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On the computational complexity of coalitional resource games

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Abstract

We study Coalitional Resource Games (CRGs), a variation of Qualitative Coalitional Games (QCGs) in which each agent is endowed with a set of resources, and the ability of a coalition to bring about a set of goals depends on whether they are collectively endowed with the necessary resources. We investigate and classify the computational complexity of a number of natural decision problems for CRGs, over and above those previously investigated for QCGs in general. For example, we show that the complexity of determining whether conflict is inevitable between two coalitions with respect to some stated resource bound (i.e., a limit value for every resource) is co-NP-complete. We then investigate the relationship between CRGs and QCGs, and in particular the extent to which it is possible to translate between the two models. We first characterise the complexity of determining equivalence between CRGs and QCGs. We then show that it is always possible to translate any given CRG into a *succinct* equivalent QCG, and that it is not always possible to translate a QCG into an equivalent CRG; we establish some necessary and some sufficient conditions for a translation from QCGs to CRGs to be possible, and show that even where an equivalent CRG exists, it may have size exponential in the number of goals and agents of its source QCG.

Keywords: Coalitional games; Resources; Computational complexity; Multi-agent systems

1. Introduction

The questions of why and how self-interested agents might choose to cooperate are central to several research areas, of which multi-agent systems is an important recent example [4,44,46]. One problem that has received particular attention is that of *coalition formation* [20,34–36,39,40]. The main question in coalition formation is that of *which coalition an agent should join*: the main answer to this question is that an agent should join a coalition that is *stable*, that is, one such that no subset of agents from the coalition would have any rational incentive to defect from it [27, p. 255].

In previous work, we introduced a model of coalitional games in which agents were assumed to cooperate with one another in order that they can mutually accomplish their goals [47]. Such *Qualitative Coalitional Games* (QCGs) seem a useful framework for modelling goal-oriented multi-agent systems. The basic idea in QCGs is that each agent

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desires to achieve one of a set of goals, and every coalition has available to it a set of choices, where each choice intuitively represents one way that the coalition could choose to cooperate. A choice is modelled as a set of goals, which would be achieved if the coalition chose to cooperate in the corresponding way. The incentive for an agent to join a coalition is that the individual choices available to this agent may not result in the satisfaction of its goals, but by cooperating, a coalition can achieve a set of goals to their mutual satisfaction. In [47], we presented a systematic survey of the complexity of decision problems associated with QCGs, and also defined an efficient representation for them, based on propositional logic.

Although QCGs seems appropriate for modelling and understanding the abstract properties of cooperation in goal-oriented multi-agent systems, they do not consider the *origin* of the choices available to coalitions. These choices are simply ascribed to coalitions via a characteristic function, in much the same way as in conventional coalitional games [27, p. 257]. In this paper, we consider a special case of QCGs, which provides one answer to the question of how these choices arise. In a *Coalitional Resource Game* (CRG), the choices available to a coalition are dependent on the *resources* available to its members and the resources required to achieve goals. Thus, in CRGs, we assume that agents have goals that they desire to achieve, exactly as in QCGs; but each agent is also assumed to have a fixed endowment of resources, while to achieve any given goal requires the expenditure of a certain profile of resources. A coalition will then form in order to pool resources to achieve a set of goals that satisfies all members of the coalition. Defined in this way, every CRG can also be understood as a QCG: given any CRG, it is possible to construct a QCG that is "equivalent", in the sense of the choices available to coalitions. Thus, given a CRG, we can ask all the questions relating to coalitions, goals sets, and QCGs generally that were studied in [47]. But it also becomes possible to ask questions relating to (for example) resource consumption (e.g., is the consumption of a given resource strictly necessary in order to satisfy the goals of a given coalition?) and resource contention (e.g., is it the case that two given coalitions cannot achieve their goals without consuming more than some stated resource bound?). In this sense, CRGs enable us to ask more *fine grained* questions about cooperation in the scenarios for which they are applicable than is possible using QCGs.

Many naturally occurring scenarios in contemporary computing and AI can be understood as CRGs. One of the most timely and important is that of *virtual organisations*, (VOs), particularly within emerging software infrastructures such as the Grid:

> VOs have the potential to change dramatically the way we use computers to solve problems, much as the Web has changed how we exchange information. [... The] need to engage in collaborative processes is fundamental to many diverse disciplines and activities. It is because of this broad applicability of VO concepts that Grid technology is important. [14]

VOs are of particular interest in collaborative science projects, where a number of partners cooperate by sharing resources (e.g., particle accelerators, super-computers or Grid networks, gene sequencers) in order to accomplish individual goals. In such situations, profit is not the motivation; the VO participants are primarily interested in accomplishing their specific goals. Such scenarios naturally map to CRGs. When the participants in a VO are software agents, then the computational questions associated with them—particularly the complexity of these questions—naturally come to the fore.

We believe that the focus on resources is very natural, given the concerns of multi-agent systems and related disciplines: resource limitations, and the need to efficiently manage and share resources in a multi-agent environment, provides one of the fundamental motivations for distributed AI and multi-agent systems [4, p. 9]. Most consideration of resources in the multi-agent systems community has been directed at the *resource allocation problem*, i.e., the problem of determining which agent or agents should have access to some scarce resource [4, p. 15]. Economic mechanisms (such as auctions) are currently the focus of much attention with respect to resource allocation [24,33]. In this paper, we focus not on the resource allocation problem, but rather on the *properties* of such allocations, and in particular, how and what coalitions may form, given a specific allocation, and what the properties of such allocations are with respect to resources.

Overall, the paper makes the following three key contributions to the computational study of games played with resources:

- First, we present ten natural decision problems associated with CRGs, and classify their computational complexity.
- Second, we investigate the relationship between CRGs and QCGs in detail. We define the notion of "equivalence" between a CRG and QCG, and show that the problem of deciding equivalence is co-NP-complete.
- Third, we investigate a number of questions associated with "translating" between CRGs and QCGs. We establish that, not only is it the case that any CRG can be represented by an equivalent QCG, but that for every CRG, there exists a *succinct* equivalent QCG. More precisely, we show that any CRG containing t resources, m goals, and n agents can be represented by a QCG of size $O(bt(n+m+b))$, where b is the number of bits used to encode endowment and resource quantity values. The proof is constructive, in that we show how to build such an equivalent QCG. With respect to translating from QCGs to CRGs, we first show that in the general case, no such translation is possible. We then define some necessary and some sufficient conditions for such a translation to be possible, and show that, even when such a translation is possible, it may result in a CRG of size exponential in $n+m$.

The paper also makes a more general contribution to the problem of how to represent coalitional games succinctly, a problem which has attracted a number of researchers over the past five years or so (cf. [3,8,17]); see Section 6.

The remainder of the paper is structured as follows. First, in Section 2, we motivate and introduce the formal framework of CRGs. Section 3 presents our main complexity results. Section 4 considers the relationship between CRGs and QCGs, Section 5 discusses variants of the CRG model in which some of the underlying assumptions are relaxed, while Section 6 presents some related work. Finally, Section 7 presents some conclusions. We begin, in the following subsection, with a summary of some key notational conventions and a *very* brief review of some relevant concepts from complexity theory.

1.1. Notation

If $f:S\to T$ is a function, then we denote the range of f by ran f, so ran $f=\{y\colon \exists x\in S \text{ such that } y=f(x)\}$.

We use the symbols $\top$ and $\bot$ as the Boolean constants for truth and falsity, respectively. In general, upper case Greek letters—Φ, Ψ, etc.—are used as meta-language variables ranging over formulae of propositional logic. In addition to the standard Boolean operations of conjunction ($\wedge$), disjunction ($\vee$), implication ($\Rightarrow$), and negation ($\neg$), some constructions will make of the binary *exclusive-or* function, which we denote by $\oplus$. For a propositional formula $\Phi(x_1,\ldots,x_n)$ defined over the variables $X_n=\langle x_1,\ldots,x_n\rangle$, given $Z\subseteq X_n$, we denote by $\Phi[Z]$ the result of evaluating Φ under the instantiation $x_i=\top$ if $x_i\in Z$, and $x_i=\bot$ if $x_i\notin Z$. Thus, $\Phi[Z]$ is equivalent to the value of $\Phi(\zeta_1,\ldots,\zeta_n)$ where the tuple $\zeta=\langle\zeta_1,\ldots,\zeta_n\rangle$ describes the characteristic vector from $\{\top,\bot\}^n$ for Z with respect to X_n. For example, given the propositional formula $\Psi(x_1,x_2,x_3)=x_1\wedge(x_2\vee x_3)$, the expression $\Psi[x_2]$ evaluates to $\bot\wedge(\top\vee\bot)$, which in turn evaluates to $\bot$. Where no ambiguity arises, we frequently omit explicit indication of conjunction, ($\wedge$), writing $\varphi\psi$ rather than $\varphi\wedge\psi$.

Much of the paper concerns the computational complexity of several questions that may be asked of CRGs: we refer the reader to [15,18,28] for introductions to this subject, and [47] for a more detailed review of the key concepts. We are concerned largely with the well-known complexity classes P, (of languages/problems that may be recognised/solved in deterministic polynomial time), and NP (of languages/problems that may be recognised/solved in non-deterministic polynomial time), and with the complement, co-NP, of NP. We also use the "difference" class, D^p [28, p. 412]: a language L is in D^p if there exist languages $L_1\in$ NP and $L_2\in$ co-NP such that $L=L_1\cap L_2$. Finally, we also make use of the slightly less well known notion of *strong* completeness [28, pp. 203–204]. Roughly speaking, a problem is said to be NP-complete in the strong sense if it remains NP-complete even when an extremely inefficient representation scheme is used for numbers appearing in instances of the problem. More formally, strong completeness for complexity classes $\mathcal{C}$ closed under polynomial time reductions is defined as follows. First, if I is an instance of a computational problem, then let *number*(I) denote the largest integer appearing in I. If Π is a computational problem, and $f:\mathbb{N}\to\mathbb{N}$ is a function from natural numbers to natural numbers, then Π_f denotes Π restricted to instances I such that *number*$(I)\leqslant f(|I|)$, where $|I|$ is the length of the encoding instance I. We then say Π is complete for $\mathcal{C}$ in the strong sense if there exists a polynomial $p:\mathbb{N}\to\mathbb{N}$ such that Π_p is $\mathcal{C}$-complete. Thus, for example, problems that are NP-complete in the strong sense remain NP-complete even if we encode instances of them using unary instead of binary for representing numbers.[1]

2. Coalitional resource games

In this section, we present the formal framework of CRGs; we begin by informally introducing and motivating the components of these structures. First, the games we study contain a (non-empty, finite) set $Ag = \{a_1, \ldots, a_n\}$ of *agents*. A *coalition*, typically denoted by C, is simply a set of agents, i.e., a subset of Ag. The *grand coalition* is the set of all agents, Ag. Each agent $i \in Ag$ is assumed to have associated with it a (finite) set G_i of *goals*, drawn from a set of overall possible goals G. The intended interpretation is that the members of G_i represent all the different ways that agent i's goals might be satisfied. That is, agent i would be happy if *any* member of G_i were achieved—but we are not concerned with preferences over individual goals. Thus, at this level of modelling, i is *indifferent* among the members of G_i: it will be *satisfied* if *at least one* member of G_i is achieved, and *unsatisfied* otherwise. Note that cases where more than one of an agent's goals are satisfied are not an issue—an agent's aim will simply be to ensure that at least one of its goals is achieved, and there is no sense of an agent i attempting to satisfy as many members of G_i as possible.

In order to bring about their goals, agents must expend *resources*. We assume a (fixed, finite, non-empty) set of resources, R, and assume that each agent is *endowed* with a (possibly zero) natural number quantity of each resource. We denote the amount of resource $r \in R$ that agent $i \in Ag$ is endowed with by $\mathbf{en}(i, r)$, thus $\mathbf{en}(i, r) \in \mathbb{N}$. Different goals may require different quantities of each resource for their achievement. We denote the amount of resource r required to achieve goal g by $\mathbf{req}(g, r)$; again, we assume that $\mathbf{req}(g, r) \in \mathbb{N}$.

Collecting these components together, we get *coalitional resource games* (CRGs).

Definition 1. A *coalitional resource game* Γ is an $(n + 5)$-tuple:

$$\Gamma = \langle Ag, G, R, G_1, \ldots, G_n, \mathbf{en}, \mathbf{req} \rangle$$

where:

- $Ag = \{a_1, \ldots, a_n\}$ is a set of *agents*;
- $G = \{g_1, \ldots, g_m\}$ is a set of *possible goals*;
- $R = \{r_1, \ldots, r_t\}$ is a set of *resources*;
- for each $i \in Ag$, $G_i \subseteq G$ is a set of goals, the intended interpretation being that any of the goals in G_i would satisfy i—but i is indifferent between the members of G_i;
- $\mathbf{en} : Ag \times R \to \mathbb{N}$ is an *endowment function*, with the intended interpretation that if $\mathbf{en}(i, r) = k$, then agent $i \in Ag$ is endowed with quantity $k \in \mathbb{N}$ of resource $r \in R$; and
- $\mathbf{req} : G \times R \to \mathbb{N}$ is a *requirement function*, with the intended interpretation that if $\mathbf{req}(g, r) = k$, then to achieve goal $g \in G$, it is necessary to expend quantity $k \in \mathbb{N}$ of resource $r \in R$.

We will assume that no goal in G is "trivially" attainable, i.e., every goal requires a non-zero expenditure of *at least one* resource. This assumption seems reasonable, since such "trivial" goals can be eliminated without altering the strategic structure of a game: since every agent can achieve such a goal, then these goals can have no effect on the formation or otherwise of specific coalitions. Formally, we assume that

$$\forall g \in G, \exists r \in R \text{ such that } \quad \mathbf{req}(g, r) > 0$$

We say a CRG is *binary* if the endowments and resources in that CRG are either zero or one, i.e., if $\operatorname{ran} \mathbf{en} \subseteq \{0, 1\}$ and $\operatorname{ran} \mathbf{req} \subseteq \{0, 1\}$. The significance of binary CRGs will become clear later, when we prove complexity results.

[1] The point being that unary is an extremely inefficient representation scheme: the natural number 1024 represented in unary is a string containing 1024 symbols, whereas the binary representation of 1024 requires only 10 symbols: binary is thus exponentially more succinct than unary. Strong completeness implies that such clumsy representations have no effect on the complexity of the problem; note that there exist computational problems that are known to be NP-complete in the ordinary sense, but which are *not* NP-complete in the strong sense—KNAPSACK is an example of such a problem [15, p. 247].

We extend the endowment function **en** to coalitions via the function $en: 2^{Ag} \times R \to \mathbb{N}$, as follows.

$$en(C, r) = \sum_{i \in C} \mathbf{en}(i, r)$$

Thus if $en(C, r) = k$, then the total amount of resource $r \in R$ available to coalition $C \subseteq Ag$ is k. Similarly, we extend the **req** function to sets of goals via the function $req: 2^G \times R \to \mathbb{N}$, as follows.

$$req(G', r) = \sum_{g \in G'} \mathbf{req}(g, r)$$

A set of goals G' *satisfies* agent i if $G' \cap G_i \neq \emptyset$; we say that G' satisfies coalition $C \subseteq Ag$ if it satisfies every member of C. We denote the set of goal sets that satisfy a coalition C by $sat(C)$.

$$sat(C) = \{G' \subseteq G: \forall i \in C, G_i \cap G' \neq \emptyset\}$$

A set of goals G' is *feasible* for coalition C if that coalition is endowed with sufficient resources to achieve all the goals in G'. We denote the set of feasible goal sets for coalition C by $feas(C)$.

$$feas(C) = \{G' \subseteq G: \forall r \in R, req(G', r) \leqslant en(C, r)\}$$

Notice that monotonically increasing coalitions have monotonically increasing feasible goal sets. That is, if $C \subseteq C'$, then $feas(C) \subseteq feas(C')$. In the terminology of [47], CRGs are thus inherently *coalition monotonic*.

Finally, we define the function $sf: 2^{Ag} \to 2^G$ to return the set of goal sets that both satisfy and are feasible for a given coalition.

$$sf(C) = sat(C) \cap feas(C)$$

Of course, unlike *feas*, *sf* is not monotonic: if $C \subseteq C'$, this does not necessarily imply $sf(C) \subseteq sf(C')$.

An example is called for.

Example 1. Consider the following CRG, which we refer to as Γ_1. We have three agents, $Ag = \{a_1, a_2, a_3\}$, with two possible goals, $G = \{g_1, g_2\}$, and two resources $R = \{r_1, r_2\}$. The goal sets for each agent are as follows.

$$G_1 = \{g_1\} \qquad G_2 = \{g_2\} \qquad G_3 = \{g_1, g_2\}$$

The endowment function **en** is defined as follows.

$$\mathbf{en}(a_1, r_1) = 2 \qquad \mathbf{en}(a_1, r_2) = 0$$
$$\mathbf{en}(a_2, r_1) = 0 \qquad \mathbf{en}(a_2, r_2) = 1$$
$$\mathbf{en}(a_3, r_1) = 1 \qquad \mathbf{en}(a_3, r_2) = 2$$

And the requirement function as follows.

$$\mathbf{req}(g_1, r_1) = 3 \qquad \mathbf{req}(g_1, r_2) = 2$$
$$\mathbf{req}(g_2, r_1) = 2 \qquad \mathbf{req}(g_2, r_2) = 1$$

There are eight possible coalitions in the game, as follows.

$$C_0 = \emptyset \qquad C_1 = \{a_1\} \qquad C_2 = \{a_2\} \qquad C_3 = \{a_1, a_2\}$$
$$C_4 = \{a_3\} \qquad C_5 = \{a_1, a_3\} \qquad C_6 = \{a_2, a_3\} \qquad C_7 = \{a_1, a_2, a_3\}$$

The endowments for these coalitions are summarised in Table 1, together with the feasible goal sets for each coalition, and the goal sets that are both feasible for and satisfy each coalition.

Table 1
Endowments, feasible goal sets, and satisfying feasible goal sets for coalitions in Example 1

	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
$en(C_x, r_1)$	0	2	0	2	1	3	1	3
$en(C_x, r_2)$	0	0	1	1	2	2	3	3
$feas(C_x)$	$\emptyset$	$\emptyset$	$\emptyset$	$\{g_2\}$	$\emptyset$	$\{\{g_1\}, \{g_2\}\}$	$\emptyset$	$\{\{g_1\}, \{g_2\}\}$
$sf(C_x)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{g_1\}$	$\emptyset$	$\emptyset$

2.1. A note on assumptions

There are several observations to make about CRGs, particularly with respect to the assumptions underlying the model.

