i and $i+1$, Linda gains (at least) $B_i - B_{i+1} + 1$ (for $1 \leq i < N-2$). This value is never negative. Let b be the minimum of $B_i - B_{i+1} + 1$ (for $s \leq i < N-2$). By Observation 2.1, if $q \leq p$, Linda's money after t turns is at least $L + Tb - \sum_{i=q+1}^{p} B_i$. Otherwise it is at least $L + Tb + \sum_{i=p}^{q-1} B_i$. In either case, Linda has enough money to cover her next bid. Similarly, if $q = N-1$ (so that Rick wins), Rick's money is at most $R - Tb - \sum_{i=p}^{N-2}(B_i + 1)$, which is negative and therefore impossible. ∎

Lemma 4.8 *Assume Linda always has the initiative. Then she has a nonadaptive strategy to prevent Rick from winning if there is an integer value k such that*

$$\begin{array}{lll} & k \geq p \quad \textit{and} & \dfrac{R+1}{N-p-1} - \dfrac{N-p}{2} \;\leq\; k \;\leq\; \dfrac{L}{p} + \dfrac{p-1}{2} \\ \textit{or} & k \leq p \quad \textit{and} & \dfrac{R+1}{N-p-1} - \dfrac{N-p}{2} \;\leq\; k \;\leq\; \dfrac{\sqrt{8L+1}-1}{2} \;. \end{array}$$

Proof: We will derive the value for k in the process of proving the lemma. Lemma 4.7 gives the following upper bound on Rick's money: $R < \sum_{i=p}^{N-2}(B_i + 1)$. In order to make the right side as large as possible we need the values to increase by exactly one. Thus, Linda can guarantee a draw, if there is an integer k such that

$$R + 1 \;\leq\; \sum_{i=p}^{N-2}(k + (i-p) + 1) \;=\; \sum_{i=1}^{N-p-1}(k+i) \;=\; (N-p-1)k + \frac{(N-p-1)(N-p)}{2} \;.$$

Solving for k gives

$$k \;\geq\; \frac{R+1}{N-p-1} - \frac{N-p}{2} \;.$$

By Lemma 4.7 we need a value s such that $L \geq \sum_{i=s}^{p} B_i$. In order to make the right side as small as possible, once again, we need the values to increase by exactly one. So, k must satisfy the condition that $L \geq \sum_{i=s}^{p} k - (p-i)$.

If $k \geq p$, then $s = 1$ giving

$$L \;\geq\; \sum_{i=1}^{p} k - (p-i) \;=\; \sum_{i=1}^{p} (k-p) + \sum_{i=1}^{p} i \;=\; p(k-p) + \frac{p(p+1)}{2}\;.$$

Solving for k gives

$$k \;\leq\; \frac{L}{p} + \frac{p-1}{2}$$

Thus Linda draws if there is an integer value k such that $k \geq p$ and

$$\frac{L}{p} + \frac{p-1}{2} \;\geq\; k \;\geq\; \frac{R+1}{N-p-1} - \frac{N-p}{2}\;.$$

If $k \leq p$, then $s = p-k+1$, giving

$$L \geq \sum_{i=p-k+1}^{p} k - (p-i) \;=\; \sum_{i=1}^{k} i \;=\; \frac{k(k+1)}{2}\;.$$

Solving for k gives

$$k \;\geq\; \frac{\sqrt{8L+1}-1}{2}\;.$$

Thus Linda draws if there is an integer value k such that $k \leq p$ and

$$\frac{R+1}{N-p-1} - \frac{N-p}{2} \;\leq\; k \;\leq\; \frac{\sqrt{8L+1}-1}{2}\;.$$

■

4.5 The Cutoffs between Linda Wins and Rick Draws and between Rick Wins and Linda Draws

The following common technique is used for converting rational inequalities from $<$ to $\leq$.

Lemma 4.9 *Let $a, b, k \in \mathsf{N}$.*

(a) If $k > \frac{a}{b}$ then $k \geq \frac{a+1}{b}$.

(b) If $k < \frac{a}{b}$ then $k \leq \frac{a-1}{b}$.

We will also need a more general technique:

Lemma 4.10 *Let $a, b, c, k \in \mathsf{N}$, where the product bc is even.*

(a) If $k > \frac{a}{b} + \frac{c}{2}$ then $k \geq \frac{a+1}{b} + \frac{c}{2}$.

(b) If $k < \frac{a}{b} + \frac{c}{2}$ then $k \leq \frac{a-1}{b} + \frac{c}{2}$.