- First, we are only concerned here with *one shot* games: we are not concerned with repeated plays of a game, where an agent must factor in concerns about reserving resources for future consumption, or where resources are potentially replenished in some way in the future. Clearly such concerns suggest themselves for future study, and some preliminary investigations into Temporal QCGs have already begun [1].
- Second, note that achieving a goal which satisfies two agents is assumed to cost no more than achieving the same goal to satisfy one agent. There is thus a fixed resource profile for achieving goals, which does not change no matter how many agents the goal satisfies.

There are some natural scenarios that cannot be directly captured within the CRG framework. Perhaps the most obvious of these are as follows.

- First, consider a scenario in which a resource, not possessed by every agent, is necessary for the accomplishment of a goal but is not consumed in achieving the goal. We cannot directly model such cases, but since we are only concerned with "one shot" CRGs (i.e., a single decision-making round, with no consideration of future resource usage), then we can capture such necessary but non-consumable resources by introducing a resource for which the requirement is 1: those agents who are endowed with the resource are given an endowment of $2^{|Ag|}$ units of it (i.e., enough for every possible coalition).
- Second, suppose there are multiple possible ways to achieve a given goal (i.e., there is not just one resource profile characterising the requirements for a given goal). Again, we can indirectly capture such scenarios by introducing multiple goals, one for each different way that a goal can be satisfied.

We comment on extensions to the basic CRGs model in Section 5.

Finally, it is worth commenting on the *representation* of CRGs. One of the main questions considered in [47] was that of how to represent QCGs, given that naive (extensive set theoretic) representations would be exponentially large in the number of agents and goals. To answer this question, a "succinct" representation of QCGs was proposed, based on propositional logic formulae. Now, it should be immediately clear that this issue *does not arise* for CRGs, because it is possible to represent the endowment function as an $n \times t$ matrix of natural numbers (where n is the number of agents and t is the number of resources), and the requirement function as a $m \times t$ matrix of natural numbers (where m is the number of goals). This representation is very obviously of size polynomial in the number of agents, goals, and resources. This raises the interesting question of whether CRGs might be used as a succinct representation for QCGs. However, as we shall see in Section 4, it is not always possible to "translate" a QCG into an equivalent CRG, and even where such a translation is in principle possible, there are cases where it inevitably results in a CRG that is exponentially large in $n + m$, i.e., translating from the original QCG to an equivalent CRG necessitates the use of exponentially many distinct resources.

3. The complexity of coalitional resource games

In this section, we present the first of our main results. We introduce and define a number of natural decision problems associated with CRGs, and then characterise their computational complexity: the results of this section are summarised in Table 2.

Before proceeding, it is worth remarking that, although we will not investigate the issue in depth until Section 4, it should be clear now that every CRG induces a QCG that is "equivalent" with respect to what it says about the choices available to coalitions, and the goal sets that coalitions can achieve. It follows that CRGs can be understood as a subset, or special case of QCGs, and hence all the questions proposed for QCGs in [47] may also be asked of CRGs. It would be legitimate to investigate whether or not there is any difference in complexity between those problems when framed in the general case, for QCGs, and the special case, for CRGs. In fact, the only problem that we investigate here that is directly equivalent to a corresponding QCG problem is SUCCESSFUL COALITION: the problem of determining

Table 2
Main complexity results relating to CRGs

Problem	Complexity	Reference
SUCCESSFUL COALITION	NP-complete	Theorem 1
MAXIMAL COALITION	co-NP-complete	Theorem 2
MAXIMAL SUCCESSFUL COALITION	D^p-complete	Theorem 3
NECESSARY RESOURCE	co-NP-complete	Theorem 4
STRICTLY NECESSARY RESOURCE	D^p-complete	Theorem 5
(C, G', r)-OPTIMAL	NP-complete	Theorem 6
R-PARETO OPTIMALITY	co-NP-complete	Theorem 7
SUCCESSFUL COALITION WITH RESOURCE BOUNDS	NP-complete	Theorem 8
CONFLICTING COALITIONS	co-NP-complete	Theorem 9
POTENTIAL GOAL SET	in P	Theorem 10

whether a given coalition would be able to cooperate in such a way as to achieve at least one goal for every member of the coalition. This problem was shown to be NP-complete in the general QCG case, and as we shall see, it is also NP-complete for the special case of CRGs. Rather than investigating again the problems we previously investigated in [47], we prefer here to focus on questions that may be asked of CRGs that cannot be asked of QCGs. However, we hypothesise that there is no difference in complexity between the QCG cases investigated in [47] for *coalition monotonic* characteristic functions, and the special case of CRGs discussed here.[2]

We divide the problems we consider into four categories, as follows:

- problems relating to coalitions;
- problems relating to resources;
- problems relating to resource bounds and resource conflicts; and
- positive results.

With respect to the first category, the first decision problem we consider is SUCCESSFUL COALITION (SC). This problem was introduced in [47] as the most fundamental question that could be asked of a QCG; as we noted above, this is the only problem from [47] that was considered in the general QCG case that we also consider for CRGs. SUCCESSFUL COALITION asks whether, given a particular CRG and a coalition in this CRG, is there a feasible choice available to the coalition that will satisfy all its members? That is, is the coalition endowed with sufficient resources to bring about a set of goals that will satisfy all its members? While we cannot be sure that a coalition will form just because it is successful, we *can* be sure that an *un*successful coalition will *not* form, because by definition such a coalition would leave at least one of its members unsatisfied. The MAXIMAL COALITION problem is that of determining whether a coalition has reached a limit, beyond which adding members will make the coalition unsuccessful; that is, the problem of checking whether every superset of a coalition is unsuccessful. Since the MAXIMAL COALITION problem does not require that the coalition in question is in fact successful itself, we also consider the MAXIMAL SUCCESSFUL COALITION problem, which considers whether, not only is the coalition maximal, but whether it is also successful.

The next set of problems we consider relate to resources. The NECESSARY RESOURCE problem is that of determining whether, given a particular CRG and a particular resource, the exploitation of that resource is necessary by every successful coalition. Knowing whether a resource is necessary can be useful in understanding the strategic structure of a scenario: for example, at least one such agent endowed with such resources will form part of *every* successful coalition. With respect to the real world, one only has to consider the role that resources such as oil play in the strategic planning of industrialised nations to understand why the consideration of necessary resources is worthwhile. Necessary resources can be thought of as being analogous to veto players in conventional coalitional games and QCGs [47]. (Indeed, it is immediately obvious that if a particular resource r is necessary, and only one agent i has a non-zero endowment of r, then i will be a veto player for every agent in the game, while if $C \subseteq Ag$ are the only agents holding a non-zero endowment of r, then every coalition must have some member of C, and so C could collectively veto

[2] If it is not already obvious, it will become clear in Section 4 why we refer here to the case of *monotonic* QCGs.

any coalition.) The STRICTLY NECESSARY RESOURCE problem considers a resource together with a coalition, and asks whether not only is the resource necessary, but that the coalition is successful—the point being that this coalition would then necessarily have to expend some of this resource, were they to form.

Next, we consider the problem of optimal resource utilisation: whether a choice represents the most efficient way of satisfying a coalition. We first consider the problem of optimality with respect to a *single* resource: the (C, G', r)-OPTIMALITY problem asks whether the goal set G' is optimal for C with respect to the usage of resource r; more precisely whether there is no way that C can succeed without expending more of resource r than they would by choosing G'. Generalising to arbitrary sets of resources, the R-PARETO EFFICIENT GOAL SET problem asks whether a particular goal set, for a particular coalition, is Pareto efficient with respect to resource usage [27, p. 122]; that is, whether or not any other goal set that represents a feasible satisfying choice for the given coalition, and which used strictly less of some resource, would inevitably use more of some other resource.

Very often, of course, a coalition will be interested in achieving some set of goals within some given resource bound—even when there is in principle more of this resource available than is allowed by the resource bound. Such scenarios are quite common when one considers the use of a finite resource (such as oil, as discussed above); here, even though we have in principle a large quantity of a resource available, we want to limit its usage now, in order that we can achieve our goals in the future. Similarly, we might wish to bound the usage of some resource for other reasons: limiting pollution emission by manufacturing plants is one obvious relevant scenario that has previously been studied in the multi-agent systems literature [23, pp. 175–211]. Here, the expenditure of a particular resource (e.g., oil) leads to the emission of pollutants, and so we seek to keep the usage of this resource within some given bound, while still satisfying our goals. The SUCCESSFUL COALITION WITH RESOURCE BOUND problem considers whether a coalition can be successful while at the same time respecting some resource bound.

Of course, where resources are bounded, resource *conflicts* may occur. This motivates us to investigate the CONFLICTING COALITIONS problem, which investigates when two given coalitions are in conflict with respect to some given resource bound. Two coalitions would be said to be in conflict with respect to a given resource bound if there is no way that they can both be successful without jointly exceeding the bound.

Finally, we consider a problem relating to goal sets in CRGs: we show that the problem of determining whether a goal set has the "potential" to be achieved within the context of a CRG—i.e., whether there is some coalition of agents such that this goal set is both feasible for and satisfies this coalition—is not only decidable by polynomial time sequential methods but also admits an efficient *parallel* realisation.[3]

3.1. Problems relating to coalitions

Recall that a coalition is *successful* if that coalition has a feasible choice that satisfies every member of the coalition.

Example 2. With reference to the CRG Γ_1 of Example 1, only one coalition is successful: namely, coalition $C_5 = \{a_1, a_3\}$. This coalition has in fact just one goal set—$\{g_1\}$—which both satisfies it and is feasible for it. To see this, simply note that $\mathbf{req}(g_1, r_1) = en(C_5, r_1)$, and $\mathbf{req}(g_1, r_2) < en(C_5, r_2)$.

Formally, the decision problem is as follows:

SUCCESSFUL COALITION: (SC)
Instance: CRG Γ and coalition C.
Answer: "Yes" if $sf(C) \neq \emptyset$.

Theorem 1. SUCCESSFUL COALITION *is* NP*-complete, even for binary* CRG*s, and hence* SUCCESSFUL COALITION *is strongly* NP*-complete.*

[3] More formally, we show "potential goal set for CRGs" is in the class NC^1 of languages recognisable by polynomial size $O(\log N)$-depth Boolean networks. One consequence of this is that potential goal set for CRGs will not (under the usual assumption $\mathrm{NC} \subset \mathrm{P}$) be P-complete, i.e., a "hardest" problem within P. For further background on parallel time complexity classes and NC we refer the reader to [19,28].

Proof. For membership, the following NP algorithm decides the problem: guess a subset G' of G and verify that both $G' \in \mathit{sat}(C)$ and $G' \in \mathit{feas}(C)$. Both steps can obviously be done in time polynomial in the size of Γ, and so the problem is in NP.

For NP-hardness, we reduce from SAT [28, p. 171]. An instance of SAT is given by a propositional logic formula $\Phi(x_1, \ldots, x_n)$, the aim being to answer "yes" if there is some valuation to the Boolean variables $x_1, \ldots, x_n$ that satisfies the formula. Without loss of generality, we assume that $\Phi(x_1, \ldots, x_n)$ is presented in Conjunctive Normal Form (CNF), i.e.,

$$\Phi(x_1, \ldots, x_n) = \bigwedge_i \psi_i$$

where each ψ_i is a clause—a disjunction of literals. We create an instance of SC as follows.

- For each literal ℓ occurring in Ψ we create a goal g_ℓ.
- For each clause ψ_i, we create an agent a_{ψ_i}, and define G_i to be the set of goals corresponding to the literals that occur in ψ_i. That is, if $\psi_i = \ell_1 \vee \cdots \vee \ell_k$, then we define $G_i = \{g_{\ell_1}, \ldots, g_{\ell_k}\}$.
- For each propositional variable x, we create a resource r_x.
- For each goal g and resource r_x, we define the quantity of r_x required for g as follows:

$$\mathbf{req}(g, r_x) = \begin{cases} 1 & \text{if } g = g_x \text{ or } g = g_{\neg x} \\ 0 & \text{otherwise.} \end{cases}$$

- We then pick the agent a_1 corresponding to the first clause, ψ_1, and define its endowment function so that for all resources r, we have $\mathbf{en}(a_1, r) = 1$. Every other agent is given an endowment of zero for every resource.

We now prove that Ag is a successful coalition in the CRG Γ_Φ if and only if $\Phi(x_1, \ldots, x_n)$ is satisfiable:

($\Rightarrow$) Assume that Ag is successful, and let G' be a goal set that is both feasible for and satisfies Ag. Then we can trivially extract a satisfying assignment for $\Phi(x_1, \ldots, x_n)$ from G': for example, if $G' = \{g_{x_1}, g_{\neg x_2}\}$, then the satisfying assignment involves assigning $\top$ to x_1 and $\bot$ to x_2. The assignment is consistent, because there is only sufficient resource corresponding to propositional variable x_i to achieve g_{x_i} or $g_{\neg x_i}$—the coalition cannot achieve both. Every clause is satisfied because every agent is satisfied, and agent i's goal set G_i corresponds to the literals in clause C_i: it must have at least one of its goals achieved, and hence must have one of these literals satisfied.

($\Leftarrow$) Assume that $\Phi(x_1, \ldots, x_n)$ is satisfiable; then from a satisfying assignment, we can extract a goal set that both satisfies the grand coalition and is feasible for this coalition: each clause must have at least one literal satisfied, and hence by construction each agent must have at least one goal satisfied; and the grand coalition must clearly have sufficient resources, since the amount of resource r_{x_i} required to achieve either g_{x_i} or $g_{\neg x_i}$ is 1, and the grand coalition has exactly this much of every resource available.

This concludes the proof that SUCCESSFUL COALITION is NP-complete in the regular sense. To see that it is strongly NP-complete, note that the CRG created in the reduction is binary: all requirements and endowments are either zero or one. Unary representations for instances of binary CRGs are clearly only polynomially larger than the size of the corresponding binary representation. □

(Note that, in the proofs that follow, we will omit the corresponding argumentation for the strong completeness results, merely noting where the reduction produces a binary CRG, which is enough to establish completeness in the strong sense.)

As is clear from the reduction used, Theorem 1 holds in the special case of C being the grand coalition. It will be useful explicitly to define this special case, independently, as a decision problem which we denote by GCS. Formally we have,

GRAND COALITION SUCCESS: (GCS)
Instance: CRG Γ.
Answer: "Yes" if $\mathit{sf}(Ag) \neq \emptyset$.

so that, from Theorem 1,

Corollary 1. GCS *is* NP*-complete, even for binary* CRG*s, and hence* GCS *is strongly* NP*-complete.*

The next problem we consider is that of whether or not a particular coalition is *maximal*; that is, whether adding any members to the coalition would lead to it being unsuccessful, in the sense of having insufficient resources to collectively satisfy its goals.

MAXIMAL COALITION: (MAXC)
Instance: CRG Γ and coalition C.
Answer: "Yes" if $\forall C'$ such that $C' \supset C$, $sf(C') = \emptyset$.

There are several points to note here. First, we require set inclusion between coalitions to be *strict* in this definition; second, the grand coalition will always be maximal; and, finally, that the formulation does *not* assume or require that C itself be successful, i.e., have $sf(C) \neq \emptyset$.

Example 3. With reference to the CRG Γ_1 of Example 1, only coalitions C_0, C_1 and C_4 are not maximal, but all others are. Since there is a successful coalition (C_5) certainly C_0 cannot be maximal. To see that the other two coalitions are not maximal, observe that there is only one successful coalition overall (C_5), and both C_1 and C_4 are strict subsets of C_5.

Theorem 2. MAXIMAL COALITION *is co*-NP*-complete, even for binary* CRG*s, and hence* MAXIMAL COALITION *is strongly co*-NP*-complete.*

Proof. For membership in co-NP, given $\langle \Gamma, C\rangle$ an instance of MAXC with $\Gamma = \langle Ag, G, R, G_1, \ldots, G_n, \mathbf{en}, \mathbf{req}\rangle$ and $C \subseteq Ag$, it suffices to check that

$$\forall C' \supset C, G' \subseteq G, \big(\forall r \in R\ en(C', r) \geqslant req(G', r)\big) \Rightarrow (\exists a_i \in C'\colon\ G_i \cap G' = \emptyset)$$

since the condition can be evaluated in deterministic polynomial-time (recall that R is part of the instance so the test $\forall r \in R\ en(C', r) \geqslant req(G', r)$ can be carried out in polynomial-time) it follows that MAXC is in co-NP.

To prove it is co-NP-hard, we show that the complementary problem—non-maximal Coalition is NP-hard using a reduction from GCS restricted to binary CRGs. Let $\Gamma = \langle Ag, G, R, G_1, \ldots, G_n, \mathbf{en}, \mathbf{req}\rangle$ be such an instance of GCS. We form an instance $\langle \Gamma', C\rangle$ of non-maximal Coalition simply by choosing $\Gamma' = \Gamma$ and $C = Ag \setminus \{a_n\}$. If Γ is accepted as an instance of GCS then $sf(Ag) \neq \emptyset$ and $C \subset Ag$, i.e., C is not a maximal coalition so that $\langle \Gamma, C\rangle$ is accepted as an instance of non-maximal Coalition. On the other hand if $\langle \Gamma, C\rangle$ is accepted as an instance of non-maximal Coalition then, since the only strict superset of C is Ag, this indicates that $sf(Ag) \neq \emptyset$, hence Γ is accepted as an instance of GCS. We note that the instance of non-maximal Coalition formed is binary giving the strong NP-completeness property claimed. □

Our formulation of MAXC does not require the coalition forming part of an instance to be successful itself. If we require C to be maximal *and* successful then the resulting decision problem—Maximal Successful Coalition (MAXSC)—becomes rather more complex.

MAXIMAL SUCCESSFUL COALITION: (MAXSC)
Instance: CRG Γ and coalition C.
Answer: "Yes" if $sf(C) \neq \emptyset$ *and* $\forall C'$ such that $C' \supset C$, $sf(C') = \emptyset$

Theorem 3. MAXSC *is* D^p*-complete, even for binary* CRG*s, and hence* MAXSC *is strongly* D^p*-complete.*

Proof. For membership in D^p we need to exhibit languages $L_1 \in$ NP and $L_2 \in$ co-NP for which the set of positive instances of MAXSC is exactly the set formed by $L_1 \cap L_2$. Choosing

$$L_1 = \{\langle \Gamma, C\rangle \colon sf(C) \neq \emptyset\}$$
$$L_2 = \{\langle \Gamma, C\rangle \colon \forall D \supset C, sf(D) = \emptyset\}$$

we have $L_1 \in$ NP via Theorem 1 and $L_2 \in$ co-NP via Theorem 2. Membership of MAXSC is now immediate from the fact that the instances that should be accepted are precisely those in the set $L_1 \cap L_2$.

For D^p-hardness we show that the problem *Critical Variable* (CV) is polynomially reducible to MAXSC. Instances of the former problem comprise a pair $\langle \Phi, z\rangle$ with Φ a CNF-formula and z a propositional variable. An instance is accepted if there is a satisfying instantiation for Φ in which $z = \top$ and no satisfying instantiation of Φ in which $z = \bot$. An easy proof that CV is D^p-complete is given in [12, Theorem 3].

Consider an instance $\langle \Phi, z\rangle$ of CV with

$$\Phi = \bigwedge_{i=1}^{m} \Psi_i$$

each Ψ_i being a clause. The instance $\langle \Gamma, C\rangle$ of MAXSC is formed exactly as the instance of SC described in the proof of Theorem 1, but with the following modifications: in addition to the agents, a_i, corresponding to the clauses Ψ_i we have an agent a_{m+1} for which $G_{m+1} = \{g_{\neg z}\}$ and for each $r \in R$, $\mathbf{en}(a_{m+1}, r) = 0$. The instance of MAXSC is then $\langle \Gamma, Ag \setminus \{a_{m+1}\}\rangle$.

Suppose that $\langle \Phi, z\rangle$ is accepted as an instance of CV. From the fact that Φ has a satisfying instantiation in which $z = \top$, we may extract a subset G' of G such that $G' \in sf(Ag \setminus \{a_{m+1}\})$ using an identical argument to that of Theorem 1, i.e., the coalition $Ag \setminus \{a_{m+1}\}$ is successful. Similarly, from the fact that there is no satisfying instantiation of Φ in which $z = \bot$ we see that Ag—the only possible strict superset of $Ag \setminus \{a_{m+1}\}$ cannot succeed: if $G' \in sf(Ag)$ then $g_{\neg z} \in G'$ (otherwise a_{m+1} fails to achieve its desired goal), and now it is not possible to identify goals for the remaining agents, i.e., literals within each clause, which can be effected using the resources available.

Conversely, if $\langle \Gamma, Ag \setminus \{a_{m+1}\}\rangle$ is accepted as an instance of MAXSC then not only is it the case that Φ is satisfiable (from the fact that $sf(Ag \setminus \{a_{m+1}\}) \neq \emptyset$), it further holds that all such satisfying instantiations can have $z = \top$: a satisfying instantiation in which $z = \bot$ would contradict the fact that $sf(Ag) = \emptyset$.

We deduce that $\langle \Phi, z\rangle$ is accepted as an instance of CV if and only if $\langle \Gamma, Ag \setminus \{a_{m+1}\}\rangle$ is accepted as an instance of MAXSC and thus that MAXSC is D^p-complete.

It remains only to observe that the CRG in the instance of MAXSC is binary, yielding strong D^p-completeness. □

3.2. Problems relating to resources

The problems above were focused primarily on issues relating to coalitions—whether they were successful, and if so, how they related to other coalitions, successful or otherwise. In this subsection, we consider questions relating instead to *resources* and their usage by successful coalitions.

The first problem we consider is NECESSARY RESOURCE. The idea of a necessary resource is somewhat analogous to that of veto players, as discussed in the context of conventional coalitional games [27, p. 187] and QCGs [47]. Thus a resource is said to be necessary if the accomplishment of any set of goals that would satisfy a coalition would imply the consumption of some of the resource. Necessary resources seem to be strategically important: a coalition cannot be successful without such a resource and thus successful coalitions require at least one member to have a positive endowment of such a resource. It follows that if a necessary resource is scarce (i.e., is possessed by only a few agents), then such agents are in a strategically important position. If only one agent has the resource, then this agent will be a veto player for *every* coalition. Formally, the decision problem is as follows.

NECESSARY RESOURCE: (NR)
Instance: CRG Γ, coalition C, and resource r.
Answer: "Yes" if $\forall G' \in sf(C)$, we have $req(G', r) > 0$.

Example 4. With reference to the CRG Γ_1 of Example 1, both resources r_1 and r_2 are necessary. To see this, note that C_5 is the only successful coalition, $sf(C_5) = \{\{g_1\}\}$: $\mathbf{req}(g_1, r_1) = 3$ and $\mathbf{req}(g_1, r_2) = 2$.

Theorem 4. NECESSARY RESOURCE *is co*-NP-*complete, even for binary* CRG*s, and hence* NECESSARY RESOURCE *is strongly co*-NP-*complete.*

Proof. For membership of co-NP, the following algorithm decides the problem:

(1) Universally select each $G' \subseteq G$;
(2) check that if $G' \in sat(C)$ and $G' \in feas(C)$ then $req(G', r) > 0$.

Since we have a single universal alternation in step (1), and step (2) can be done in polynomial time, the algorithm can be realised in co-NP.

To see that the problem is co-NP-hard, we show that its complementary version—the problem *redundant resource* (RR) whose instances are triples $\langle \Gamma, C, r\rangle$, accepted if there is some $G' \in sf(C)$ for which $req(G', r) \leqslant 0$—to be NP-hard. We note that RR *is* the complement of NR as can be seen by the following observation concerning the languages L_{NR} and L_{RR} of instances accepted

$$L_{\mathrm{NR}} = \big\{\langle \Gamma, C, r\rangle\colon\ \forall G' \subseteq G(G' \in sf(C)) \Rightarrow (req(G', r) > 0)\big\}$$
$$L_{\mathrm{RR}} = \big\{\langle \Gamma, C, r\rangle\colon\ \exists G' \subseteq G(G' \in sf(C)) \text{ and } (req(G', r) \leqslant 0)\big\}$$

That RR is NP-hard is easily shown by employing an reduction from GCS, again restricted to binary CRGs.

Given an instance, $\Gamma = \langle Ag, G, R, G_1, \ldots, G_n, \mathbf{en}, \mathbf{req}\rangle$ of the former, we construct an instance $\langle \Gamma', Ag, s\rangle$ of RR in which

$$\Gamma' = \big\langle Ag, G \cup \{g_s\}, G_1 \cup \{g_s\}, \ldots, G_n \cup \{g_s\}, R \cup \{s\}, \mathbf{en}', \mathbf{req}'\big\rangle$$

with s a new resource and g_s a new goal. For each $a \in Ag$, we fix,

$$\mathbf{en}'(a, r) = \begin{cases} \mathbf{en}(a_i, r) & \text{if } r \neq s \\ 0 & \text{if } r = s \end{cases}$$

Similarly, for $g \in G \cup \{g_s\}$ and $r \in R \cup \{s\}$ we fix

$$\mathbf{req}'(g, r) = \begin{cases} \mathbf{req}(g, r) & \text{if } r \neq s \text{ and } g \neq g_s \\ 0 & \text{if } r \neq s \text{ and } g = g_s \\ 0 & \text{if } r = s \text{ and } g \neq g_s \\ 1 & \text{if } r = s \text{ and } g = g_s \end{cases}$$

Suppose that $\langle \Gamma', Ag, s\rangle \in L_{\mathrm{RR}}$. Then there is some $G' \in sf(Ag)$ for some $G' \subseteq G \cup \{g_s\}$ and with $req'(G', s) \leqslant 0$. Since $\mathbf{req}'(g_s, s) > 0$ we must have $g_s \notin G'$, i.e., $G' \subseteq G$ with $G' \in sf(Ag)$: thus Γ is a positive instance of GCS.

On the other hand suppose that Γ is accepted as an instance of GCS. Consider the goal set G' witnessing this. In the CRG Γ' this goal set is such that $req'(G', s) = 0$ and has $G' \in sf(Ag)$, i.e., $\langle \Gamma', Ag, s\rangle \in L_{\mathrm{RR}}$.

It follows that RR is NP-complete, and therefore its complement problem, NR, is co-NP-complete as claimed. Note that the CRG produced in the reduction is binary if the instance of GCS is binary, and hence the problem is strongly co-NP-complete. □

Of course, the fact that a resource is necessary does not in fact mean that it will be used, because it could be that the coalition in question is in fact unsuccessful (in which case, every resource is trivially necessary). This concept motivates the following problem, in which we ask whether a resource really *is* necessary: that is, the coalition can succeed, and in order for it to do so, it must make use of the resource.

STRICTLY NECESSARY RESOURCE: (SNR)
Instance: CRG Γ, coalition C, and resource r.
Answer: “Yes” if $sf(C) \neq \emptyset$ and $\forall G' \in sf(C)$, we have $req(G', r) > 0$.

Example 5. With reference to the CRG Γ_1 of Example 1, resources r_1 and r_2 are strictly necessary with respect to the coalition C_5 (in addition to simply being necessary).

Theorem 5. SNR *is* D^p*-complete, even for binary* CRG*s, and hence* SNR *is strongly* D^p*-complete.*

Proof. To establish membership in D^p, we must find two languages, L_1 and L_2, such that: (i) $L_1 \in$ NP; (ii) $L_2 \in$ co-NP; and (iii) STRICTLY NECESSARY RESOURCE $= L_1 \cap L_2$ [28, pp. 412–415]. The language of the problem SUCCESSFUL COALITION immediately suggests itself as a candidate for L_1, but instances of this problem are constituted by a CRG and a coalition, with no resource present. We therefore create an AUXILIARY problem, with input instances being triples consisting of a CRG, a coalition, and a resource (i.e., taking the same form as instances of NECESSARY RESOURCE). The acceptance condition for AUXILIARY, however is the same as that of SUCCESSFUL COALITION: the additional input item is ignored. Clearly, the AUXILIARY problem is NP-complete, by the same argument as Theorem 1. We then define the language L_1 by

$$L_1 = \{x\colon \text{AUXILIARY}(x)\}$$

Since AUXILIARY is NP-complete, $L_1 \in$ NP as desired. We then define L_2 by

$$L_2 = \{x\colon \text{NECESSARY RESOURCE}(x)\}$$

By Theorem 4, $L_2 \in$ co-NP. Now by definition,

$$L_1 \cap L_2 = \{x\colon \text{STRICTLY NECESSARY RESOURCE}(x)\}$$

Thus the problem is in D^p.

To see that the problem is D^p-hard, we use a reduction from the problem SAT-UNSAT, instances of which are a pair of CNF formulae, $\langle \Phi_1, \Phi_2\rangle$, with an instance accepted if Φ_1 is satisfiable *and* Φ_2 is unsatisfiable. Notice that, without loss of generality, we may assume that the propositional variables of Φ_1 are disjoint from the propositional variables of Φ_2.

Given an instance, $\langle \Phi_1, \Phi_2\rangle$, of SAT-UNSAT for which

$$\Phi_1 = \bigwedge_{i=1}^{p} C_i = \bigwedge_{i=1}^{p} \bigvee_{j=1}^{k_i} x_{i,j}; \qquad \Phi_2 = \bigwedge_{i=1}^{q} D_i = \bigwedge_{i=1}^{q} \bigvee_{j=1}^{l_i} y_{i,j}$$

with $x_{i,j}$ a literal over the propositional variables $\{x_1, \ldots, x_n\}$ and $y_{i,j}$ a literal over the propositional variables $\{y_1, \ldots, y_n\}$, consider the CNF formula Ψ of the $2n+1$ propositional variables

$$\{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$$

given by

$$\Psi = \bigwedge_{i=1}^{p} (\neg z \vee C_i) \wedge \bigwedge_{i=1}^{q} (z \vee D_i) = \bigwedge_{i=1}^{p+q} E_i$$

Let Γ_Ψ be the CRG constructed from Ψ following the reduction of Theorem 1, but in which the goal sets G_i (which we recall mapped to literal in clauses in the earlier reduction) now being given by

$$G_i = \begin{cases} \{g_w\colon \ w \text{ is a literal in the clause } C_i\} & \text{if } E_i = \neg z \vee C_i \\ \{g_w\colon \ w \text{ is a literal in the clause } z \vee D_i\} & \text{if } E_i = z \vee D_i \end{cases}$$

The only other modifications we make to the reduction are to introduce a resource r_{null} with $\mathbf{req}(g, r_{null}) = 1$ if $g = g_{\neg z}$ and 0 otherwise; similarly $\mathbf{en}(a_{p+q}, r_{null}) = 1$. In addition for the resource r_z we fix

$$\mathbf{req}(g_w, r_z) = \begin{cases} 1 & \text{if } w = z \\ 0 & \text{otherwise} \end{cases}$$

Notice that this now gives $\mathbf{req}(g_{\neg z}, r_z) = 0$ instead of 1.

We claim that $\langle \Gamma_\Psi, Ag, r_z\rangle$ is accepted as an instance of SNR if and only if $\langle \Phi_1, \Phi_2\rangle$ is accepted as an instance of SAT-UNSAT.

(⇒) Suppose $\langle \Gamma_\Psi, Ag, r_z\rangle$ is a positive instance of SNR. Consider any $G' \in sf(Ag)$, noting that $sf(Ag) \neq \emptyset$, and the instantiation of Ψ induced by G'. For $i \in \{1, \ldots, p\}$, the goal satisfied for a_i in G_i must correspond to one of the literals in C_i from which we deduce $sf(Ag) \neq \emptyset$ implies that Φ_1 is satisfiable. Furthermore, from the fact that $req(G', r_z) > 0$ we must have $g_z \in G'$ for every $G' \in sf(Ag)$. It follows that we cannot form a subset G'' of $G \setminus \{g_z\}$ that satisfies each agent a_i with $i \in \{p+1, \ldots, p+q\}$ *and* is feasible, i.e., has $en(Ag, r) \leqslant req(G'', r)$ for each r. This suffices to prove via the reduction in Theorem 1 that Φ_2 must be unsatisfiable.

($\Leftarrow$) Suppose that Φ_1 is satisfiable and Φ_2 is unsatisfiable. Then every satisfying instantiation of Ψ must set the literal z to take the value $\top$. Letting G'' be the goals corresponding to satisfied literals defined from a satisfying instantiation of Φ_1, we see that $G' = G'' \cup \{g_z\}$ is in $sf(Ag)$ thus this set is non-empty. Furthermore, it must be the case that every $G' \in sf(Ag)$ has $g_z \in G'$ for otherwise the unsatisfiability of Φ_2 is contradicted. It therefore follows that for all $G' \in sf(Ag)$ we have $req(G', r_z) > 0$, and thus $\langle \Gamma_\Psi, Ag, r_z \rangle$ is a positive instance of SNR.

Note that the CRGs produced by this reduction are binary, giving strong D^p-completeness as claimed. □

Next, we consider the issue of *minimising* resource usage. Our starting point is this: Suppose a coalition C has two feasible and satisfying goal sets, G_1 and G_2. Then clearly, the coalition will be happy with either of these goal sets being achieved: if satisfaction was the only concern, then all other things being equal, the coalition would be indifferent about which of G_1 or G_2 was actually achieved. However, it is unlikely that satisfaction will be the only concern. In particular, we might expect coalitions to desire to *minimise* resource usage—the availability of unused resources will enable coalitions to achieve goals in the future. This suggests the following problem.

(C, G', r)-OPTIMALITY: (CGRO)
Instance: CRG Γ, coalition C, and goal set G' with $G' \in sf(C)$, and resource r.
Answer: "Yes" if $\forall G'' \in sf(C)$, we have $req(G'', r) \geqslant req(G', r)$.

Notice that if a coalition is unsuccessful, then any goal set is optimal with respect to any resource for that coalition. Also, if a coalition has only one choice available that both satisfies and is feasible for that coalition, then that goal set is optimal for all resources for that coalition.

Example 6. With reference to the CRG Γ_1 of Example 1, goal set $G_1 = \{g_1\}$ is optimal for coalition C_5 with respect to all resources.

Theorem 6. (C, G', r)-OPTIMALITY *is co-*NP*-complete, even for binary* CRG*s, and hence* (C, G', r)-OPTIMALITY *is strongly co-*NP*-complete.*

Proof. To see that CGRO is in co-NP it suffices to note that an instance $\langle \Gamma, C, G', r \rangle$ is accepted if and only if for all $G'' \subseteq G$,

$$\big(G'' \in sf(C)\big) \Rightarrow \big(req(G'', r) \geqslant req(G', r)\big)$$

The predicate described can be evaluated in time polynomial in the size of the instance.

To prove co-NP-hardness we consider the complementary problem, (C, G', r)-*sub-optimality*. Instances for this are a 4-tuple, $\langle \Gamma, C, G', r \rangle$ as before, but with these accepted if and only if

$$\exists G'' \subseteq G: \quad G'' \in sf(C) \quad \text{and} \quad \mathbf{req}(G'', r) < req(G', r)$$

Denoting this problem CGRSO, we show that GCS for binary CRGs is polynomial reducible to CGRSO.

Given such an instance $\Gamma = \langle Ag, G, R, G_1, \ldots, G_n, \mathbf{en}, \mathbf{req} \rangle$ of GCS we form an instance of $\langle \Gamma', C, G', s \rangle$ of CGRSO in which

$$\Gamma' = \big\langle Ag, G \cup \{g_s\}, G_1 \cup \{g_s\}, \ldots, G_n \cup \{g_s\}, R \cup \{s\}, \mathbf{en}', \mathbf{req}' \big\rangle$$

where $g_s \notin G$ is a new goal and $s \notin R$ a new resource. In Γ', $\mathbf{req}'(g_s, s) = 1$ and $\mathbf{req}'(g_s, r) = 0$ for all $r \in R$ with $\mathbf{req}'(g, s) = 0$ for each $g \in G$; in addition $\mathbf{en}'(a_i, s) = 1$ for each $a_i \in Ag$. We complete the instance $\langle \Gamma', C, G', s \rangle$ by fixing $C = Ag$ and $G' = \{g_s\}$.