Proof:

(a) Assume, by way of contradiction, that $\frac{a}{b} + \frac{c}{2} < k < \frac{a+1}{b} + \frac{c}{2}$. Then

$$a + \frac{bc}{2} < bk < a + 1 + \frac{bc}{2} \ ,$$

which is impossible.

(b) Follows from essentially the same proof.

∎

Lemma 4.11 *Let $a, b \in \mathsf{R}$. Assume that for all integers k*

$$a \leq k \quad \textit{or} \quad k \leq b \ .$$

(a) There exists an integer k such that $a - 1 \leq k \leq b$.

(b) There exists an integer k such that $a \leq k \leq b + 1$.

(Note that we do not claim any relation between a and b.)

Proof: Let k be the unique integer such that $b - 1 < k \leq b$ (so that $k = \lfloor b \rfloor$). Then $k + 1 > b$, so $a \leq k + 1$, and $a - 1 \leq k$. Thus, $a - 1 \leq k \leq b$, which is Part (a). Part (b) follows immediately. ∎

Theorem 4.12 *Assume Linda always has the initiative. Then she wins if and only if*

$$\left\lceil \frac{R+1}{N-p-1} \right\rceil - 1 \ \leq \left\lfloor \frac{L}{p} \right\rfloor .$$

Furthermore, if the condition holds Linda wins with a nonadaptive strategy (independent of Rick's strategy), and if the condition does not hold Rick can force at least a draw with a nonadaptive strategy (independent of Linda's strategy).

Proof: By Lemma 4.2 Linda wins with a nonadaptive strategy (independent of Rick's strategy) if there exists an integer k such that

$$\frac{R+1}{N-p-1} - 1 \leq k \leq \frac{L}{p}$$

and by Lemma 4.4 Rick at least draws with a nonadaptive strategy (independent of Linda's strategy) if there exists an integer k such that

$$\frac{L+1}{p} \leq k \leq \frac{R}{N-p-1} \ .$$

We need to show that these two conditions cover all of the cases.

Assume the condition for Linda winning does not hold. Then

$$\begin{aligned}
& \neg\, \exists k \in \mathsf{N}, \quad \frac{R+1}{N-p-1} - 1 \le k \le \frac{L}{p} \\
\Longrightarrow \quad & \forall k \in \mathsf{N}, \quad \frac{R+1}{N-p-1} - 1 > k \quad \text{or} \quad k > \frac{L}{p} \\
\Longrightarrow \quad & \forall k \in \mathsf{N}, \quad \frac{L}{p} < k \quad \text{or} \quad k < \frac{R+1}{N-p-1} - 1 \\
\Longrightarrow \quad & \forall k \in \mathsf{N}, \quad \frac{L+1}{p} \le k \quad \text{or} \quad k \le \frac{R}{N-p-1} - 1 \qquad \text{by Lemma 4.9.}
\end{aligned}$$

By Lemma 4.11b

$$\frac{L+1}{p} \le k \le \frac{R}{N-p-1}\ .$$

So Rick draws by Lemma 4.4.

∎

Lemma 4.13 *Let $L \in \mathsf{N}$. There is no integer strictly between*

$$\frac{\sqrt{8L+1}-1}{2} \qquad \textit{and} \qquad \frac{\sqrt{8L+9}-1}{2}$$

Proof: Assume there were such an integer k. Then

$$\begin{aligned}
& \frac{\sqrt{8L+1}-1}{2} \;<\; k \;<\; \frac{\sqrt{8L+9}-1}{2} \\
\Longrightarrow \quad & \sqrt{8L+1}-1 \;<\; 2k \;<\; \sqrt{8L+9}-1 \\
\Longrightarrow \quad & \sqrt{8L+1} \;<\; 2k+1 \;<\; \sqrt{8L+9} \\
\Longrightarrow \quad & 8L+1 \;<\; (2k+1)^2 \;<\; 8L+9 \\
\Longrightarrow \quad & 8L+1 \;<\; 4k^2+4k+1 \;<\; 8L+9 \\
\Longrightarrow \quad & 8L \;<\; 4k^2+4k \;<\; 8L+8 \\
\Longrightarrow \quad & L \;<\; \frac{k(k+1)}{2} \;<\; L+1
\end{aligned}$$