If $\langle \Gamma', Ag, \{g_s\}, s \rangle$ is accepted as an instance of CGRSO then there is some $G' \subseteq G \cup \{g_s\}$ such that $G' \in sf(Ag)$ and $req(G', s) < req(\{g_s\}, s)$, from which if follows that $g_s \notin G'$ and thus, G' witnesses to Γ being a positive instance of GCS.

Conversely if Γ is accepted as an instance of GCS, then there is some $G' \subseteq G$ with which $G' \in sf(Ag)$. Clearly $G' \subset G \cup \{g_s\}$ and (from $\mathbf{req}'(g,s) = 0$ for each $g \in G$),

$$req(G', s) = 0 < req(\{g_s\}, s) = 1$$

so that $\langle \Gamma', Ag, \{g_s\}, s\rangle$ is a positive instance of CGRSO.

This completes the proof that CGRSO is NP-complete and thus its complement—the problem CGRO—is co-NP-complete.

Note that if the instance of GCS is binary then the CRG produced in the reduction is also binary, giving strong co-NP-completeness as claimed. □

(C, G', r)-OPTIMALITY measures how good a goal set is with respect to the usage of *one* resource. An obvious question is how this generalises to the set of *all* resources. To measure the optimality of a goal set with respect to the set of all resources, we use the idea of *Pareto efficiency*. Pareto efficiency is often considered in the context of outcomes to bargaining—it is one of the three properties uniquely characterising the Nash solution to bargaining problems [27, p. 305]. However, the concept of Pareto efficiency has also found wide currency in other economic settings.

In general, an outcome ω is said to be Pareto efficient with respect to a set of agents if there is no other outcome ω' that is preferred at least as much as ω by every agent, and is strictly preferred over ω by at least one of the agents. If an outcome is not Pareto efficient, then intuitively, the agents can select another solution such that nobody would be worse off, and some would be strictly better off. In our setting, an "outcome" equates to a goal set for some coalition, and we consider whether an outcome is better or worse with respect to *resource usage*. Thus if G' is a goal set and C is a coalition, then we say a goal set G'' which is both feasible for and satisfies every member of C, and which requires no more of every resource and strictly less of some resource than G' is said to be an *R-Pareto improvement* over G'. A goal set G' is said to be *R-Pareto efficient* (with respect to C) if there does not exist any goal set G'' which represents R-Pareto improvement over G, i.e., if every other goal set that is both feasible for an satisfies all members of C requires *at least as much* of every resource, and *strictly more* of some resource as G'.

Formally, goal set G' is R-Pareto efficient with respect to some coalition C iff:

$$\begin{aligned}
&\forall G'' \in sf(C):\\
&\quad \big[\exists r_1 \in R\colon\ req(G'', r_1) < req(G', r_1)\big] \Rightarrow\\
&\quad \big[\exists r_2 \in R\colon\ req(G'', r_2) > req(G', r_2)\big]
\end{aligned}$$

The decision problem is then as follows.

R-PARETO EFFICIENT GOAL SET: (RPOGS)
Instance: CRG Γ, coalition C, and goal set G'.
Answer: "Yes" if G' is R-Pareto efficient for C.

(As an aside, notice that the formulation of this problem does *not* require that $G' \in sf(C)$.) Although the statement of the R-Pareto efficiency condition seems somewhat involved, it turns out that there is an elegant reduction which establishes its complexity.

Theorem 7. *The R-PARETO EFFICIENT GOAL SET problem is strongly co-NP-complete.*

Proof. Membership is witnessed by the following algorithm:

(1) Universally select all $G'' \subseteq G$;
(2) Check that if $G'' \in sf(C)$, then G'' is not an R-Pareto improvement over G'.

Since the algorithm contains a single universal alternation, and step (2) can be checked in polynomial time, the problem can be decided in co-NP.

For hardness, we reduce GCS for binary CRGs to the complement of the problem, i.e., the problem of checking whether there exists some $G'' \in sf(C)$ such that G'' represents an R-Pareto improvement over G'. More formally, the complement problem asks whether the following formula is true with respect to a given Γ, C, and G':

$$\exists G'' \in sf(C){:}$$
$$\big[\exists r_1 \in R{:}\ req(G'', r_1) < req(G', r_1)\big] \wedge$$
$$\big[\forall r_2 \in R{:}\ req(G'', r_2) \leqslant req(G', r_2)\big]$$

Given an instance $\Gamma = \langle Ag, G, R, G_1, \ldots, G_n, \mathbf{en}, \mathbf{req}\rangle$ of GCS we form an instance $\langle \Gamma', C, G'\rangle$ of co-RPOGS using

$$\Gamma' = \big\langle Ag, G \cup \{g_s\}, G_1 \cup \{g_s\}, \ldots, G_n \cup \{g_s\}, R \cup \{s\}, \mathbf{en}', \mathbf{req}'\big\rangle$$

with $g_s \notin G$ and $s \notin R$ a new goal and new resource. For $\mathbf{en}'$ and $\mathbf{req}'$ we use

$$\mathbf{en}'(a_i, r) = \begin{cases} \mathbf{en}(a_i, r) & \text{if } r \neq s \\ 1 & \text{if } r = s \end{cases}$$

$$\mathbf{req}'(g, r) = \begin{cases} 1 & \text{if } g = g_s \text{ and } r = s \\ req(G, r) & \text{if } g = g_s \text{ and } r \neq s \\ 0 & \text{if } g \neq g_s \text{ and } r = s \\ \mathbf{req}(g, r) & \text{if } g \neq g_s \text{ and } r \neq s \end{cases}$$

To complete the instance we set $C = Ag$ and $G' = \{g_s\}$

If $\langle \Gamma', Ag, \{g_s\}\rangle$ is a positive instance of co-RPOGS then we can identify $G'' \subseteq G \cup \{g_s\}$ and some $r' \in R \cup r_s$ with $G'' \in sf(Ag)$ and for which

$$req'(G'', r') < req'\big(\{g_s\}, r'\big) \quad \text{and}$$
$$\forall r \in R \cup \{s\} req'(G'', r) \leqslant req'\big(\{g_s\}, r\big)$$

First observe that any such G'' cannot contain g_s: for otherwise we have $req'(G'', s) = 1 = req(\{g_s\}, s)$ and, in addition for every $r \neq s$

$$req(G'', r) = req\big(G'' \setminus \{g_s\}, r\big) + req\big(\{g_s\}, r\big) \geqslant req\big(\{g_s\}, r\big)$$

so that the requirement that strictly less of some resource be expended in realising G'' would fail to hold. We deduce that $G'' \subseteq G$ with $G'' \in sf(Ag)$ and hence, from $\langle \Gamma', Ag, \{g_s\}\rangle$ being a positive instance of co-RPOGS, Γ is a positive instance of GCS as required.

Conversely, suppose there is some $G'' \subseteq G$ such that $G'' \in sf(Ag)$, i.e., that Γ is accepted as an instance of GCS. Consider the corresponding set of goals within the CRG, Γ'. From the fact that $g_s \notin G''$ we obtain,

$$req(G'', s) = 0 < 1 = req\big(\{g_s\}, s\big)$$

Furthermore, for every resource other than s, since $G'' \subseteq G$

$$req(G'', r) \leqslant req(G, r) = req\big(\{g_s\}, r\big)$$

Thus there is a resource (s) requiring strictly less usage in realising G'' in comparison with realising $\{g_s\}$ and no resource requires strictly greater usage. In total these properties indicate that $\langle \Gamma', Ag, \{g_s\}\rangle$ is a positive instance of co-RPOGS.

We note that the CRG, Γ', constructed in this reduction is *not* (necessarily) binary if Γ is. We may deduce strong NP-hardness for co-RPOGS by observing that (when Γ is binary) we may represent $\mathbf{en}'$ and $\mathbf{req}'$ in unary whilst retaining the polynomial-time bound on the reduction: the only non-binary element in Γ' arises in the form of $\mathbf{req}'(g_s, r)$ for $r \neq s$. This, however, is $req(G, r)$, which when Γ is binary is at most $|G|$. Hence, were co-RPOGS efficiently decidable using unary representation, our reduction would give an efficient decision procedure for deciding GCS restricted to binary CRGs. □

3.3. Problems relating to resource bounds and resource conflicts

So far, we have said nothing about resource bounds: whether it is possible for a coalition to achieve a particular set of goals using at most a given amount of each resource.

A *resource bound* is a function $\mathbf{b} : R \to \mathbb{N}$, with the intended interpretation that if $\mathbf{b}(r) = k$, then only k units of resource r are available in total, for all coalitions. As with CRGs in general, we say a resource bound is binary if the bound for every resource is either zero or one. We say a goal set G' *respects* a resource bound $\mathbf{b}$ (with respect to some given CRG) if the total amount of every resource required to achieve all goals in G' is less than or equal to the associated bound for that resource. We define a two place predicate $respects(G', \mathbf{b})$ to capture this idea.

$$respects(G', \mathbf{b}) \quad \text{iff} \quad \forall r \in R, req(G', r) \leqslant \mathbf{b}(r)$$

The obvious decision problem, corresponding to SUCCESSFUL COALITION is as follows.

SUCCESSFUL COALITION WITH RESOURCE BOUND: (SCRB)
Instance: CRG Γ, coalition C, and resource bound $\mathbf{b}$.
Answer: "Yes" if $\exists G' \in sf(C)$ s.t. $respects(G', \mathbf{b})$.

Example 7. With reference to the CRG Γ_1 of Example 1, consider resource bounds $\mathbf{b}_1$, where $\mathbf{b}_1(r_1) = 3$ and $\mathbf{b}_1(r_2) = 2$, and $\mathbf{b}_2$, where $\mathbf{b}_2(r_1) = 2$ and $\mathbf{b}_2(r_2) = 2$. Then coalition C_5 is successful with respect to $\mathbf{b}_1$ but not $\mathbf{b}_2$.

Theorem 8. SUCCESSFUL COALITION WITH RESOURCE BOUND *is* NP*-complete, even for binary* CRG*s and resource bounds, and hence* SUCCESSFUL COALITION WITH RESOURCE BOUND *is strongly* NP*-complete.*

Proof. Almost identical to that of Theorem 1. The membership argument is basically the same: verifying the additional constraint (that $respects(G', \mathbf{b})$) can clearly be done in polynomial time. For NP-hardness, the same reduction from SAT used in Theorem 1 works—in the instance of SCRB that we create, we set the resource bound to be 1, for every resource. Notice that both the CRG and the resource bound that we create in this reduction are binary, giving strong NP-completeness as claimed. □

Resource contention—when two agents or coalitions desire to use some resource that cannot be used by both—is perhaps the paradigm source of conflict in real and artificial social systems. Resource contention is a classic problem in distributed and concurrent systems, for example where an operating system must enforce mutual exclusion over resources such as memory, printers, and communication channels [2, p. 15]. And of course, the problem remains in multi-agent systems, where the resources in question are typically more complex artifacts. In our framework, we consider conflicts between sets of coalitions with respect to some resource bound.

First, let us say that two goal sets over some CRG are *in conflict* with respect to some resource bound if the goal sets are individually achievable within this bound, and yet are not jointly achievable within it. We define a three place predicate $cgs(\ldots)$, where $cgs(G_1, G_2, \mathbf{b})$ is intended to mean that goal sets G_1 and G_2 are in conflict with respect to resource bound $\mathbf{b}$.

$$cgs(G_1, G_2, \mathbf{b}) \quad \text{iff} \quad \big(\big(respects(G_1, \mathbf{b}) \text{ and } respects(G_2, \mathbf{b})\big) \text{ and not } respects(G_1 \cup G_2, \mathbf{b})\big).$$

Notice that the predicate $cgs(\cdots)$ can of course be computed in polynomial time. Given this predicate, we can define what it means for two coalitions to be in conflict. We will say two coalitions are in conflict with respect to some given resource bound if there is no way that they can both be satisfied and jointly respect the resource bound. The decision problem is as follows.

CONFLICTING COALITIONS: (CC)
Instance: CRG Γ, coalitions C_1, C_2, and resource bound $\mathbf{b}$.
Answer: "Yes" if $\forall G_1 \in sf(C_1), \forall G_2 \in sf(C_2)$, we have $cgs(G_1, G_2, \mathbf{b})$.

Theorem 9. CC *is co-*NP*-complete, even for binary* CRG*s, and hence* CC *is strongly co-*NP*-complete.*

Proof. Membership in co-NP is immediate from the fact that an instance $\langle \Gamma, C_1, C_2, \mathbf{b}\rangle$ is accepted if and only if

$$\forall G_1, G_2 \subseteq G \quad \big(G_1 \in sf(C_1) \text{ and } G_2 \in sf(C_2)\big) \Rightarrow cgs(G_1, G_2, \mathbf{b})$$

To see that CC is co-NP-hard, consider the complement problem in which an instance $\langle \Gamma, C_1, C_2, \mathbf{b}\rangle$ is accepted if and only if

$$\exists G_1, G_2 \subseteq G \quad \big(G_1 \in sf(C_1) \text{ and } G_2 \in sf(C_2) \text{ and } \big(\forall r \in R\ req(G_1 \cup G_2, r) \leqslant b(r)\big)\big)$$

That this problem is NP-hard, is a trivial reduction from SCRB: given an instance $\langle \Gamma, C, \mathbf{b}\rangle$ of the latter, simply form the instance $\langle \Gamma, C, C, \mathbf{b}\rangle$. If $\langle \Gamma, C, \mathbf{b}\rangle$ is accepted as an instance of SCRB, via some goal set G', then $G_1 = G_2 = G'$ satisfies

$$\big(G_1 \in sf(C) \text{ and } G_2 \in sf(C) \text{ and } \big(\forall r \in R\ req(G_1 \cup G_2, r) \leqslant b(r)\big)\big)$$

On the other hand if there are sets G_1, G_2 in $sf(C)$ for which every $r \in R$ has $req(G_1 \cup G_2, r) \leqslant b(r)$ then $req(G_1, r) \leqslant b(r)$ and thus $\langle \Gamma, C, \mathbf{b}\rangle$ is accepted as an instance of SCRB. It follows that the complementary problem to CC is NP-hard and hence CC is co-NP-hard. □

3.4. Positive results

Although the results above indicate that decision problems involving questions about coalitional properties in CRGs tend to be hard, in the case of questions about properties of *goal sets* it is possible to find efficient algorithmic solutions. For example, consider the problem,

POTENTIAL GOAL SET: (PGS)
Instance: CRG Γ, and set of goals G'.
Answer: "Yes" if $\exists C \subseteq Ag$: $G' \in sf(C)$.

Theorem 10. PGS *is polynomial-time decidable.*

Proof. Given $\Gamma = \langle Ag, G, R, G_1, \ldots, G_n, \mathbf{en}, \mathbf{req}\rangle$ and $G' \subseteq G$, consider the coalition, $C_{\max} \subseteq Ag$ defined as

$$C_{\max} = \bigcup_{g \in G'} \{a_i\colon\ g \in G_i\}$$

If $G' \in sf(D)$ for some $D \subseteq Ag$, then it must be the case that $D \subseteq C_{\max}$: for every $a_j \notin C_{\max}$ we have $G_j \cap G' = \emptyset$, hence $D \not\subseteq C_{\max}$ is inconsistent with $G' \in sf(D)$. It follows that in order to decide if $\langle \Gamma, G'\rangle$ should be accepted it suffices to test if $G' \in sf(C_{\max})$, i.e., if for every $r \in R$, we have $en(C_{\max}, r) \leqslant req(G', r)$. These tests can clearly be performed in polynomial-time. □

We can, in fact, obtain a rather stronger result, indicating that PGS is decidable by an efficient *parallel* algorithm.

Theorem 11. PGS $\in$ NC^1*, i.e., there is a uniform family of N-input Boolean combinational logic networks,*[4] $\langle S_N\rangle$ *which: given an instance of* PGS *encoded as a binary N-tuple, S_N returns $\top$ if and only if the instance should be accepted; the size of S_N is polynomially-bounded* (*as a function of N*); *and the* (*parallel*) *run-time of S_N is* $O(\log N)$.[5]

Proof. The result follows directly from the translation given in proving Theorem 14. The details are presented following this. □

[4] A family of N-input Boolean combinational logic networks is *uniform* if there is a polynomial-time Turing machine program that given 1^N as input constructs the encoding of the Nth member of the family.

[5] The formal definition of "parallel run-time" is as the "depth" of the combinational logic network, see, e.g. [11, p. 20].

4. CRGs and QCGs

In this section, we consider the relationship between CRGs, as introduced in this paper, and QCGs, as introduced in [47].[6] Recall that a QCG is a structure

$$Q = \langle G, Ag, G_1, \dots, G_n, \Psi \rangle$$

in which the Ag, G, and G_i components have exactly the same intended interpretation as in CRGs, and Ψ is a formula of propositional logic, which represents the *characteristic function* of the game. The characteristic function of a QCG captures the different ways in which coalitions in the game can cooperate, and in particular, the different sets of goals that coalitions can bring about. More precisely, Ψ is a formula of propositional logic over two sets of propositional variables, corresponding to agents and goals, respectively. The idea is that $\Psi[C, G'] = \top$ iff the goal set $G' \subseteq G$ is a feasible choice for coalition $C \subseteq Ag$, that is, if one of the ways that coalition C can cooperate will result in all the goals G' being achieved.

4.1. CRG-QCG equivalence: basic definitions and complexity

We say that a CRG Γ and a QCG Q are *comparable* if their Ag, G, and G_i components are the same, i.e., if they contain the same agents and possible goal sets, and the goal sets associated with each agent are the same in both. In the decision problems that follow, in which we compare CRGs and QCGs, we will assume that unless explicitly stated otherwise, the CRG and QCG in question are comparable.

The first obvious formal question to ask is, given a comparable CRG and QCG, whether the two are *equivalent*, in the sense that they agree on the choices available to coalitions. Formally, given a CRG $\Gamma = \langle Ag, G, R, G_1, \dots, G_n, \mathbf{en}, \mathbf{req} \rangle$ and QCG $Q = \langle G, Ag, G_1, \dots, G_n, \Psi \rangle$, we say that they are C, G'*-equivalent* if

$$\underbrace{G' \in feas(C)}_{\text{CRG}} \quad \text{iff} \quad \underbrace{\Psi[C, G'] = \top}_{\text{QCG}}$$

We indicate that CRG Γ and QCG Q are C, G' equivalent by writing $\Gamma \equiv_{C,G'} Q$. We simply write $\Gamma \equiv Q$ to indicate that $\Gamma \equiv_{C,G'} Q$ for all $C \subseteq Ag$, $G' \subseteq G$. We now come to the first decision problem that relates CRGs to QCGs.