But $\frac{k(k+1)}{2}$ is an integer. ∎

Lemma 4.14 *Assume Linda always has the initiative. Rick wins if and only if either*

$$\max\left(p, \left\lceil \frac{L}{p} + \frac{p-1}{2} \right\rceil\right) \;\le\; \left\lfloor \frac{R}{N-p-1} - \frac{N-p}{2} \right\rfloor + 1 \tag{6}$$

$$\textit{or} \qquad \left\lceil \frac{\sqrt{8L+9}-1}{2} \right\rceil \;\le\; \min\left(p, \left\lfloor \frac{R}{N-p-1} - \frac{N-p}{2} \right\rfloor + 1\right)\ . \tag{7}$$

Furthermore, if the condition holds Rick wins with a nonadaptive strategy (independent of Linda's strategy), and if the condition does not hold Linda can force at least a draw with a nonadaptive strategy (independent of Rick's strategy).

Proof: By Lemma 4.6 Rick wins if there is an integer value $k \geq p$ such that

$$\frac{L+1}{p} + \frac{p-1}{2} \;\leq\; k \;\leq\; \frac{R}{N-p-1} - \frac{N-p}{2} + 1$$

(which implies Equation (6)), or there is an integer value $k \leq p$ such that

$$\frac{\sqrt{8L+9}-1}{2} \;\leq\; k \;\leq\; \frac{R}{N-p-1} - \frac{N-p}{2} + 1$$

(which implies Equation (7)). By Lemma 4.8 Linda draws if there is an integer value $k \geq p$ such that

$$\frac{R+1}{N-p-1} - \frac{N-p}{2} \;\leq\; k \;\leq\; \frac{L}{p} + \frac{p-1}{2}\;,$$

or there is an integer value $k \leq p$ such that

$$\frac{\sqrt{8L+9}-1}{2} \;\leq\; k \;\leq\; \frac{R}{N-p-1} - \frac{N-p}{2} + 1\;.$$

We need to show that these two conditions cover all of the cases.

Assume the condition for Rick winning does not hold. Then there is no integer k such that

$$\begin{aligned} &\left(k \geq p \quad \text{and} \quad \frac{L+1}{p} + \frac{p-1}{2} \;\leq\; k \;\leq\; \frac{R}{N-p-1} - \frac{N-p}{2} + 1\right) \\ \text{or} \quad &\left(k \leq p \quad \text{and} \quad \frac{\sqrt{8L+9}-1}{2} \;\leq\; k \;\leq\; \frac{R}{N-p-1} - \frac{N-p}{2} + 1\right)\;, \end{aligned}$$

which implies for all integers k

$$\begin{aligned} &\left(k < p \quad \text{or} \quad \frac{L+1}{p} + \frac{p-1}{2} \;>\; k \quad \text{or} \quad k \;>\; \frac{R}{N-p-1} - \frac{N-p}{2} + 1\right) \\ \text{and} \quad &\left(k > p \quad \text{or} \quad \frac{\sqrt{8L+9}-1}{2} \;>\; k \quad \text{or} \quad k \;>\; \frac{R}{N-p-1} - \frac{N-p}{2} + 1\right)\;. \end{aligned}$$

Case 1: Assume $k \geq p$. Then for all integers k

$$\begin{aligned} &\frac{L+1}{p} + \frac{p-1}{2} \;>\; k \quad \text{or} \quad k \;>\; \frac{R}{N-p-1} - \frac{N-p}{2} + 1 \\ \Longrightarrow\quad &\frac{R}{N-p-1} - \frac{N-p}{2} + 1 \;<\; k \quad \text{or} \quad k < \frac{L+1}{p} + \frac{p-1}{2} \\ \Longrightarrow\quad &\frac{R+1}{N-p-1} - \frac{N-p}{2} + 1 \;\leq\; k \quad \text{or} \quad k \;\leq\; \frac{L}{p} + \frac{p-1}{2} \qquad \text{by Lemma 4.10}\;. \end{aligned}$$

By Lemma 4.11a,

$$\frac{R+1}{N-p-1} - \frac{N-p}{2} \;\leq\; k \;\leq\; \frac{L}{p} + \frac{p-1}{2}\;.$$

By Lemma 4.8 Linda draws.

Case 2: Assume $k \leq p$. Then for all integers k

$$\frac{\sqrt{8L+9}-1}{2} \;>\; k \quad \text{or} \quad k \;>\; \frac{R}{N-p-1} - \frac{N-p}{2} + 1 \;.$$

Switching the inequalities around gives

$$\frac{R}{N-p-1} - \frac{N-p}{2} + 1 \;<\; k \quad \text{or} \quad k \;<\; \frac{\sqrt{8L+9}-1}{2} \;.$$

But

$$\begin{aligned} & k < \frac{\sqrt{8L+9}-1}{2} \implies k \leq \frac{\sqrt{8L+1}-1}{2} && \text{by Lemma 4.13} \\ \text{and} \quad & \frac{R}{N-p-1} - \frac{N-p}{2} + 1 < k \implies \frac{R+1}{N-p-1} - \frac{N-p}{2} + 1 \leq k && \text{by Lemma 4.9 .} \end{aligned}$$

Hence we have

$$\frac{R+1}{N-p-1} - \frac{N-p}{2} + 1 \;\leq\; k \quad \text{or} \quad k \;\leq\; \frac{\sqrt{8L+1}-1}{2} \;.$$

By Lemma 4.11a there exists an integer k such that

$$\frac{R+1}{N-p-1} - \frac{N-p}{2} \;\leq\; k \;\leq\; \frac{\sqrt{8L+1}-1}{2} \;.$$

By Lemma 4.8 Linda draws. ∎

4.6 Final Formulas and Interpretation

Theorem 4.15 *Assume Linda always has the initiative. Linda wins if*

$$\left\lfloor \frac{L}{p} \right\rfloor \geq \left\lceil \frac{R+1}{N-p-1} \right\rceil - 1 \;.$$

Rick wins if either

$$\begin{aligned} & \max\left(p, \left\lceil \frac{L}{p} + \frac{p+1}{2} \right\rceil\right) \;\leq\; \left\lfloor \frac{R}{N-p-1} - \frac{N-p}{2} \right\rfloor + 1 \\ \text{or} \quad & \left\lceil \frac{\sqrt{8L+9}-1}{2} \right\rceil \;\leq\; \min\left(p, \left\lfloor \frac{R}{N-p-1} - \frac{N-p}{2} \right\rfloor + 1\right) \;. \end{aligned}$$

Otherwise the game is a draw.

Calling the floor -1/2 and the ceiling +1/2 gives the approximation that Linda wins if

$$\frac{L}{p} \;\geq\; \frac{R}{N-p-1} \;.$$

If we furthermore let L be large enough relative to p so that $p \leq \left\lceil \frac{L}{p} + \frac{p+1}{2} \right\rceil$, then Rick wins if

$$\frac{L}{p} + \frac{N+1}{2} \;\leq\; \frac{R}{N-p-1} \;.$$

So the game is a draw if

$$\frac{L}{p} \;<\; \frac{R}{N-p-1} \;<\; \frac{L}{p} + \frac{N+1}{2} \;.$$

This means that Rick needs about an extra half dollar for each node to win compared with Linda winning; or, to say it another way, draws have a range of about $\frac{N+1}{2}$.

We can do the calculations for when the players alternate initiative using the formulas from Appendix C. Depending on the exact conditions (the parities of p and $N-p-1$, and who has the initial initiative) the game is a draw if approximately

$$\frac{L}{p} - \frac{N+1}{4} \;<\; \frac{R}{N-p-1} \;<\; \frac{L}{p} + \frac{N+1}{4} \;.$$

So draws still have a range of about $\frac{N+1}{2}$, but the range is shifted by $\frac{N}{4}$ (which makes the game almost completely fair).

4.7 Appendix B

Lemma 4.16 *Assume that* $N = 4$, $\frac{L}{p} > \frac{R}{N-p-1}$, *and Linda always has the initiative. If* $L \geq (2^p - 1)R$ *then Linda crushes Rick and wins in* p *turns. if* $p = 1$ *she wins in*

$$2\left\lceil \log_4\left(\frac{L+R}{2(2L-R)}\right)\right\rceil + 1$$

turns, and if $p = 2$ *she wins in*

$$2\left\lceil \log_4\left(\frac{L+R}{L-2R}\right)\right\rceil$$

turns.

Proof: If $L \geq (2^p - 1)R$ then Linda crushes Rick and wins in p turns. Otherwise we proceed as follows.