EQUIVALENCE: (EQUIV)
Instance: CRG Γ and QCG Q.
Answer: “Yes” if $\Gamma \equiv Q$.

Theorem 12. EQUIVALENCE *is co*-NP*-complete, even for binary* CRG*s, and hence* EQUIVALENCE *is strongly co*-NP*-complete.*

Proof. Membership of co-NP follows from the fact that the following algorithm decides the problem:

(1) Universally select each $C \subseteq Ag$ and $G' \subseteq G$.
(2) Check that $\Gamma \equiv_{C,G'} Q$.

To establish that the problem is co-NP-hard, we reduce from TAUT, the problem of determining whether a given formula $\Phi(x_1, \dots, x_n)$ of propositional logic is satisfied by *every* valuation of its propositional variables. Given an instance $\Phi(x_1, \dots, x_n)$ of this, we form an instance of EQUIV as follows. First, let $Ag = \{x_1, \dots, x_n\}$, i.e., we create an agent for each propositional variable. We then create a single “dummy” goal and set $G = \{g_{dummy}\}$ and $G_i = G$ for all $i \in Ag$.

[6] For completeness, we give the relevant technical definitions relating to QCGs, but our presentation is brief, and in particular we do not reproduce any of the motivating discussion from [47].

For the CRG Γ_Φ that forms part of the instance of EQUIV, we define a single resource $R = \{r_{dummy}\}$, and let $\mathbf{req}(g_{dummy}, r_{dummy}) = 1$; we then define $\mathbf{en}(i, r_{dummy}) = 1$, for each agent $i \in Ag$. We note that this construction satisfies

$$feas(C) = \begin{cases} \{\{g_{dummy}\}, \emptyset\} & \text{if } C \neq \emptyset \\ \{\emptyset\} & \text{if } C = \emptyset \end{cases}$$

Finally, for the QCG instance Q_Φ we create, we define the characteristic function formula Ψ by

$$\Psi(x_1, \ldots, x_n, g_{dummy}) = \Phi(x_1, \ldots, x_k) \wedge \left(\neg g_{dummy} \vee \bigvee_{i=1}^{n} x_i\right)$$

We now claim that $\Phi(x_1, \ldots, x_n)$ is a tautology if and only if $\Gamma_\Phi \equiv Q_\Phi$.

($\Rightarrow$) Assume $\Phi(x_1, \ldots, x_n)$ is a tautology; then

$$\Psi[C, G'] = \begin{cases} \top & \text{if } C \neq \emptyset \\ \top & \text{if } C = \emptyset \text{ and } G' = \emptyset \\ \bot & \text{if } C = \emptyset \text{ and } G' \neq \emptyset \end{cases}$$

From which it follows that $G' \in feas(C)$ (for the CRG Γ_Φ) if and only if $\Psi[C, G'] = \top$ (in the QCG Q_Φ). That is, if $\Phi(x_1, \ldots, x_n)$ is a positive instance of TAUT then $\Gamma_\Phi \equiv Q_\Phi$.

($\Leftarrow$) Assume $\Gamma_\Phi \equiv Q_\Phi$. Then for all $G' \subseteq G$, $C \subseteq Ag$, we have $G' \in feas(C)$ if and only if $\Phi[C, G'] = \top$. By the construction we have $G' \in feas(C)$ for $G' = \{g_{dummy}\}$ and C any *non-empty* subset of Ag. Similarly $G' \in feas(\emptyset)$ for $G' = \emptyset$. From the construction $\Psi[C, \emptyset] \equiv \Phi[C]$, thus, since $\emptyset \in feas(C)$ from the assumption that $\Gamma_\Phi \equiv Q_\Phi$, we deduce that $\Phi[C] = \top$ for every choice of C, i.e., that $\Phi(x_1, \ldots, x_n)$ is a tautology.

Note that the CRGs produced in the reduction are binary, giving strong co-NP-completeness as claimed. □

This result suggests an obvious question: are CRGs and QCGs equivalent in expressive power? That is, can we "translate" an arbitrary CRG into an equivalent QCG, and vice versa? Thus, our main interest in subsequent subsections concerns the following four questions:

(1) Given a CRG, Γ, is there always a QCG, Q_Γ such that $Q_\Gamma \equiv \Gamma$?
(2) Given a QCG, Q, is there always a CRG, Γ_Q such that $\Gamma_Q \equiv Q$?
(3) How "efficiently" can a given CRG be expressed as an equivalent QCG in those cases where such an equivalent structure exists?
(4) How "efficiently" can a given QCG be expressed as an equivalent CRG in those cases where such an equivalent structure exists?

The first two questions are answered by the following.

Theorem 13. QCG*s are strictly more expressive than* CRG*s. More precisely*:

(a) *For every* CRG Γ *there exists a* QCG Q_Γ *such that* $\Gamma \equiv Q_\Gamma$.
(b) *It is not the case that for every* QCG Q *there exists a* CRG Γ_Q *such that* $\Gamma_Q \equiv Q$.

Proof. The first part is proved in Theorem 14. For the second part, it suffices to note that any QCG corresponding to a given CRG must be *coalition monotonic* [47]. But as we noted in Section 2, there exist QCGs that do not have this property, which hence have no equivalent CRG. □

4.2. Translating CRGs to QCGs

Theorem 13 tells us that we can always translate a CRG into an equivalent QCG. Now CRGs and QCGs are based on entirely different representations—in QCGs we use the propositional logic representation of a game's characteristic function, while we have no need of such a representation in CRGs, since the feasible choices available to a coalition are implicit within the requirement and endowment functions. The fact that we can translate from CRGs to QCGs means that, in principle, when dealing with scenarios modelled by CRGs, we have a choice of representations available: we can work with either the requirements and endowments representation of CRGs, or the propositional logic representation of QCGs. The latter representation would make it possible to use the extensive technology of propositional logic reasoning developed within AI when reasoning about CRGs. However, the fact that we can translate a CRG to an equivalent CRG does not imply that such a translation can be done *efficiently*. If the best translation we can find leads to a propositional logic QCG representation that is (say) exponential in the size of the CRG, then the fact that we can in principle translate one to the other is likely to be of little practical value. Indeed, there is an "obvious" translation of CRGS to QCGs which does indeed lead to an exponentially sized characteristic function formula. However, it turns out that we can *always* do the translation efficiently. We prove in Theorem 14 that, not only is it the case that every CRG may be described by an equivalent QCG, but also that this translation produces a QCG characteristic function formula that has an $O(b^2t)$ overhead over the size of the original CRG, where b is the maximum number of bits required to encode the endowment and requirement values, and t is the number of resources.

To make this precise, we first need to define our notions of "efficiency" and, in particular, the manner in which the "sizes" of CRGs and QCGs are captured. In what follows, recall that $m = |G|$ is the number of goals $n = |Ag|$ is the number of agents, and $t = |R|$ is the number of resources. Now, in both formalisms we have $\langle G, Ag, G_1, \ldots, G_n\rangle$ as elements in common, within CRGs the systems of feasible goal sets for coalitions are described via $\langle R, \mathbf{en}, \mathbf{req}\rangle$ and for QCGs via some propositional formula, Ψ over propositional variables Ag and G. For a CRG the significant contributing factor is, therefore, the space required to encode the $n \times t$ matrix of *endowments* and the $m \times t$ matrix of *requirements*. Thus, how "efficiently" some system of feasible goal sets for coalitions is described within in a CRG is determined by two measures: t, the number of resources employed; and the *number of bits* used in representing the values $\mathbf{en}(a, r)$, $\mathbf{req}(g, r)$. For $\Gamma = \langle Ag, G, R, G_1, \ldots, G_n, \mathbf{en}, \mathbf{req}\rangle$, with $|Ag| = n$, $|G| = m$, and $|R| = t$, fix

$$b = 1 + \max_{r\in R,\ a\in A,\ g\in G} \left\{ \lfloor \log_2 \mathbf{en}(a, r)\rfloor, \lfloor \log_2 \mathbf{req}(g, r)\rfloor \right\}$$

From which we may assume that each value of $\mathbf{en}(a, r)$, $\mathbf{req}(g, r)$ is represented in binary using exactly b bits.

We can now define the *size* of a CRG, $\Gamma = \langle Ag, G, R, G_1, \ldots, G_n, \mathbf{en}, \mathbf{req}\rangle$, denoted $M^{\mathrm{CRG}}(\Gamma)$, as

$$M^{\mathrm{CRG}}(\Gamma) = b|R|\big(|Ag| + |G|\big) = bt(n + m)$$

In contrast, for a representation via a QCG, we define the size of a QCG, Q, denoted $M^{\mathrm{QCG}}(Q)$, as $|\Psi|$, i.e., the total number of occurrences of literals used in presenting the formula Ψ, where only binary Boolean operations and negation may be used in forming Ψ. (Note that we consider some general issues respecting the measures M^{CRG} and M^{QCG} at the end of this sub-section.)

Theorem 14. *For every* CRG, $\Gamma = \langle Ag, G, R, G_1, \ldots, G_n, \mathbf{en}, \mathbf{req}\rangle$*, with* $|Ag| = n$*,* $|G| = m$*, and* $|R| = t$*, there exists a* QCG, $Q_\Gamma = \langle G, Ag, G_1, \ldots, G_n, \Psi\rangle$ *such that* $\Gamma \equiv Q_\Gamma$ *and for which*

$$|\Psi| = O\big(bt(n + m + b)\big) = O\big(M^{\mathrm{CRG}}(\Gamma)\big) + O\big(b^2t\big)$$

Proof. The proof is constructive. Given $\Gamma = \langle Ag, G, R, G_1, \ldots, G_n, \mathbf{en}, \mathbf{req}\rangle$ we form an equivalent QCG, Q_Γ, as follows. Since Ag, G, and $\langle G_1, \ldots, G_m\rangle$ are the same sets in Γ and Q_Γ, the major part of the construction is concerned with building the formula Ψ within Q_Γ so that for $C \subseteq Ag$, and $G' \subseteq G$, we have $G' \in feas(C)$ if and only if $\Psi[C, G'] = \top$.

The formula Ψ comprises t sub-formulae, $\Phi_k(Ag, G)$ where $1 \leqslant k \leqslant t$, there being one such formula for each distinct resource $r_k \in R$. The formula Ψ is formed from these, via

$$\Psi = \bigwedge_{k=1}^{t} \Phi_k(Ag, G)$$

The role of the sub-formula $\Phi_k(Ag, G)$ is to describe the conditions under which for $C \subseteq Ag$, and $G' \subseteq G$, the coalition C has a sufficient endowment of the resource r_k to satisfy the total requirement of this needed to realise G'. In other words, $\Phi_k(Ag, G)$ will be such that

$$\Phi_k[C, G'] = \top \Leftrightarrow \sum_{a_i \in C} \mathbf{en}(a_i, r_k) \geqslant \sum_{g_j \in G'} \mathbf{req}(g_j, r_k)$$

Recalling that the values $\mathbf{en}(a_i, r_k)$ and $\mathbf{req}(g_j, r_k)$ can be represented using exactly b bits, $\Phi_k(Ag, G)$ can be considered as a formula having $b \times (n + m)$ arguments,

$$\langle X_1^k, X_2^k, X_3^k, \ldots, X_n^k, Y_1^k, Y_2^k, \ldots, Y_m^k \rangle$$

with

$$X_i^k = \langle x_{i,b-1}^k, x_{i,b-2}^k, \ldots, x_{i,1}^k, x_{i,0}^k \rangle$$
$$Y_j^k = \langle y_{j,b-1}^k, y_{j,b-2}^k, \ldots, y_{j,1}^k, y_{j,0}^k \rangle$$

Of course, in order to implement Ψ, each of these $x_{i,p}^k$ and $y_{j,q}^k$ must be specified as some $a \in A$, $g \in G$, or a *constant* value.

Suppose that the b-bit binary representation of $\mathbf{en}(a_i, r_k)$ is $e_{i,b-1}^k e_{i,b-2}^k \ldots e_{i,1}^k e_{i,0}^k$ and that of $\mathbf{req}(g_j, r_k)$ is $d_{j,b-1}^k d_{j,b-2}^k \ldots d_{j,1}^k d_{j,0}^k$. So that

$$\mathbf{en}(a_i, r_k) = \sum_{p=0}^{b-1} (e_{i,p}^k \times 2^p); \qquad \mathbf{req}(g_j, r_k) = \sum_{q=0}^{b-1} (d_{j,q}^k \times 2^q)$$

For the formula $\Phi_k(Ag, G)$ the appropriate substitutions are given by

$$x_{i,p}^k := \begin{cases} a_i & \text{if } e_{i,p}^k = 1 \\ 0 & \text{if } e_{i,p}^k = 0 \end{cases}$$
$$y_{j,q}^k := \begin{cases} g_j & \text{if } d_{j,q}^k = 1 \\ 0 & \text{if } d_{j,q}^k = 0 \end{cases}$$

The sub-formula $\Phi_k(Ag, G)$ contains three separate parts:

1. $Endow_k(X_1^k, X_2^k, \ldots, X_n^k)$ which computes $u_{z-1}^k u_{z-2}^k \ldots u_1^k u_0^k$ the $z = (b + 1 + \lfloor \log_2(nm) \rfloor)$-bit *binary* representation resulting by adding the n b-bit binary values described by the instantiation of $\langle X_1^k, X_2^k, \ldots, X_n^k \rangle$.
2. $Require_k(Y_1^k, Y_2^k, \ldots, Y_m^k)$ which computes $v_{z-1}^k v_{z-2}^k \ldots v_1^k v_0^k$ the $z = (1 + b + \lfloor \log_2(nm) \rfloor)$-bit *binary* representation resulting by adding the m b-bit binary values described by the instantiation of $\langle Y_1^k, Y_2^k, \ldots, Y_m^k \rangle$.
3. $Compare_k(u_{z-1}^k, \ldots, u_0^k, v_{z-1}^k, \ldots, v_0^k)$ which returns the value $\top$ if and only if

$$\sum_{l=0}^{z-1} (u_l^k \times 2^l) \geqslant \sum_{l=0}^{z-1} (v_l^k \times 2^l)$$

Before analysing the size of the formula $\Psi = \bigwedge_{k=1}^t \Phi_k(Ag, G)$ we first establish the correctness of our construction. For this it suffices to show that $\Phi_k[C, G'] = \top$ if and only if $\sum_{a_i \in C} \mathbf{en}(a_i, r_k) \geqslant \sum_{g_j \in G'} \mathbf{req}(g_j, r_k)$. Thus consider any $C \subseteq Ag$ and $G' \subseteq G$. For the substitutions of $x_{i,p}^k$ by a_i or 0, and of $y_{j,q}^k$ by g_j or 0 described above we see that the following holds of $\Phi_k[C, G']$. If $a_i \notin C$ then X_i^k is the b-bit representation of the value 0: a_i does not contribute anything to satisfying the total requirement for r_k. On the other hand, if $a_i \in C$, then X_i^k is the b-bit representation of the value $\mathbf{en}(a_i, r_k)$. Similarly, if $g_j \notin G'$ then Y_j^k is the b-bit representation of the value 0; if $g_j \in G'$ then Y_j^k is the b-bit representation of the value $\mathbf{req}(g_j, r_k)$. It follows that the value computed by $Endow_k[C, G']$ is exactly $\sum_{a_i \in C} \mathbf{en}(a_i, r_k)$ (with z bits sufficing to represent this value in binary). In the same way, the value computed by

$Require_k[C, G']$ is exactly $\sum_{g_j \in G'} \mathbf{req}(g_j, r_k)$ (again with z bits sufficing to represent this value in binary). In total, from the description of $Compare_k()$ this yields

$$\begin{aligned}\Phi_k[C, G'] &= Compare_k\big(Endow_k[C, G'], Require_k[C, G']\big) \\ &= \left(\sum_{a_i \in C} \mathbf{en}(a_i, r_k) \geqslant \sum_{g_j \in G'} \mathbf{req}(g_j, r_k)\right)\end{aligned}$$

Thus the formula $\Psi = \bigwedge_{k=1}^{t} \Phi_k(Ag, G)$ correctly describes the feasible goal sets (G') for each coalition (C) within the CRG, Γ.

Regarding the size of this formula, it is clear that $|\Psi|$ equals

$$\sum_{k=1}^{t} \left|\Phi_k(Ag, G)\right| \leqslant t \times \left|Compare_1\big(Endow_1(Ag, G), Require_1(Ag, G)\big)\right|$$

The formulae $Endow_1(Ag, G)$ and $Require_1(Ag, G)$ implement addition of n (respectively m) b-bit values, each returning the answer using z bits. Formulae of size $O(nb)$ and $O(mb)$ for these operations are described in [21], see e.g., [11, p. 115]. Finally, the formula $Compare_1()$ involves 2 sets of z arguments, $\langle u^1_{z-1}, u^1_{z-2}, \ldots, u^1_1, u^1_0\rangle$ and $\langle v^1_{z-1}, v^1_{z-2}, \ldots, v^1_1, v^1_0\rangle$, returning the answer $\top$ if and only if

$$\sum_{p=0}^{z-1}\big(u^1_p \times 2^p\big) \geqslant \sum_{p=0}^{z-1}\big(v^1_p \times 2^p\big)$$

This holds if and only if one of the following is true:

(1) For each $0 \leqslant p \leqslant z-1$, $u^1_p = v^1_p$, i.e., $Endow_1(Ag, G) = Require_1(Ag, G)$
(2) There is an index (l) for which $u^1_l = 1$, $v^1_l = 0$, and for each $l+1 \leqslant p \leqslant z-1$ $u^1_p = v^1_p$, i.e., $Endow_1(Ag, G) > Require_1(Ag, G)$.

These conditions are described by the formula

$$\left(\bigwedge_{p=0}^{z-1}\big(u^1_p \equiv v^1_p\big)\right) \vee \bigvee_{l=0}^{z-1}\left(u^1_l \wedge \big(\neg v^1_l\big) \wedge \bigwedge_{p=l+1}^{z-1}\big(u^1_p \equiv v^1_p\big)\right)$$

An easy analysis showing this to have size $O(z^2)$ and thus $Compare_1()$ can be realised in size $O(b^2 + (\log n)^2 + (\log m)^2)$. In total Ψ has size $O(t(bn + bm + b^2 + (\log n)^2 + (\log m)^2))$ giving the bound $O(bt(n + m + b))$, via the observations $(\log n)^2 = o(n)$ and $(\log m)^2 = o(m)$. □

Using the construction in the proof of Theorem 14 we can now establish that PGS $\in$ NC^1, i.e., present the

Proof of Theorem 11. Recall that an instance of PGS consists of a CRG, $\Gamma = \langle Ag, G, R, G_1, \ldots, G_n, \mathbf{en}, \mathbf{req}\rangle$ together with a subset G' of G, with the instance accepted if there is some coalition $C \subseteq Ag$ for which $G' \in sf(C)$.

Consider the propositional formula, Ψ_Γ, constructed in the proof of Theorem 14. The sub-formula, $\Phi_k(Ag, G)$, was initially described as having $n + m$ "blocks" of b propositional arguments—X^k_i (with $1 \leqslant i \leqslant n$) and Y^k_j (with $1 \leqslant j \leqslant m$)—with the individual arguments in each block being set to constant values or propositional variables: a_i (for X^k_i), g_j (for Y^k_j). In order to obtain an NC^1 algorithm for PGS, we modify the mechanism used to replace the variables X^k_i: the substitution of variables in G for variables in Y^k_j remains unaltered.

Suppose the b-bit representation of $\mathbf{en}(a_i, r_k)$ is $e^k_{i,b-1}e^k_{i,b-1}\cdots e^k_{i,1}e^k_{i,0}$. The variable $x^k_{i,p}$ of X^k_i is replaced by the formula:

$$x^k_{i,p} := \begin{cases} \bigvee_{g \in G_i} g & \text{if } e^k_{i,p} = 1 \\ 0 & \text{if } e^k_{i,p} = 0 \end{cases}$$

Let $\Psi_{\text{PGS}}(G)$ (which is defined solely in terms of propositional variables G) be the resulting formula. Choose any $G' \subseteq G$ and consider $\Psi_{\text{PGS}}[G']$. The b-bit value described by X_i^k will be $\mathbf{en}(a_i, r_k)$ if $G' \cap G_i \neq \emptyset$ and 0 otherwise. It follows, from the analysis in Theorem 14, that $\Psi_{\text{PGS}}[G']$ will output $\top$ if and only if:

$$\forall r \in R \; en(C_{\max}, r) \geqslant req(G', r)$$

where $C_{\max} = \{a_i\colon\ G_i \cap G' \neq \emptyset\}$. As we observed in the proof of Theorem 10, if $C \subseteq Ag$ is such that $G' \in sf(C)$, then it must be the case that $C \subseteq C_{\max}$, and this suffices to establish the correctness of $\Psi_{\text{PGS}}(G)$. To complete the proof that PGS $\in$ NC^1, first observe that $|\Psi_{\text{PGS}}(G)|$ is polynomially-bounded in, N, the number of bits required to encode an instance of PGS. Since Ψ_{PGS} is a propositional formula, using the standard construction of [5], (described in [11, pp. 68–69]) yields a parallel algorithm with run-time $O(\log|\Psi_{\text{PGS}}|) = O(\log N)$. □

4.3. Translating QCGs to CRGs

Next, we consider the more difficult issue of translating QCGs to CRGs. We know from Theorem 13 that there are QCGs for which no equivalent CRG exists. This raises the question of exactly what conditions characterise QCGs that *do* have an equivalent CRG. Our next results give a partial characterisation of those QCGs for which an equivalent CRG exists:

- First, Theorem 15 gives four *necessary* conditions on QCGs for the existence of an equivalent CRG. Intuitively, these conditions state that (i) monotonically increasing coalitions have monotonically increasing feasible goal sets; (ii) if a coalition can choose a goal set G', then that coalition can choose every subset of G'; (iii) the empty coalition has no choices; and (iv) disjoint coalitions have superadditive choices—they can achieve everything together that they could achieve apart.[7]
- Second, Theorem 16 defines some *sufficient* conditions on QCGs for the existence of an equivalent CRG. Roughly, these conditions state that equivalent CRGs exist for QCGs with characterictic function formulae corresponding to certain types of monotonic Boolean functions.
- Third, Theorem 17(a–c) establishes three properties regarding the relative efficiency of describing characteristic functions via QCGs and CRGs: that the size of a QCG need be (at worst) only quadratically larger than the size of an equivalent CRG; that there are instances for which CRG representations are possible, but any such representation is exponentially larger than a minimal equivalent QCG; and, finally, that there are instances for which the smallest QCG representation has size exponential in the number of goals, (and thus, an equivalent CRG is also exponential size).

First, then, we will give some necessary conditions on QCGs for the existence of an equivalent CRG.

We start by recalling that the abstraction underlying both the concepts of CRGs and QCGs concerns the relationship between subsets of Ag (i.e., coalitions, C) and sets of goals, G', deemed as "feasible" for these. Thus the fundamental constructs being modelled are mappings

$$\mathcal{F} : 2^{Ag} \to 2^{2^G}$$

associating each $C \subseteq Ag$ with a set of subsets of G: the sets of goals that are *feasible* for C.

The formalism provided by QCGs is based on the observation that $\mathcal{F}$ can be viewed as a propositional logic function via its characteristic function $\chi_{\mathcal{F}}(Ag, G)$, and therefore, using a propositional formula Ψ for which $\Psi[C, G'] = \top$ if and only if $\chi_{\mathcal{F}}[C, G'] = \top$ obviates any need explicitly to enumerate $\mathcal{F}(C)$ for each $C \subseteq Ag$ in order to describe $\mathcal{F}$.

The formalism offered by CRGs takes a rather different approach arising from the fact that for *some* $\mathcal{F}$ we can characterise each of the sets $\mathcal{F}(C)$ in terms of some choice, $\langle R, \mathbf{en}, \mathbf{req}\rangle$, of resources, agent endowments, and goal requirements, so that $\mathcal{F}$ is effectively described, not by a propositional logic formula, but through two matrices whose values are whole numbers, i.e., the $n \times t$ matrix of agent endowments, and the $m \times t$ matrix of goal requirements. Of course while it is self-evident that *every* system $\mathcal{F}$ can be modelled by a QCG, in consequence of Theorem 13(b) there are systems that *cannot* be described by a CRG. This motivates the following definition,

[7] Readers with an interest in cooperation logics may wish to note that this last condition is essentially the same as axiom S in Pauly's Coalition Logic [30, p. 54], which in turn follows from the superadditivity property of playable games [30, p. 30].

Definition 2. We say that the system $\mathcal{F}: 2^{Ag} \to 2^{2^G}$ is *resource definable for* $\langle Ag, G\rangle$ if there is a CRG, $\Gamma = \langle Ag, G, R, G_1, \ldots, G_n, \mathbf{en}, \mathbf{req}\rangle$ in which for all $C \subseteq Ag$ and $G' \subseteq G$,

$$\big(\forall r \in R\ en(C, r) \geqslant req(G', r)\big) \Leftrightarrow \big(G' \in \mathcal{F}(C)\big)$$

Over the next series of results we describe some sufficient conditions for $\mathcal{F}$ to be resource definable. These are expressed in terms of properties of the characteristic function $\chi_{\mathcal{F}}(Ag, G)$.

Theorem 15. *Let $\mathcal{F} \in \{2^{Ag} \to 2^{2^G}\}$. The system $\mathcal{F}$ is resource definable for $\langle Ag, G\rangle$ only if it satisfies* all *of the following conditions.*

(C1) $\mathcal{F}$ *is* coalition monotonic*, that is*:

$$\big(C \subseteq D \text{ and } G' \in \mathcal{F}(C)\big) \Rightarrow G' \in \mathcal{F}(D)$$

(C2) $\mathcal{F}$ *is* goal anti-monotonic*, that is*:

$$\big(G'' \subseteq G' \text{ and } G' \in \mathcal{F}(C)\big) \Rightarrow G'' \in \mathcal{F}(C)$$

(C3) $\mathcal{F}(\emptyset) = \{\emptyset\}$.

(C4) $\mathcal{F}$ *is* superadditive*, in the sense that if $C \cap D = \emptyset$ and both $G' \in \mathcal{F}(C)$, $G'' \in \mathcal{F}(D)$ then $G' \cup G'' \in \mathcal{F}(C \cup D)$.*

Proof. Both (C1) and (C2) are trivial consequences of our formulation of CRG. (C3) is immediate from the fact that given any resource r in a CRG, $en(\emptyset, r) = 0$ and $req(\emptyset, r) = 0$. Finally, that (C4) must also hold of $\mathcal{F}$ follows from the observation that in any CRG, Γ, should $en(C, r) \geqslant req(G', r)$ and $en(D, r) \geqslant req(G'', r)$ and $C \cap D = \emptyset$, then C's expenditure of r in realising G'' does not affect the quantity of resource r that is available to D, hence $en(C \cup D, r) \geqslant req(G' \cup G'', r)$. $\square$

We make two initial observations concerning the statement of this theorem. Firstly, we note the form of condition (C3): $\mathcal{F}(\emptyset) = \{\emptyset\}$ rather than $\emptyset \in \mathcal{F}(\emptyset)$. The latter would allow the possibility for *non-empty* $G' \subseteq G$ to be *feasible* for the empty coalition of agents, however such would arise only in the event of there being goals, $g \in G$ for which $\mathbf{req}(g, r) = 0$ for each r. As mentioned in our introductory formulation of CRGs we are assuming that "trivial" goals of this nature are not present. It follows that the *only* goal set which is feasible for $C = \emptyset$ is $G' = \emptyset$.

As a second point, we note that $G \in \mathcal{F}(Ag)$ is *not* a necessary condition for a system to be resource definable as the following simple example shows. Let

$$Q = \big\langle \{g_1, g_2\}, \{a\}, \{g_1, g_2\}, \Psi(a, g_1, g_2)\big\rangle$$

where

$$\Psi(a, g_1, g_2) \equiv (\neg g_1)(\neg g_2) \vee a(\neg g_1 \vee \neg g_2)$$

An equivalent CRG, Γ_Q is formed using $R = \{r_1\}$, $\mathbf{en}(a, r_1) = 1$, $\mathbf{req}(g_1, r_1) = \mathbf{req}(g_2, r_1) = 1$. For these we have $\Psi[a, \{g_1, g_2\}] = \bot$, however each of the three feasible sets for a, i.e., $\{\emptyset, \{g_1\}, \{g_2\}\}$ in Q is also a feasible set for a in Γ_Q. Similarly, $\{g_1, g_2\}$ is not a feasible set for a in Γ_Q.

In order to state some sufficient conditions, we must first recall some basic definitions and results concerning the class of *monotone* propositional functions.

Definition 3. Let $X_n = \{x_1, \ldots, x_n\}$ be set of n propositional variables. Given two propositional functions $f(X_n)$ and $g(X_n)$, we write $f \leqslant g$ if for all $Y \subseteq X_n$ it holds that $f[Y] \leqslant g[Y]$, where the ordering of constant values is $\bot < \top$.

The function $f(X_n)$ is monotone *increasing* if

$$\forall Y \subset Z \subseteq X_n \quad f[Y] \leqslant f[Z]$$

Similarly, $f(X_n)$ is monotone *decreasing* if

$$\forall Y \subset Z \subseteq X_n \quad f[Z] \leqslant f[Y]$$

Fact 4. $f(x_1, \ldots, x_n)$ is monotone increasing

(a) if and only if there is a *unique* set

$$\mathcal{P}(f) = \{P_1, P_2, \ldots, P_k\} \subset 2^{X_n}$$

for which $(P_i \subseteq P_j) \Leftrightarrow (i = j)$ and with

$$f(X_n) \equiv \bigvee_{i=1}^{k} \bigwedge_{x_j \in P_i} x_j$$

We call $\mathcal{P}(f)$ the *product set* for f.

(b) if and only if there is a *unique* set

$$\mathcal{Q}(f) = \{Q_1, Q_2, \ldots, Q_l\} \subset 2^{X_n}$$

for which $(Q_i \subseteq Q_j) \Leftrightarrow (i = j)$ and with

$$f(X_n) \equiv \bigwedge_{i=1}^{l} \bigvee_{x_j \in Q_i} x_j$$

We call $\mathcal{Q}(f)$ the *clause set* for f.

(c) if and only if $f(x_1, \ldots, x_n) \in \{\top, \bot\}$ (i.e., f is a *constant* function) *or* there is a propositional *formula*, $\Psi(x_1, \ldots, x_n)$, employing *only* the binary operations $\{\wedge, \vee\}$ (i.e., negation, $\neg$, is not used in Ψ) for which f_Ψ, the propositional function represented by Ψ, is equivalent to f.

(d) if and only if $f(\neg x_1, \ldots, \neg x_n)$ is a monotone decreasing propositional function of $\{x_1, \ldots, x_n\}$.

For proofs of these properties the reader is referred to any standard monograph on Boolean function complexity, e.g., [11, pp. 15–17].

We need a minor development of the ideas described in Definition 3 and Fact 4 to capture the concept of a propositional function being monotone (increasing or decreasing) with respect to a *subset* of its arguments.

We say that a propositional formula $\Psi(X_n)$ is in *standard form* if Ψ is defined using the binary operations $\{\wedge, \vee\}$ and unary negation $(\neg)$ with $\neg$ applied *only* to propositional variables, e.g., $(\neg x_1 \vee x_2)$ is in standard form, however, its equivalent $\neg(x_1 \wedge \neg x_2)$ is not. It is well-known that every propositional function can be represented by a formula in standard form.[8] It is also the case that any formula, Φ (defined using binary operations) may be translated to an equivalent one, Ψ, with the latter in standard form and having $|\Psi(X_n)| = \mathrm{O}(|\Phi(X_n)|^{2+c})$ (where $c < 0.1$ is a constant value), cf. the construction presented in [31].

Definition 5. Let $f(X_n)$ be a propositional function and Y a subset of X_n. We say that f is monotone increasing *with respect to the subset* Y if for every $Z \subseteq X_n \setminus Y$

$$\forall U \subset W \subseteq Y \quad f[Z \cup U] \leqslant f[Z \cup W]$$

Similarly, f is monotone *decreasing* with respect to the subset Y if for every $Z \subseteq X_n \setminus Y$

$$\forall U \subset W \subseteq Y \quad f[Z \cup W] \leqslant f[Z \cup U]$$

It is straightforward to obtain the following development of Fact 4.

Fact 6.

(a) $f(X_n)$ is monotone increasing with respect to a subset Y of X_n if and only if there is a propositional formula $\Psi(Y, X_n \setminus Y)$ in standard form with unary negation applied *only* to variables in $X_n \setminus Y$ and for which $f_\Psi(X_n)$—the propositional function represented by Ψ—is equivalent to $f(X_n)$.

[8] For example, representations of propositional functions using CNF or DNF are examples of "standard form" formulae.

(b) $f(X_n)$ is monotone decreasing with respect to a subset Y of X_n if and only if there is a propositional formula $\Psi(Y, X_n \setminus Y)$ in standard form with *no* variables in Y occurring in positive (i.e., *un-negated*) form within Ψ and for which $f_\Psi(X_n)$—the propositional function represented by Ψ—is equivalent to $f(X_n)$.

(c) $f(X_n)$ is monotone increasing with respect to a subset Y *and* monotone decreasing with respect to the subset $X_n \setminus Y$ if and only if there is a propositional formula $\Psi(Y, X_n \setminus Y)$ in standard form with no variable in Y occurring in negated form *and* no variable in $X_n \setminus Y$ occurring in positive (i.e., un-negated) form and for which $f_\Psi(X_n)$—the propositional function represented by Ψ—is equivalent to $f(X_n)$.

Combining Fact 4(a–b) with Fact 6(b) one consequence that results is that for monotone *decreasing* functions $f(X_n)$ the product (respectively, clause) sets $\mathcal{P}(f)$ (respectively, $\mathcal{Q}(f)$) will be defined by subsets of $2^{\{\neg x_1, \neg x_2, \ldots, \neg x_n\}}$, i.e., as sets of *negated* variables. This interpretation will prove useful subsequently.

Given these definitions, the following result gives some sufficient conditions on QCGs for the existence of an equivalent CRG.

Theorem 16. *$\mathcal{F}$ is resource definable for $\langle Ag, G\rangle$ for both of the following cases.*

(a) $$\chi_\mathcal{F}(Ag, G) \equiv \left(\bigwedge_{a \in Ag} a\right) \wedge h(G) \vee \left(\bigwedge_{g \in G} \neg g\right)$$

where $h(G)$ is any monotone decreasing function of G with $h[\emptyset] = \top$.

(b) $$\chi_\mathcal{F}(Ag, G) \equiv f(Ag) \vee \left(\bigwedge_{g \in G} \neg g\right)$$

where $f(Ag)$ is any monotone increasing function of Ag with $f[\emptyset] = \bot$.

Proof. Our proof is constructive. We show how $\mathcal{F}$ may be translated to $\langle R, \mathbf{en}, \mathbf{req}\rangle$ in such a way that

$$\chi_\mathcal{F}[C, G'] \equiv \top \Leftrightarrow \left(\forall r \in R\ en(C, r) \geqslant req(G', r)\right)$$

For part (a), with $|Ag| = n$, let $\mathcal{F} \in \{2^{Ag} \to 2^{2^G}\}$ be such that

$$\chi_\mathcal{F}(Ag, G) \equiv \left(\bigwedge_{a \in Ag} a\right) \wedge h(G) \vee \left(\bigwedge_{g \in G} \neg g\right)$$

where $h[\emptyset] = \top$ and $h(G)$ is a monotone decreasing function of G. We define $\langle R, \mathbf{en}, \mathbf{req}\rangle$ so that $\chi_\mathcal{F}[C, G'] = \top$ if and only if for each $r \in R$, $en(C, r) \geqslant req(G', r)$. Let

$$\mathcal{Q}(h) = \{E_1, E_2, \ldots, E_p\}$$

be the (unique) set of subsets of $\{\neg g_1, \ldots, \neg g_m\}$ defining the *clause set* of h.