Linda will bet everything at node 1 and Rick will bet everything at node 2, so $\alpha_1 = 1$ and $\alpha_2 = 0$. By Formula (5) Linda's advantage increases by a factor of more than four every time the token crosses back-and-forth across the edge between nodes 1 and 2. Thus

$$\frac{L'}{p} - \frac{R'}{N-p-1} \quad \geq \quad 4^T\left(\frac{L}{p} - \frac{R}{N-p-1}\right) \ .$$

Linda will win in one more turn if $q = 1$ and $L' \geq R'$, which is equivalent to saying that Linda has at least half of the money. Otherwise, $L' < (L+R)/2$ and $R' > (L+R)/2$, so Linda's advantage at node 1 is

$$\frac{L'}{q} - \frac{R'}{N-q-1} \quad < \quad \frac{(L+R)/2}{q} - \frac{(L+R)/2}{N-q-1} \quad = \quad \frac{L+R}{4} \ .$$

So Linda will win in one more turn if $p = 1$ and

$$4^T\left(\frac{L}{p} - \frac{R}{N-p-1}\right) \quad \geq \quad \frac{L+R}{4} \ .$$

Substituting $N = 4$ and $p = 1$, and solving gives

$$T \quad \geq \quad \log_4\left(\frac{L+R}{2(2L-R)}\right) \ .$$

The game will be over in $2T + 1$ turns, so the total number of turns t is

$$2\left\lceil \log_4\left(\frac{L+R}{2(2L-R)}\right)\right\rceil + 1 \ .$$

If $p = 2$, in one turn the token moves to node 1. From there Linda's starting money is $L - R$ and Rick's starting money is $2R$. So Linda wins in exactly

$$2\left\lceil \log_4\left(\frac{L+R}{2(2(L-R)-2R)}\right)\right\rceil + 1 + 1 \ = \ 2\left\lceil \log_4\left(\frac{L+R}{L-2R}\right)\right\rceil$$

turns. ∎

4.8 Appendix C

4.9 Alternating Initiative, Discrete Version

We give the following formulas, without proof, for Linda winning when the players alternate initiative. The derivations are similar to the case where Linda always has the initiative. For Rick winning just switch L and R and switch P and $N-p-1$ in the formulas.

Need something like $L \geq p/2$ and $R \geq (N-p-1)/2$.

Assume Linda has the initiative on odd numbered turns (and Rick has it on even numbered turns). If p and $N-p-1$ are even then Linda wins if

$$\left\lceil \frac{R+1}{N-p-1} + \frac{N-p-1}{4} - 1 \right\rceil \leq \left\lfloor \frac{L}{p} - \frac{p}{4} \right\rfloor .$$

If p is even and $N-p-1$ is odd then Linda wins if

$$\left\lceil \frac{R+1}{N-p-1} + \frac{((N-p-1)-1)((N-p-1)-3)}{4(N-p-1)} \right\rceil \leq \left\lfloor \frac{L}{p} - \frac{p}{4} \right\rfloor .$$

If p is odd and $N-p-1$ is even then Linda wins if

$$\left\lceil \frac{R+1}{N-p-1} + \frac{N-p-1}{4} - 1 \right\rceil \leq \left\lfloor \frac{L}{p} - \frac{p^2-1}{4p} \right\rfloor .$$

If p and $N-p-1$ are odd then Linda wins if

$$\left\lceil \frac{R+1}{N-p-1} + \frac{((N-p-1)-1)((N-p-1)-3)}{4(N-p-1)} \right\rceil \leq \left\lfloor \frac{L}{p} - \frac{p^2-1}{4p} \right\rfloor .$$

Assume Rick has the initiative on odd numbered turns (and Linda has it on even numbered turns). If p and $N-p-1$ are even then Linda wins if

$$\left\lceil \frac{R+1}{N-p-1} + \frac{1}{2}(\frac{N-p-1}{2} - 1) \right\rceil \quad \leq \quad \left\lfloor \frac{L}{p} - \frac{1}{2}(\frac{p}{2} - 1) \right\rfloor .$$

If p is even and $N-p-1$ is odd then Linda wins if

$$\left\lceil \frac{R+1}{N-p-1} + \frac{((N-p-1)-1)^2}{4(N-p-1)} \right\rceil \quad \leq \quad \left\lfloor \frac{L}{p} - \frac{1}{2}(\frac{p}{2} - 1) \right\rfloor .$$

If p is odd and $N-p-1$ is even then Linda wins if

$$\left\lceil \frac{R+1}{N-p-1} + \frac{1}{2}(\frac{N-p-1}{2} - 1) \right\rceil \quad \leq \quad \left\lfloor \frac{L}{p} - \frac{(p-1)^2}{4p} \right\rfloor .$$

If p and $N-p-1$ are odd then Linda wins if

$$\left\lceil \frac{R+1}{N-p-1} + \frac{((N-p-1)-1)^2}{4(N-p-1)} \right\rceil \quad \leq \quad \left\lfloor \frac{L}{p} - \frac{(p-1)^2}{4p} \right\rfloor .$$

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