We call a set of goals, $S \subseteq G$ a *forbidden goal set* if there is some $E \in \mathcal{Q}(h)$ for which

$$\{\neg g\colon\ g \in S\} = E$$

If S is any forbidden set (or a superset of such a set) it is certainly the case that $\chi_\mathcal{F}[Ag, S] = \bot$ since there is some $E_i \in \mathcal{Q}(h)$ for which

$$h[S] \equiv E_i[S] \equiv \left(\bigvee_{\neg g_j \in E_i} \neg g_j\right)[S] \equiv \bot$$

In addition, however, $\chi_\mathcal{F}[Ag, T] = \top$ for any *strict* subset T of S. For if we suppose that $g \in S \setminus T$, then $E_i[T] \equiv (\neg g)[T] \equiv \top$, and from Fact 4(b), it follows that $E_j[T] = \top$ for every E_j: were $E_j[T] = \bot$ then this would imply $E_j \subset E_i$.

We now have a sufficient basis for constructing the required components $\langle R, \mathbf{en}, \mathbf{req}\rangle$.

Associated with each $g_k \in G$ we have a resource, $r_{g,k}$ for which

$$\mathbf{en}(a_i, r_{g,k}) = 1$$

$$\mathbf{req}(g_j, r_{g,k}) = \begin{cases} n & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

These settings suffice to ensure that $\mathit{feas}[C] = \{\emptyset\}$ for every $C \subset Ag$ since each g_j requires exactly $en(Ag, r_{g,j})$ units of the resource $r_{g,j}$ to be expended.

For each forbidden set $S \subseteq G$, there is a resource $r_S \in R$, having

$$\mathbf{en}(a_i, r_S) = |S| - 1$$
$$\mathbf{req}(g_j, r_S) = \begin{cases} n & \text{if } g_j \in S \\ 0 & \text{if } g_j \notin S \end{cases}$$

We note that since $h(G) \not\equiv \bot$, every forbidden set has *at least* one element, i.e., the empty clause is not in the clause set $\mathcal{Q}(h)$. Furthermore if $\mathcal{Q}(h) = \emptyset$, then we have $h[G] \equiv \top$, i.e., $\chi_{\mathcal{F}}[Ag, G'] = \top$ for every $G' \subseteq G$. In this case, there are no forbidden set resources needed and the collection $\langle R, \mathbf{en}, \mathbf{req} \rangle$ with $R = \{r_{g,j}\colon 1 \leqslant j \leqslant m\}$ suffices to form an equivalent CRG.

We now show that

$$\forall G' \subseteq G \begin{cases} \chi_{\mathcal{F}}[Ag, G'] = \top \Rightarrow \forall r \in R \;\; en(Ag, r) \geqslant req(G', r) \\ \chi_{\mathcal{F}}[Ag, G'] = \bot \Rightarrow \exists r \in R \;\; en(Ag, r) < req(G', r) \end{cases}$$

Suppose first that $\chi_{\mathcal{F}}[Ag, G'] = \bot$. From our analysis we must have $S \subseteq G'$ for some forbidden set S, since $h[G'] = \bot$ indicates that some $E_j \in \mathcal{Q}(h)$ evaluates to $\bot$. Consider the resource r_S. We have

$$req(G', r_S) = \sum_{g \in G'} \mathbf{req}(g, r_S) = \sum_{g \in S} \mathbf{req}(g, r_S) = n|S|$$

However, $en(Ag, r_S) = n(|S| - 1)$, and thus $G' \notin \mathit{feas}(Ag)$.

On the other hand, suppose that $\chi_{\mathcal{F}}[Ag, G'] = \top$. It is certainly the case that $en(Ag, r_{g,k}) \geqslant req(G', r_{g,k})$ for each $1 \leqslant k \leqslant m$: if $g_k \notin G'$ then $req(G', r_{g,k}) = 0$; if $g_k \in G'$ then $req(G', r_{g,k}) = n = en(Ag, r_{g,k})$.

Since $\chi_{\mathcal{F}}[Ag, G'] = \top$ it must hold that $G' \subset S$ for *every* forbidden set S: in each $E \in \mathcal{Q}(h)$ there must be at least one literal $\neg g_j$ for which $g_j \notin G'$. Consider any of the forbidden set resources $r_S \in R$, then

$$req(G', r_S) = \sum_{g \in G'} \mathbf{req}(g, r_S) \leqslant n\big(|S| - 1\big)$$

(since $\mathbf{req}(g, r) \leqslant n$ for every $g \in G$ and $r \in R$).

Now, since $en(Ag, r_S) = n(|S| - 1)$ for all $r_S \in R$, it follows that $G' \in \mathit{feas}(Ag)$. This completes the proof of part (a).

For part (b), let

$$\chi_{\mathcal{F}}(Ag, G) \equiv f(Ag) \vee \left(\bigwedge_{g \in G} \neg g \right)$$

where $f(Ag)$ is any monotone *increasing* function of Ag with $f[\emptyset] = \bot$. We first observe that the class of QCGs described may be viewed as indicating the *minimal* coalitions C that must form in order to bring about any non-empty set of goals. For such C, with any (non-empty) subset G' of G, $\chi_{\mathcal{F}}[C, G'] = \top$ but $\chi_{\mathcal{F}}[D, G''] = \bot$ for any $D \subset C$ and any non-empty $G'' \subseteq G$. In addition, $\chi_{\mathcal{F}}[C, \emptyset] = \top$ for every $C \subseteq Ag$. Let $\mathcal{Q}(f) = \{E_1, \ldots, E_k\}$ be the set of subsets of Ag defined by the clause set of $f(Ag)$. Via the convention that the empty *disjunction* is equivalent to $\bot$, if $f(a_1, \ldots, a_n) \equiv \bot$ then $\mathcal{Q}(f) = \{\emptyset\}$, i.e., $\mathcal{Q}(f)$ contains *exactly one* set.

We form an equivalent CRG as follows. For each clause $E \in \{E_1, \ldots, E_k\}$ there is a corresponding resource $r_E \in R$. For the resource r_E,

$$\mathbf{en}(a_i, r_E) = \begin{cases} m|E| & \text{if } a_i \in E \\ 0 & \text{if } a_i \notin E \end{cases}$$
$$\mathbf{req}(g_j, r_E) = |E|$$

We claim that

$$\forall C \subseteq Ag \quad \begin{cases} \chi_{\mathcal{F}}[C, G] = \top \Rightarrow \forall r \in R \; en(C, r) \geqslant req(G, r) \\ \chi_{\mathcal{F}}[C, g_j] = \bot \Rightarrow \exists r \in R \; en(C, r) < req(g_j, r) \end{cases}$$

Thus, suppose that $C \subseteq Ag$ is such that $\chi_{\mathcal{F}}[C, G] = \top$. It must be the case that $f[C] = \top$ and thus for each $E \in \{E_1, \ldots, E_k\}$ we have $C \cap E \neq \emptyset$. From this property, for each $r_E \in R$,

$$en(C, r_E) \geqslant m|E| = req(G, r_E)$$

and hence $G \in feas(C)$ for the CRG formed.

On the other hand, suppose that $\chi_{\mathcal{F}}[C, g_j] = \bot$. In this case we have $f[C] = \bot$, implying that is at least one $E \in \{E_1, \ldots, E_k\}$ for which $C \cap E = \emptyset$. Consider the resource $r_E \in R$. For each $g_j \in G$ we have $\mathbf{req}(g_j, r_E) = |E|$, however, since no member of C occurs in E, these can contribute nothing towards meeting the quantity of resource r_E needed, i.e.,

$$en(C, r_E) = \sum_{a_i \in C} \mathbf{en}(a_i, r_E) = 0 < |E|$$

It follows that $\{g_j\} \notin feas(C)$.

This completes the proof that for any QCG of the form in the theorem statement, an equivalent CRG exists. □

Aside from the specialisation in the latter result, we note one significant difference in the respective constructions of Theorem 14 and Theorem 16 is that in the former case the translation from CRGs to QCGs operates directly on the sub-structure $\langle R, \mathbf{en}, \mathbf{req}\rangle$, whereas the latter does not operate on the particular propositional *formula*, but an equivalent representation whose structure is determined from the propositional *function* characterised by $\chi_{\mathcal{F}}(Ag, G)$. This equivalent representation may, of course, be exponentially large in m.

Next, we turn to the issue of how efficiently (or otherwise) a QCG may be translated to a CRG, given that such a translation is possible. In order to develop these results, we need to extend our definitions of M^{QCG} and M^{CRG}. The starting point for these extensions is the observation that given a resource definable system $\mathcal{F}$ its representation as either a QCG or CRG is not unique.

We have introduced already a notion of "equivalence" between CRGs and QCGs. It is straightforward to formulate the idea of two QCGs,

$$Q = \langle G, Ag, G_1, \ldots, G_n, \Psi\rangle$$
$$Q' = \langle G, Ag, G_1, \ldots, G_n, \Phi\rangle$$

being equivalent simply by defining this to be the case if the propositional functions, f_Ψ and f_Φ described by the formulae Ψ and Φ are equivalent.

It follows that we may define equivalence between CRGs,

$$\Gamma = \langle Ag, G, R, G_1, \ldots, G_n, \mathbf{en}, \mathbf{req}\rangle$$
$$\Gamma' = \langle Ag, G, R', G_1, \ldots, G_n, \mathbf{en}', \mathbf{req}'\rangle$$

via Γ is equivalent to Γ' if and only if: for all $C \subseteq Ag$, $G' \subseteq G$

$$G' \in feas(C) \text{ for the CRG } \Gamma$$
$$\Leftrightarrow$$
$$G' \in feas(C) \text{ for the CRG } \Gamma'$$

In summary each resource definable $\mathcal{F}$ for $\langle Ag, G\rangle$ induces equivalence classes $[\mathcal{F}]^{\mathrm{QCG}}$ (respectively, $[\mathcal{F}]^{\mathrm{CRG}}$) corresponding to the QCGs (respectively, CRGs) that could be used to represent $\mathcal{F}$.

Having developed these concepts, we can now present the extensions of M^{QCG} and M^{CRG} (which were couched in terms of specific representations Ψ, $\langle R, \mathbf{en}, \mathbf{req}\rangle$), to measures defined on the equivalence classes of QCGs and CRGs arising via the definitions above. We will denote by $\mu^{\mathrm{QCG}}(\mathcal{F})$ the size of the smallest QCG that is equivalent to $\mathcal{F}$, and by $\mu^{\mathrm{CRG}}(\mathcal{F})$ the smallest CRG that is equivalent to $\mathcal{F}$. Formally,

Definition 7. Let $\mathcal{F}$ be resource definable for $\langle Ag, G\rangle$. The measures, $\mu^{\mathrm{QCG}}(\mathcal{F})$ and $\mu^{\mathrm{CRG}}(\mathcal{F})$ are defined as,

$$\mu^{\mathrm{QCG}}(\mathcal{F}) = \min\{M^{\mathrm{QCG}}(Q)\colon\ Q \in [\mathcal{F}]^{\mathrm{QCG}}\}$$
$$\mu^{\mathrm{CRG}}(\mathcal{F}) = \min\{M^{\mathrm{CRG}}(\Gamma)\colon\ \Gamma \in [\mathcal{F}]^{\mathrm{CRG}}\}$$

One consequence of using the translation described in Theorem 16(b), in which there is a distinct resource r_q for each clause $q \in \mathcal{Q}(f)$, is that the resulting CRG could be significantly larger than an "optimal" equivalent CRG. As a very simple example of such behaviour consider the following system involving n agents and a *single* goal, g,

$$\mathcal{F}^{\mathrm{MAJ}}(C) = \begin{cases} \{g\} & \text{if } |C| \geqslant n/2 \\ \emptyset & \text{if } |C| < n/2 \end{cases}$$

The system $\mathcal{F}^{\mathrm{MAJ}}$ is resource definable, as it corresponds to the propositional *majority* function $\mathrm{MAJ}(Ag) \vee \neg g$ where MAJ is the n-argument function whose value is $\top$ if and only if at least $n/2$ of its arguments are assigned the value $\top$: since this function is monotone (increasing), an equivalent CRG can be constructed using the process described in the proof of Theorem 16(b). The clause set, $\mathcal{Q}(\mathrm{MAJ})$ of MAJ, however, has

$$\left|\mathcal{Q}(\mathrm{MAJ})\right| = \begin{cases} \binom{n}{\frac{n-2}{2}} & \text{if } n \text{ is even} \\ \binom{n}{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases} = \Omega\left(\frac{2^{n/2}}{\sqrt{n}}\right)$$

and thus the CRG formed would have exponentially many resources, for a propositional function that has polynomial size formulae, see, e.g. [11, p. 331].[9] The system $\mathcal{F}^{\mathrm{MAJ}}$, however, is easily captured by the CRG, Γ^{MAJ} with agent set Ag, goals $G = \{g\}$, and a *single* resource, r, for which

$$\begin{aligned} \mathbf{req}(g, r) &= \lceil n/2 \rceil \\ \mathbf{en}(a, r) &= 1 \quad \forall a \in Ag \end{aligned}$$

Although the translation from QCGs to CRGs can be extremely inefficient, there are nonetheless, cases of "simple" resource definable systems that can be represented by linear (in $n + m$) size propositional formulae but can only be described by exponential size CRGs: an example of such a system is presented in Theorem 17(b).

More generally this theorem gives us some upper and lower bounds on the size of the QCGs and CRGs for cases where they exist. In particular, we prove that:

- for any resource definable $\mathcal{F}$, the minimum size of the QCG representing $\mathcal{F}$ is no more than quadratically larger than the size of the smallest CRG representing $\mathcal{F}$ (Theorem 17(a));
- there are resource definable $\mathcal{F}$ for which a *linear* size QCG representation of $\mathcal{F}$ is possible, but for which the smallest CRG representing $\mathcal{F}$ has size exponential in the number of agents (Theorem 17(b)); and
- there are resource definable $\mathcal{F}$ for which the smallest QCG that represents $\mathcal{F}$ is of size exponential in the number of goals in $\mathcal{F}$ (Theorem 17(c)).

We observe that by combining Theorem 15 with the first two results above it follow that not only are QCGs more expressive than CRGs (Theorem 15) but, in addition, QCGs can be exponentially more succinct than the smallest equivalent CRG (Theorem 17(b)) while, at worst, only quadratically larger than the smallest equivalent CRG (Theorem 17(a)). Formally, we have the following.

Theorem 17.

(a) *For all resource definable $\mathcal{F}$*

$$\mu^{\mathrm{QCG}}(\mathcal{F}) = \mathrm{O}\big(\big(\mu^{\mathrm{CRG}}(\mathcal{F})\big)^2\big)$$

(b) *There exist resource definable $\mathcal{F}$ for which*

$$\begin{aligned} \mu^{\mathrm{QCG}}(\mathcal{F}) &= \mathrm{O}(n) \\ \mu^{\mathrm{CRG}}(\mathcal{F}) &= \Omega\big(2^n\big) \end{aligned}$$

[9] This holds even if the formula basis comprises only the operations, $\{\wedge, \vee\}$ as has been demonstrated by the construction of Valiant [42].

(c) *There exist resource definable $\mathcal{F}$ for which,*

$$\mu^{\mathrm{QCG}}(\mathcal{F}) = \Omega\left(\frac{2^m}{\log m\sqrt{m}}\right)$$

Proof. (a) This is immediate from Theorem 14, noting that we have conservatively bounded the translation of $bt(n+m)$ (used in the minimal size CRG) to $\mathrm{O}(bt(n+m+b))$ (arising in the proof of Theorem 14) by over-compensating for the term b^2t to give $\mathrm{O}((bt(n+m))^2)$ as the upper bound.

(b) Our proof is constructive in that we present a specific resource-definable system with the properties claimed. For any $n \geqslant 1$, let Ag be a set of $2n$ agents

$$A \cup B = \{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\}$$

and the goal set G consist of a single goal: $G = \{g\}$. The system $\mathcal{F}_n^{\mathrm{PAIR}}$ is given by

$$\mathcal{F}_n^{\mathrm{PAIR}}(C) = \begin{cases} \{g\} & \text{if } \exists 1 \leqslant i \leqslant n\text{: } \{a_i, b_i\} \subseteq C \\ \emptyset & \text{otherwise} \end{cases}$$

First observe that $\mathcal{F}_n^{\mathrm{PAIR}}$ is represented by the QCG $Q = \langle \{g\}, Ag, G_1, \ldots, G_{2n}, \Psi\rangle$ with

$$\Psi\big(Ag, \{g\}\big) = \bigvee_{i=1}^{n} (a_i \wedge b_i) \vee \neg g$$

a propositional formula of size $2n+1$. Furthermore, since, from Fact 6(a), the sub-formula $\vee_{i=1}^{n}(a_i \wedge b_i)$ describes a monotone increasing propositional function of Ag, it follows from Theorem 16(b) that $\mathcal{F}_n^{\mathrm{PAIR}}$ is resource-definable.

Consider any $\Gamma \in [\mathcal{F}_n^{\mathrm{PAIR}}]^{\mathrm{CRG}}$, i.e., in the equivalence class of CRGs representing the system $\mathcal{F}_n^{\mathrm{PAIR}}$. We now show that $M^{\mathrm{CRG}}(\Gamma) \geqslant 2^{n-1}$ by arguing that any such CRG must employ a resource set of at least this size.

Let

$$R^{\Gamma} = \{r_1, r_2, \ldots, r_t\}$$

be the set of resources specified in Γ. For $a_i \in A$, $b_i \in B$ denote by $a_{i,k}$ and $b_{i,k}$ the endowment values $\mathbf{en}(a_i, r_k)$ and $\mathbf{en}(b_i, r_k)$. Similarly, we denote by g_k the requirement value $\mathbf{req}(g, r_k)$ in the CRG Γ.

It is easy to see that with each $C \subseteq A$ we can associate a *unique, maximal* subset $\mathit{unpaired}(C)$ of B with which

$$\mathcal{F}_n^{\mathrm{PAIR}}\big(C \cup \mathit{unpaired}(C)\big) = \emptyset$$

This set is given by,

$$\mathit{unpaired}(C) = \{b_i\text{: } a_i \notin C\}$$

We first observe that, since Γ represents the system $\mathcal{F}_n^{\mathrm{PAIR}}$ the following inequalities must hold,

$$\forall i,\ \forall k \quad a_{i,k} + b_{i,k} \geqslant g_k$$
$$\forall C \subseteq A,\ \exists k \quad \sum_{a_i \in C} a_{i,k} + \sum_{b_j \in \mathit{unpaired}(C)} b_{j,k} < g_k$$

For each $C \subseteq A$ define the subset $\mathit{fail}(C)$ of R^{Γ} by

$$\mathit{fail}(C) = \left\{ r_k\text{: } \sum_{a_i \in C} a_{i,k} + \sum_{b_j \in \mathit{unpaired}(C)} b_{j,k} < g_k \right\}$$

Suppose, for the sake of contradiction, that $|R^{\Gamma}| \leqslant 2^{n-1} - 1$. Then since there are 2^n subsets of A, it must be the case that there is some resource, r_k, that occurs in at least

$$\left\lceil \frac{2^n}{2^{n-1} - 1} \right\rceil = 3$$

different choices of $\mathit{fail}(C)$, i.e., there are distinct choices C, D, E as subsets of A for which

$$\exists r \in R^{\Gamma}\text{: } r \in \mathit{fail}(C) \cap \mathit{fail}(D) \cap \mathit{fail}(E)$$

We will show that we may choose two of these sets in $\{C, D, E\}$—X and Y say—with the property:

$$|X \setminus Y \cup Y \setminus X| = \big|(X \cup Y) \setminus (X \cap Y)\big| \geqslant 2$$

To see this, assume without loss of generality, that $|C| \geqslant |D| \geqslant |E|$ and that the difference in size between any two is at most 1: if we have $|X| - |Y| \geqslant 2$ then the property sought follows immediately. Suppose that both,

$$\big|(C \cup D) \setminus (C \cap D)\big| = 1$$
$$\big|(C \cup E) \setminus (C \cap E)\big| = 1$$

(since C, D, and E are distinct, neither size can be zero; if either case has size at least 2 then the property is already established).

From the assumption $|C| \geqslant |D| \geqslant |E|$ we deduce that,

$$C = D \cup \{a_i\} \quad \text{and} \quad C = E \cup \{a_j\}$$

where $a_i \notin D$ and $a_j \notin E$. Now,

$$\big(a_i \notin D \text{ and } C = E \cup \{a_j\}\big) \Rightarrow\ a_i \in E$$
$$\big(a_j \notin E \text{ and } C = D \cup \{a_i\}\big) \Rightarrow\ a_j \in D$$

$$D \neq E \Rightarrow a_i \neq a_j$$

so that, $a_i \in E \setminus D$ and $a_j \in D \setminus E$, i.e.,

$$\{a_i, a_j\} \subseteq D \setminus E \cup E \setminus D = (D \cup E) \setminus (D \cap E)$$

giving the property claimed of C, D, and E.

To summarise, we have two subsets—X and Y—of A and a resource $r_k \in R^{\Gamma}$ with which, $|X \setminus Y \cup Y \setminus X| \geqslant 2$, and

$$\text{P1.} \quad \sum_{a_i \in X} a_{i,k} + \sum_{b_j \in unpaired(X)} b_{j,k} < g_k$$
$$\text{P2.} \quad \sum_{a_i \in Y} a_{i,k} + \sum_{b_j \in unpaired(Y)} b_{j,k} < g_k$$

Again without loss of generality suppose, $\{a_1, a_2\} \subseteq (X \cup Y) \setminus (X \cap Y)$ and consider the set

$$X \cup unpaired(X) \cup Y \cup unpaired(Y)$$

From $a_1 \in (X \cup Y) \setminus (X \cap Y)$ we deduce that either $a_1 \notin X$ or $a_1 \notin Y$, and therefore from the definition of *unpaired*,

$$b_1 \in unpaired(X) \cup unpaired(Y)$$

Similarly, we deduce $b_2 \in unpaired(X) \cup unpaired(Y)$, so that in total

$$\text{P3.} \quad \{a_1, a_2, b_1, b_2\} \subseteq X \cup unpaired(X) \cup Y \cup unpaired(Y)$$

Combining P1 and P2 gives,

$$\sum_{a_i \in X \cup Y} a_{i,k} + \sum_{b_j \in unpaired(X) \cup unpaired(Y)} b_{j,k} < 2g_k$$

From P3 and the definition of the system $\mathcal{F}_n^{\text{PAIR}}$, however, we get in contradiction

$$\sum_{a_i \in X \cup Y} a_{i,k} + \sum_{b_j \in unpaired(X) \cup unpaired(Y)} b_{j,k} \geqslant a_{1,k} + b_{1,k} + a_{2,k} + b_{2,k} \geqslant 2g_k$$

In consequence the assumption that $\Gamma \in [\mathcal{F}_n^{\text{PAIR}}]^{\text{CRG}}$ may be chosen with $|R^{\Gamma}| < 2^{n-1}$ cannot be valid, and thus $\mu^{\text{CRG}}(\mathcal{F}_n^{\text{PAIR}}) = \Omega(2^n)$ as claimed.

(c) Theorem 16 indicates that each monotone propositional function of m arguments yields a resource definable set. The lower bound now follows via the counting argument of Riordan and Shannon [32], ([11, pp. 272–274])[10] and the fact that for $Q(m)$ the class of monotone propositional functions of m arguments,

$$\left|Q(m)\right| \geqslant 2^{\binom{m}{\lceil m/2 \rceil}} \qquad \square$$

5. Possible extensions of the CRG model

The model of CRGs as developed over the preceding sections makes a number of simplifying assumptions concerning the nature of agents and goals within the system. We now, briefly, consider one possible development of the basic model and highlight some specific problems that might arise with this.

Our model associates with each goal $g \in G$ and resource $r \in R$ a quantity of that resource which must be used in order to achieve the goal: the requirement function $\mathbf{req}: G \times R \to \mathbb{N}$. Given $R = \{r_1, \ldots, r_k\}$ we can present these requirements via a k-tuple,

$$\underline{\mathbf{req}}(g_i) = \big\langle \mathbf{req}(g_i, r_1), \ldots, \mathbf{req}(g_i, r_j), \ldots, \mathbf{req}(g_i, r_k)\big\rangle$$

describing a single "*profile*" of resource usage that suffices to bring about g_i, i.e., if $\underline{x} \in \mathbb{N}^k$ and $\underline{x} \geqslant \underline{\mathbf{req}(g_i)}$[11] then expending (for each j) x_j of resource r_j will realise g_i.

In a number of applications, particularly in negotiation contexts, it is possible that there are a number of *incomparable* methods of bringing about the *same* goal, e.g. with $R = \{r_1, r_2\}$ *both* of the profiles $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$ may be effective methods of realising g.

In summary, instead of the description $\Gamma = \langle Ag, G, R, G_1, \ldots, G_n, \mathbf{en}, \mathbf{req}\rangle$ presented in Definition 1, the component $\mathbf{req}$ describes for each $g \in G$ the set of *effective* profiles, $\Pi(g)$, any one of which would suffice to bring about g, i.e.,

$$\Pi(g) = \{\underline{x}\text{: expending, for each } j, x_j \text{ units of } r_j, \text{ realises } g\}$$

Let

$$\text{M-CRG} = \langle Ag, G, G_1, \ldots, G_n, R, \Pi_1, \Pi_2, \ldots, \Pi_m, \mathbf{en}\rangle$$

be a *multiple profile* CRG. One suitable description for a profile $\underline{x}$ would be simply to present $\underline{x}$ as a vector in $\mathbb{N}^k$. It is trivial to see that, using this enriched model and associated representation, the basic successful coalition problem remains NP-complete. We note that, in such multiple profile models and in contrast to the basic CRG model, the question of whether a given coalition $C \subseteq Ag$ has a sufficient collective endowment in order to bring about $G' \subseteq G$ could become rather more complicated, e.g. suppose $|\Pi(g)| = 2$ for each $g \in G'$ so that, in principle, there are $2^{|G'|}$ alternative mechanisms for realising G'. It may be that this decision problem can, in fact, be efficiently decided, (e.g., using suitable integer programming formulations), however this remains open.

Another obvious extension is to assume agents are in addition equipped with sets of "capabilities", modelled as functions *cap* which transform a multiset of resources (i.e., a bag of resources) into another multiset of resources. Here, we are close to the realm of production planning systems, and operational research techniques may be useful for understanding and reasoning about them. We note that one technical difficulty in understanding the complexity of reasoning about such systems lies in the representation of capabilities.

Finally, another possible extension would be to consider cases where an agent is interested in trying to *maximise* the number of goals it achieves. There is an obvious decision problem associated with this extension, where in addition to a CRG Γ and a coalition $C \subseteq Ag$, we are given a *target* $\tau_i \leqslant |G_i| \in \mathbb{N}$ for each agent $i \in C$, representing the number of goals that we desire i to accomplish. We are then asked whether there exists some $G' \in sf(C)$ such that for all $i \in C$, we have $|G' \cap G_i| \geqslant \tau_i$, i.e., whether there is some feasible goal set for C in which every agent $i \in C$ satisfies its target number of goals τ_i. It is easy to see that this problem is NP-complete, and hence no worse than the general SUCCESSFUL COALITION problem: SUCCESSFUL COALITION is the special case of the "target" problem where we set $\tau_i = 1$ for every $i \in C$.

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[10] In very informal terms, this gives a lower bound on the formula size of propositional functions belonging to "large enough" classes $Q(n)$, the bound being expressed in terms of $|Q(n)|$.

[11] For $\underline{v}$. and $\underline{w}$ in $\mathbb{N}^k$ we write $\underline{v} \geqslant \underline{w}$ whenever $v_i \geqslant w_i$ for each $1 \leqslant i \leqslant k$.

6. Discussion and related work

Before discussing related work, it may be worth clarifying the extent to which CRGs (and indeed QCGs) may be considered as *games*, as opposed to simply *optimisation problems* [29]. For example, one domain for studying coordination in multi-agent systems, which is currently receiving much attention, is the *distributed constraint optimisation* (DCOP) scenario of Modi et al. [25]. In DCOP, each agent in a system is assumed to control a set of variables; the goal is for the agents to assign values to their variables in such a way that some overall objective function over these variables is minimised. DCOP problems are not usefully considered as games because there are no *strategic* concerns: every agent benevolently tries to minimise the global objective function, and there is no concept of self interest, in the sense of an agent attempting to do the best for itself. DCOP problems represent an important and useful abstraction of many coordinated problem solving settings [13], whence the current level of interest.

In QCGs and CRGs, however, the issue is not simply one of optimisation, in the sense of (for example) minimising resource usage. While such considerations may come into play, they are *secondary* to an agent's primary goal of *satisfying at least one of its goals*. In order to accomplish one of its goals, an agent must typically cooperate with other agents because it does not have sufficient resources on its own. Strategic considerations arise in CRGs because an agent must decide *which* other agents to cooperate with, and in doing so, must consider that these other agents will also engage in reasoning of the same type.

With respect to work that relates to CRGs and QCGs in general, we note that [47] provides a detailed review, and also give more detailed references to the role of cooperative games and coalition formation in the multi-agent systems community. Shehory and Kraus developed algorithms for coalition structure formation in which agents were modelled as having different capabilities, and were assumed to benevolently desire some overall task to be accomplished, where this task had some complex (plan-like) structure [37–39]. Sandholm et al [34] investigate the coalition structure generation problem (i.e., the problem of partitioning a set of agents into teams). Conitzer and Sandholm also investigated the complexity of determining non-emptiness of the core [8], while Bilbao and colleagues survey the complexity of a number of problems in cooperative game theory [3], and Tennenholtz and Moses investigated the *cooperative goal achievement* problem [26,41], showing it to be PSPACE-complete.

One of the concerns that pre-occupies both this paper and its predecessor [47] is that of *succinctly representing coalitional games*. Recall that the classic model of a coalitional game (with transferable payoff) is as a pair $\langle Ag, \nu\rangle$, where Ag is a set of agents and ν is a characteristic function, $\nu : 2^{Ag} \to \mathbb{R}$, which assigns to every possible coalition a numeric value, the idea being that this value can then be distributed between members of the coalition [27, p. 257]. If we wish to consider the computational problems associated with reasoning about such games, (such as determining whether the core of a given game is non-empty), then the question of how to *represent* the game, and in particular, the characteristic function ν, becomes extremely significant. The main issue is that a naive, *extensive* representation of ν—as a set of pairs $\{(C, x) \mid C \subseteq Ag, x = \nu(C)\}$—is both infeasible and entirely unrealistic, because it will be of size exponential in the number of agents. While we may get results indicating that for this representation, a particular problem is tractable, the assumptions upon which these results are based—an unworkable representation of characteristic functions—render such results meaningless [10, p. 258].[12]

Faced with this problem, there are two obvious lines of attack:

- Give the characteristic function a *specific interpretation in terms of combinatorial structures*. This is the approach adopted in [10,17], and indeed the present paper. The advantage of such an approach is that the representation can always be guaranteed to be succinct; the disadvantage is that not all characteristic functions can be represented.
- Try to find a *succinct general representation* for ν, i.e., a succinct way of representing characteristic functions in general. For example, this is roughly the approach adopted in [47], where a representation of QCG characteristic functions based on formulae of propositional logic was proposed. This representation is indeed general (in that it can completely capture all QCG characteristic functions), although it is not always guaranteed to be succinct [47, pp. 36–37].

[12] Similar issues of course arise in many areas of AI, such as planning [16, p. 59].

With respect to the former approach, Deng and Papadimitriou undertook arguably the first systematic investigation of the complexity of solution concepts in coalitional games [10]. They used a representation based on weighted graphs. To represent a coalitional game with agents Ag, they used an undirected graph on Ag, with integer weights $w_{i,j}$ between nodes $i, j \in Ag$. The value of a coalition C was then defined to be $\sum_{\{i,j\}\subseteq C} w_{i,j}$, i.e., the value of a coalition $C \subseteq Ag$ is defined to be the weight of the subgraph induced by C. Given this representation, Deng and Papadimitriou showed that the problem of determining emptiness of the core was NP-complete, while the problem of checking whether a specific imputation was in the core of such a game was co-NP-complete [10, p. 260]; they also showed that these problems could be solved in polynomial time for graphs with non-negative weights [10, p. 261].

With respect to the latter approach, Conitzer and Sandholm consider a modular representation of coalitional games, where a characteristic function is represented as a collection of sub-games [9]; under this representation, they showed that checking non-emptiness of the core is co-NP-complete. In related work, Ieong and Shoham propose a representation of coalitional games called *marginal contribution nets* [17]. In this representation, a characteristic function over a set Ag of agents is represented as a set of rules, with the structure

$$\text{pattern} \longrightarrow \text{value}$$

The pattern is a conjunction of agents, and such a rule is said to *apply* to a group of agents S if S is a superset of the agents in the pattern conjunction. The value of a coalition in the marginal contribution network representation is then the sum over the values of all the rules that apply to the coalition. The following example, from [17], illustrates this in more detail. Suppose the marginal contribution net is defined by the following rules:

$$a \wedge b \longrightarrow 5$$

$$b \longrightarrow 2$$

Then given this representation, we would have $\nu(\{a\}) = 0$, $\nu(\{b\}) = 2$, and $\nu(\{a, b\}) = 7$. Ieong and Shoham show that, under this representation, checking whether an imputation is in the core is co-NP-complete, while checking whether the core is non-empty is co-NP-hard.[13] They also show that their representation can capture that of Conitzer and Sandholm [9].

Moving more specifically to work that is related to the resource-based framework of CRGs, we note some similarity between our ideas and that of the classic model of a *market economy* [27, p. 260]. In a market economy, there are assumed to be k different goods, each agent i is endowed with some number of each different good, and each agent is also associated with a production function, which transforms bundles of goods into other bundles of goods. The classic question studied with respect to exchange economies is that of finding a *competitive equilibrium*, i.e., a feasible allocation of bundles of goods to agents at a particular price such that this allocation maximises each agent's profit, with respect to it's production function, and the respective costs of input and output goods. Wellman's seminal WALRAS system used an auction approach to compute competitive equilibria in a computational market framework [45]. Although there are similarities between CRGs and market economies, there is no distinction in the classic model of a market economy between goals and resources: this framework works well when the domain at hand fits with market model of prices and transferrable goods, but is less well suited to goal-oriented systems, or systems where goods are not transferrable.

We also note some similarities between our motivations and those of Chakrabati et al on *resource interfaces* [7]. The idea of a resource interface is to represent the resources required by a particular process over time. The motivating example used by Chakrabati is that of the power requirements of a process controlling part of a mobile robot. In order to see whether it is possible to combine two such processes, we try to find a winning strategy to schedule execution of the two processes such that the combined processes never exceed a particular resource bound. There are clearly similarities between the two models, although the concerns are rather different; one key idea in the work of Chakrabati et al is that of resource requirements over time—see conclusions below.

We should also point to the work of von Martial on the first-order axiomatisation of a coordination theory for multi-agent plans [43]. von Martial explicitly allows for multiple resources, for agents to be endowed with different quantities of each resource, and for actions to require the consumption of different quantities of resources, very much as in CRGs [43, pp. 90–94]. However, von Martial's main focus was on the axiomatisation of his theory: he did not

[13] As noted in [17] whether non-emptiness of the core can be carried out by a co-NP algorithm using their representation is an open problem.

investigate any computational properties of his model. Related work by Buzing et al. [6] develops a more rigorous model of multi-agent coordination, and considers the computational complexity of coordination in this setting. The model of Buzing et al allows for resources ("capabilities" in their parlance). The emphasis is again slightly different, as they do not focus on the coalitional game properties of their model.

7. Conclusions

We have presented Coalitional Resource Games, (CRGs), a specialisation of Qualitative Coalitional Games (QCGs) built around the notion of expending resources to accomplish goals. We investigated the computational complexity of a range of natural decision problems associated with such games, and also investigated the extent to which CRGs and QCGs could be translated to one-another.

There are many issues for future work. Perhaps the most interesting, suggested by the work of [7], and also by emerging work on dynamic coalition formation [22], is that of games (QCGs and CRGs) played over a period of time. It would be interesting to see what analogues our concepts have in such games, and how efficiently solutions could be computed.

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